Correlation-based imaging in random media

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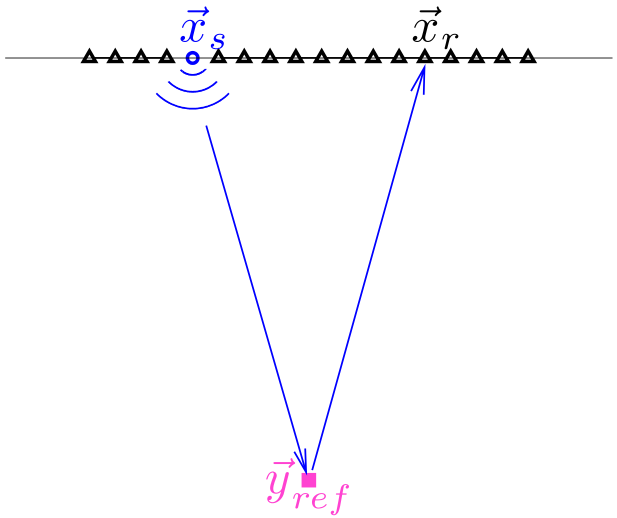
In this talk: correlation-based imaging is useful when the medium is scattering.

Conventional reflector imaging through a homogeneous medium



• Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}_{\rm ref}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver. Measured data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \ldots, N_{\rm r}, s = 1, \ldots, N_{\rm s}\}$.

• Mathematical model:

$$\Big(\frac{1}{c_0^2} + \frac{1}{c_{\rm ref}^2} \mathbf{1}_{B_{\rm ref}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}_{\rm ref}) \Big) \frac{\partial^2 u}{\partial t^2}(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\rm s}) - \Delta_{\vec{\boldsymbol{x}}} u(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\rm s}) = f(t) \delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_{\rm s})$$

• Purpose of imaging: using the measured data, build an imaging function $\mathcal{I}(\vec{\boldsymbol{y}}^S)$ that would ideally look like $\frac{1}{c_{\rm ref}^2} \mathbf{1}_{B_{\rm ref}}(\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{y}}_{\rm ref})$, in order to extract the relevant information $(\vec{\boldsymbol{y}}_{\rm ref}, B_{\rm ref}, c_{\rm ref})$ about the reflector.

- Classical imaging functions:

1) Least-Squares imaging: minimize the quadratic misfit between measured data and synthetic data obtained by solving the wave equation with a candidate $(\vec{\boldsymbol{y}}, B, c)_{\rm test}$.

2) Linearized Least-Squares imaging: simplify Least-Squares imaging by "linearization" of the forward problem (Born).

3) Reverse Time imaging: simplify Linearized Least-Squares imaging by forgetting the normal operator.

4) Kirchhoff Migration: simplify Reverse Time imaging by substituting travel time migration for full wave equation.

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4) Kirchhoff Migration: simplify Reverse Time imaging by substituting travel time migration for full wave equation.

• Kirchhoff Migration function:

$$\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S) = \sum_{r=1}^{N_{\rm r}} \sum_{s=1}^{N_{\rm s}} u\Big(\frac{|\vec{\boldsymbol{x}}_s - \vec{\boldsymbol{y}}^S|}{c_0} + \frac{|\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}_r|}{c_0}, \vec{\boldsymbol{x}}_r ; \vec{\boldsymbol{x}}_s \Big)$$

It forms the image with the superposition of the backpropagated traces.
$|\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}|/c_0$ is the travel time from $\vec{\boldsymbol{x}}$ to $\vec{\boldsymbol{y}}^S$.

[1] H. Ammari, J. Garnier, and K. Sølna, *Waves in Random and Complex Media* **22**, 40 (2012).

Kirchhoff Migration:

$$\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S) = \sum_{r=1}^{N_{\rm r}} \sum_{s=1}^{N_{\rm s}} u\big(\mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_r), \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s\big)$$

- Resolution analysis:
- Lateral resolution: $\lambda L/a$, where λ is the central wavelength, L is the distance from the array to the reflector, and a is the array diameter (paraxial regime $\lambda \ll a \ll L$).
- Range resolution: c_0/B, where c_0 is the background velocity and B is the bandwidth.

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- Range resolution: c_0/B, where c_0 is the background velocity and B is the bandwidth.

- Stability analysis:
- Very robust with respect to additive measurement noise [1].
- Sensitive to medium noise: If the medium is scattering, then Kirchhoff Migration (usually) does not work.

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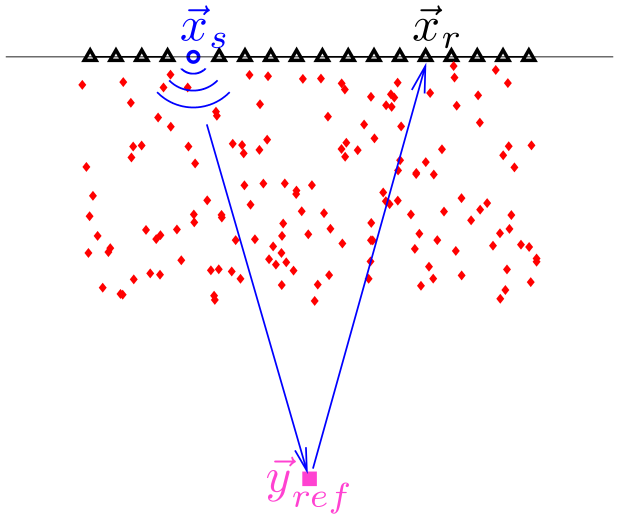
Conventional reflector imaging through a scattering medium



- Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}_{\rm ref}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.

Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \dots, N_{\rm r}, s = 1, \dots, N_{\rm s}\}$.

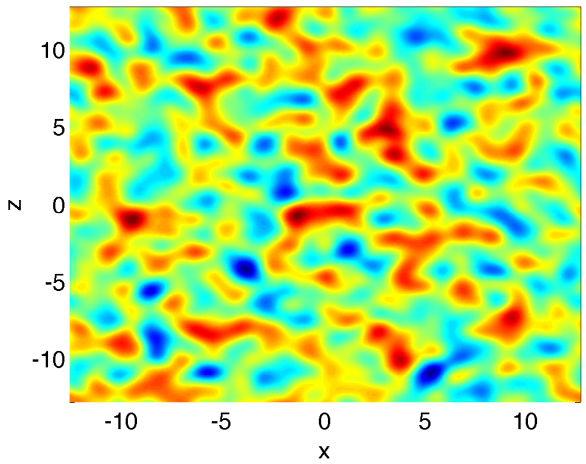
$$\Big(\frac{1}{c^2(\vec{\boldsymbol{x}})} + \frac{1}{c_{\rm ref}^2} \mathbf{1}_{B_{\rm ref}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}_{\rm ref}) \Big) \frac{\partial^2 u}{\partial t^2}(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\rm s}) - \Delta_{\vec{\boldsymbol{x}}} u(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\rm s}) = f(t)\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_{\rm s})$$

- Random medium model:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_0^2}\big(1 + \mu(\vec{\boldsymbol{x}})\big)$$

c_0 is a reference speed,

$\mu(\vec{\boldsymbol{x}})$ is a zero-mean random process.



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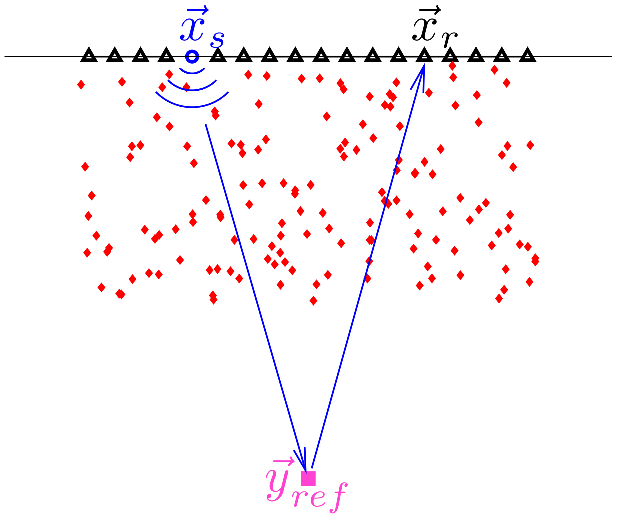

• Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}_{\rm ref}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.
Data: $\{\hat{u}(\omega, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \dots, N_{\rm r}, s = 1, \dots, N_{\rm s}\}$.

$$\omega^2 \Big(\frac{1}{c^2(\vec{\boldsymbol{x}})} + \frac{1}{c_{\rm ref}^2} \mathbf{1}_{B_{\rm ref}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}_{\rm ref}) \Big) \hat{u}(\omega, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\rm s}) + \Delta_{\vec{\boldsymbol{x}}} \hat{u}(\omega, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\rm s}) = -\hat{f}(\omega) \delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_{\rm s})$$

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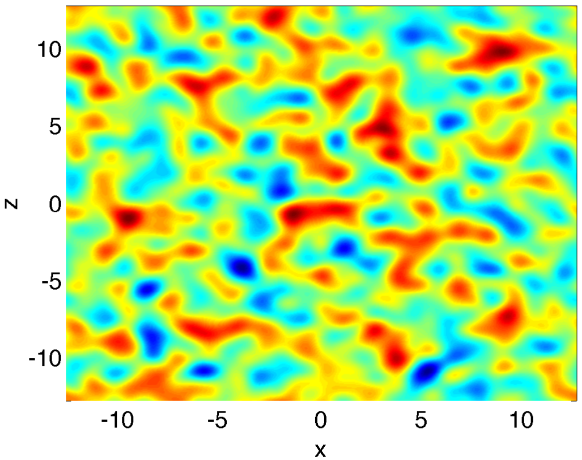

Strategy: Stochastic and multiscale analysis

- Remark: The medium noise $\hat{u} - \hat{u}_0$ (where $\hat{u}_0$ is the data that would be obtained in a homogeneous medium) is very different from an additive measurement noise !

- A detailed analysis is possible in different regimes of separation of scales (small wavelength, large propagation distance, small correlation length, ...).

$\hookrightarrow$ Analysis of the moments of $\hat{u}$.

- Compute the mean and variance of an imaging function $\mathcal{I}(\vec{\boldsymbol{y}}^S)$.

$\hookrightarrow$ resolution and stability analysis.

- The resolution analysis of the mean imaging function $\mathbb{E}\big[\mathcal{I}(\vec{\boldsymbol{y}}^S)\big]$ gives lateral and range resolutions.
- Criterium for statistical stability:

$$\text{SNR} := \frac{\mathbb{E}\big[\mathcal{I}(\vec{\boldsymbol{y}}^S)\big]}{\text{Var}\big(\mathcal{I}(\vec{\boldsymbol{y}}^S)\big)^{1/2}} > 1$$

$\hookrightarrow$ design the imaging function to get good trade-off between stability and resolution.

• General results obtained by a stochastic analysis:

• The mean (coherent) wave is small.

$\Longrightarrow$ The Kirchhoff Migration function (or Reverse Time imaging function) is unstable in randomly scattering media.

$$\frac{\mathbb{E}\left[\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S)\right]}{\mathrm{Var}\left(\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S)\right)^{1/2}} \ll 1$$

• The wave fluctuations at nearby points and nearby frequencies are correlated.

The wave correlations carry information about the medium and the reflector.

$\Longrightarrow$ One should use local cross correlations for imaging.

Wave propagation in the random paraxial regime

- Consider the time-harmonic form of the scalar wave equation ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$)

$$(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}\big(1 + \mu(\boldsymbol{x}, z)\big)\hat{u} = 0.$$

Consider the paraxial regime "$\lambda \ll l_c \ll L$":

$$\omega \to \frac{\omega}{\varepsilon^4}, \qquad \mu(\boldsymbol{x}, z) \to \varepsilon^3 \mu\big(\frac{\boldsymbol{x}}{\varepsilon^2}, \frac{z}{\varepsilon^2}\big).$$

The function $\hat{\phi}^\varepsilon$ (slowly-varying envelope of a plane wave) defined by

$$\hat{u}^\varepsilon(\omega, \boldsymbol{x}, z) = e^{i\frac{\omega z}{\varepsilon^4 c_0}} \hat{\phi}^\varepsilon\big(\omega, \frac{\boldsymbol{x}}{\varepsilon^2}, z\big)$$

satisfies

$$\varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \left(2i\frac{\omega}{c_0}\partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_0^2}\frac{1}{\varepsilon}\mu\big(\boldsymbol{x}, \frac{z}{\varepsilon^2}\big)\hat{\phi}^\varepsilon\right) = 0.$$

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- In the regime $\varepsilon \ll 1$, the forward-scattering approximation in direction z is valid and $\hat{\phi} = \lim_{\varepsilon \to 0} \hat{\phi}^\varepsilon$ satisfies the Itô-Schrödinger equation [1]

$$2i\frac{\omega}{c_0}\partial_z \hat{\phi} + \Delta_\perp \hat{\phi} + \frac{\omega^2}{c_0^2}\dot{B}(\boldsymbol{x}, z)\hat{\phi} = 0$$

with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}') \min(z, z')$,
$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\boldsymbol{0}, 0)\mu(\boldsymbol{x}, z)]dz$.

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$$d\hat{\phi} = \frac{ic_0}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_0}\hat{\phi} \circ dB(\boldsymbol{x}, z)$$

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with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}') \min(z, z')$,
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- We introduce the fundamental solution $\hat{G}\big(\omega, (\boldsymbol{x}, z), (\boldsymbol{x}_0, z_0)\big)$:

$$d\hat{G} = \frac{ic_0}{2\omega}\Delta_\perp \hat{G} dz + \frac{i\omega}{2c_0}\hat{G} \circ dB(\boldsymbol{x}, z)$$

starting from $\hat{G}\big(\omega, (\boldsymbol{x}, z = z_0), (\boldsymbol{x}_0, z_0)\big) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$.

- In a homogeneous medium ($B \equiv 0$) the fundamental solution is

$$\hat{G}_0\big(\omega, (\boldsymbol{x}, z), (\boldsymbol{x}_0, z_0)\big) = \frac{\exp\left(\frac{i\omega|\boldsymbol{x}-\boldsymbol{x}_0|^2}{2c_0|z-z_0|}\right)}{2i\pi c_0 \frac{|z-z_0|}{\omega}}.$$

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- In a random medium,

$$\mathbb{E}\big[\hat{G}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\big] = \hat{G}_0\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\exp\Big(-\frac{\gamma(\mathbf{0})\omega^2|z-z_0|}{8c_0^2}\Big),$$

where $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty}\mathbb{E}[\mu(\mathbf{0},0)\mu(\boldsymbol{x},z)]dz$.

- Strong damping of the coherent wave if $|z-z_0| > z_{\rm sca} := 8c_0^2/(\gamma(\mathbf{0})\omega^2)$.

$\Longrightarrow$ Coherent imaging methods (such as Kirchhoff migration) fail.

- In a random medium,

$$\mathbb{E}\Big[\hat{G}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{G}\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\Big]$$
$$= \hat{G}_0\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{G}_0\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\exp\Big(-\frac{\gamma_2(\boldsymbol{x}-\boldsymbol{x}')\omega^2|z-z_0|}{4c_0^2}\Big),$$

where $\gamma_2(\boldsymbol{x})=\int_0^1\gamma(\mathbf{0})-\gamma(\boldsymbol{x}s)ds$ (note $\gamma_2(\mathbf{0})=0$).
If $|z-z_0|>z_{\rm sca}:=8c_0^2/(\gamma(\mathbf{0})\omega^2)$, then

$$\mathbb{E}\Big[\hat{G}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{G}\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\Big]$$
$$\simeq \hat{G}_0\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{G}_0\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\exp\Big(-\frac{\bar{\gamma}_2\omega^2|z-z_0|}{12c_0^2}|\boldsymbol{x}-\boldsymbol{x}'|^2\Big),$$

where $\gamma(\boldsymbol{x})=\gamma(\mathbf{0})-\bar{\gamma}_2|\boldsymbol{x}|^2+o(|\boldsymbol{x}|^2)$ for small $|\boldsymbol{x}|$.

- The fields at nearby points (closer than $X_{\rm c}:=\sqrt{12}c_0/(\sqrt{\bar{\gamma}_2}\omega)$) are correlated.
- Same results in frequency: The fields at nearby frequencies are correlated.

$\Longrightarrow$ One should migrate local cross correlations for imaging.

- In a random medium,

$$\mathbb{E}\big[\hat{G}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{G}\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\big]$$
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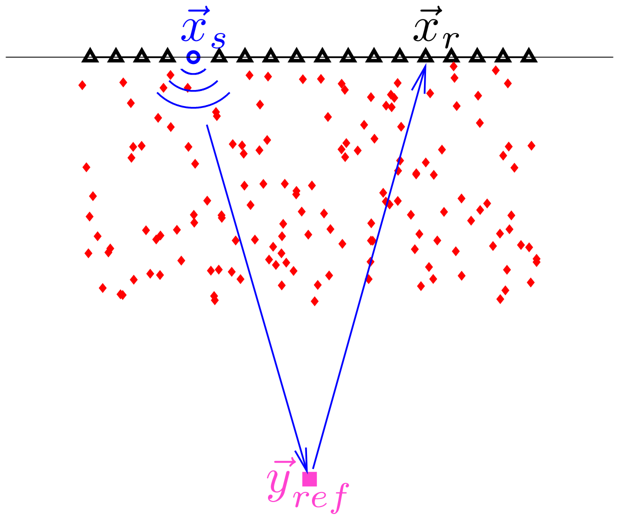
- In a random medium, one can write a closed-form equation for the n-th order moment.

Depending on the statistics of the random medium, the wave fluctuations may have Gaussian statistics or not [1].

[1] J. Garnier and K. Sølna, Comm. Part. Differ. Equat. 39 (2014), 626.

Imaging through a scattering medium



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Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \ldots, N_{\rm r}, s = 1, \ldots, N_{\rm s}\}$.

If the medium is scattering, then Kirchhoff migration does not work:

$$\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S) = \sum_{s=1}^{N_{\rm s}} \sum_{r=1}^{N_{\rm r}} u\Big(\frac{|\vec{\boldsymbol{x}}_s - \vec{\boldsymbol{y}}^S|}{c_0} + \frac{|\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}_r|}{c_0}, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s\Big)$$

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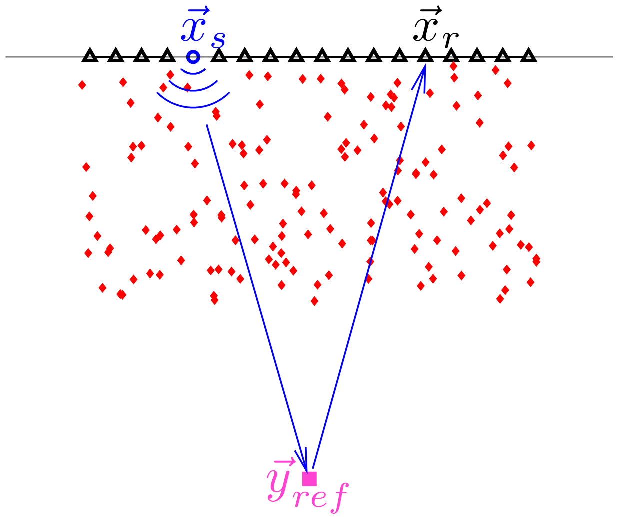

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$$\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S) = \sum_{s=1}^{N_{\rm s}}\sum_{r=1}^{N_{\rm r}}\int \overline{\hat{u}(\omega,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)}\exp\Big\{i\omega\Big[\frac{|\vec{\boldsymbol{x}}_s-\vec{\boldsymbol{y}}^S|}{c_0}+\frac{|\vec{\boldsymbol{y}}^S-\vec{\boldsymbol{x}}_r|}{c_0}\Big]\Big\}d\omega$$

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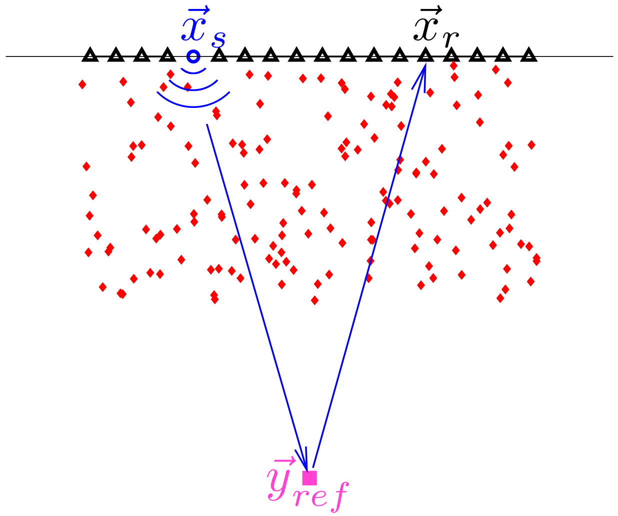


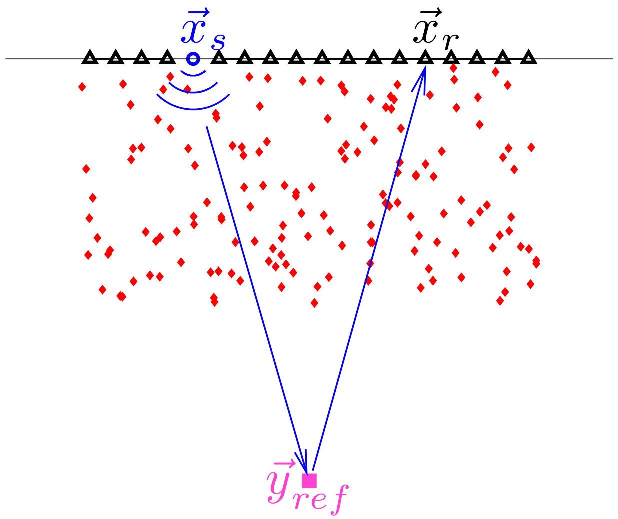
Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}_{\rm ref}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.
Data: $\{u(t,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s), r=1,\ldots,N_{\rm r}, s=1,\ldots,N_{\rm s}\}$.

If the medium is scattering, then full migration of cross correlations does not work:

$$\begin{aligned}
\mathcal{I}_{\rm fullCC}(\vec{\boldsymbol{y}}^S) &= \sum_{s,s'=1}^{N_{\rm s}} \sum_{r,r'=1}^{N_{\rm r}} \iint d\omega d\omega' \, \hat{u}(\omega,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)\overline{\hat{u}(\omega',\vec{\boldsymbol{x}}_{r'};\vec{\boldsymbol{x}}_{s'})} \\
&\quad \times \exp\Big\{ -i\omega\Big[\frac{|\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{y}}^S|}{c_0}+\frac{|\vec{\boldsymbol{x}}_s-\vec{\boldsymbol{y}}^S|}{c_0}\Big] + i\omega'\Big[\frac{|\vec{\boldsymbol{x}}_{r'}-\vec{\boldsymbol{y}}^S|}{c_0}+\frac{|\vec{\boldsymbol{x}}_{s'}-\vec{\boldsymbol{y}}^S|}{c_0}\Big]\Big\} \\
&= \Big| \sum_{s=1}^{N_{\rm s}} \sum_{r=1}^{N_{\rm r}} \int \overline{\hat{u}(\omega,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)} \exp\Big\{ i\omega\Big[\frac{|\vec{\boldsymbol{x}}_s-\vec{\boldsymbol{y}}^S|}{c_0}+\frac{|\vec{\boldsymbol{y}}^S-\vec{\boldsymbol{x}}_r|}{c_0}\Big]\Big\} d\omega \Big|^2 = \big|\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S)\big|^2
\end{aligned}$$

If one migrates all cross correlations, one gets the same image as with Kirchhoff !

Imaging through a scattering medium



Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}_{\rm ref}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.
Data: $\{u(t,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s), r=1,\ldots,N_{\rm r}, s=1,\ldots,N_{\rm s}\}$.

If the medium is scattering, then use Coherent Interferometric Imaging (CINT):

$$\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S) = \sum_{\substack{s,s'=1\\ |\vec{\boldsymbol{x}}_s-\vec{\boldsymbol{x}}_{s'}|\leq X_{\rm d}}}^{N_{\rm s}} \sum_{\substack{r,r'=1\\ |\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{x}}_{r'}|\leq X_{\rm d}}}^{N_{\rm r}} \iint_{|\omega-\omega'|\leq\Omega_{\rm d}} d\omega d\omega'\, \hat{u}(\omega,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)\overline{\hat{u}(\omega',\vec{\boldsymbol{x}}_{r'};\vec{\boldsymbol{x}}_{s'})}$$

$$\times \exp\Big\{-i\omega\Big[\frac{|\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{y}}^S|}{c_0}+\frac{|\vec{\boldsymbol{x}}_s-\vec{\boldsymbol{y}}^S|}{c_0}\Big]+i\omega'\Big[\frac{|\vec{\boldsymbol{x}}_{r'}-\vec{\boldsymbol{y}}^S}{c_0}+\frac{|\vec{\boldsymbol{x}}_{s'}-\vec{\boldsymbol{y}}^S)|}{c_0}\Big]\Big\}$$

It forms the image with the superposition of the backpropagated local cross correlations of the traces.

[1] L. Borcea, J. Garnier, G. Papanicolaou, and C. Tsogka, *Inverse Problems* **27**, 085004 (2011).

Coherent Interferometric Imaging (CINT):

$$\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S) = \sum_{\substack{s,s'=1 \\ |\vec{\boldsymbol{x}}_s - \vec{\boldsymbol{x}}_{s'}| \leq X_{\rm d}}}^{N_{\rm s}} \sum_{\substack{r,r'=1 \\ |\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{x}}_{r'}| \leq X_{\rm d}}}^{N_{\rm r}} \iint_{|\omega-\omega'|\leq \Omega_{\rm d}} d\omega d\omega' \, \hat{u}(\omega, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s) \overline{\hat{u}(\omega', \vec{\boldsymbol{x}}_{r'}; \vec{\boldsymbol{x}}_{s'})}$$

$$\times \exp\Big\{ - i\omega \big[\mathcal{T}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S) \big] + i\omega' \big[\mathcal{T}(\vec{\boldsymbol{x}}_{r'}, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_{s'}, \vec{\boldsymbol{y}}^S) \big] \Big\}$$

- Resolution analysis:

Lateral resolution: $\lambda L / X_{\rm d}$ (for $X_{\rm d} < a$, where a is the array diameter).

Range resolution: $c_0 / \Omega_{\rm d}$ (for $\Omega_{\rm d} < B$, where B is the bandwidth).

[1] L. Borcea, J. Garnier, G. Papanicolaou, and C. Tsogka, *Inverse Problems* **27**, 085004 (2011).

Coherent Interferometric Imaging (CINT):

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$$\times \exp\Big\{ -i\omega\big[\mathcal{T}(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{x}}_s,\vec{\boldsymbol{y}}^S)\big] + i\omega'\big[\mathcal{T}(\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{x}}_{s'},\vec{\boldsymbol{y}}^S)\big]\Big\}$$

- Resolution analysis:

Lateral resolution: $\lambda L/X_{\rm d}$ (for $X_{\rm d} < a$, where a is the array diameter).
Range resolution: $c_0/\Omega_{\rm d}$ (for $\Omega_{\rm d} < B$, where B is the bandwidth).

- Statistical stability:

$$\mathrm{SNR}_{\rm CINT} := \frac{\mathbb{E}\big[\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S)\big]}{\mathrm{Var}\big(\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S)\big)^{1/2}} > 1 \text{ when } \frac{X_{\rm d}}{X_{\rm c}} < 1, \ \frac{a}{X_{\rm c}} > 1 \text{ and/or } \frac{\Omega_{\rm d}}{\Omega_c} < 1, \ \frac{B}{\Omega_{\rm c}} > 1$$

where $X_{\rm c}$ is the decoherence length (distance between sensors beyond which the signals are not correlated) and $\Omega_{\rm c}$ is the decoherence frequency (frequency gap beyond which the frequency components of the recorded signals are not correlated).

[1] L. Borcea, J. Garnier, G. Papanicolaou, and C. Tsogka, *Inverse Problems* **27**, 085004 (2011).

Coherent Interferometric Imaging (CINT):

$$\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S) = \sum_{\substack{s,s'=1 \\ |\vec{\boldsymbol{x}}_s-\vec{\boldsymbol{x}}_{s'}|\leq X_{\rm d}}}^{N_{\rm s}} \sum_{\substack{r,r'=1 \\ |\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{x}}_{r'}|\leq X_{\rm d}}}^{N_{\rm r}} \iint_{|\omega-\omega'|\leq\Omega_{\rm d}} d\omega d\omega' \, \hat{u}(\omega,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)\overline{\hat{u}(\omega',\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{x}}_{s'})}$$

$$\times \exp\Big\{ -i\omega\big[\mathcal{T}(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{x}}_s,\vec{\boldsymbol{y}}^S)\big] + i\omega'\big[\mathcal{T}(\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{x}}_{s'},\vec{\boldsymbol{y}}^S)\big]\Big\}$$

- Resolution analysis:

Lateral resolution: $\lambda L/X_{\rm d}$ (for $X_{\rm d} < a$, where a is the array diameter).
Range resolution: $c_0/\Omega_{\rm d}$ (for $\Omega_{\rm d} < B$, where B is the bandwidth).

- Statistical stability:

$${\rm SNR}_{\rm CINT} := \frac{\mathbb{E}\big[\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S)\big]}{{\rm Var}\big(\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S)\big)^{1/2}} > 1 \text{ when } \frac{X_{\rm d}}{X_{\rm c}} < 1, \ \frac{a}{X_{\rm c}} > 1 \text{ and/or } \frac{\Omega_{\rm d}}{\Omega_c} < 1, \ \frac{B}{\Omega_{\rm c}} > 1$$

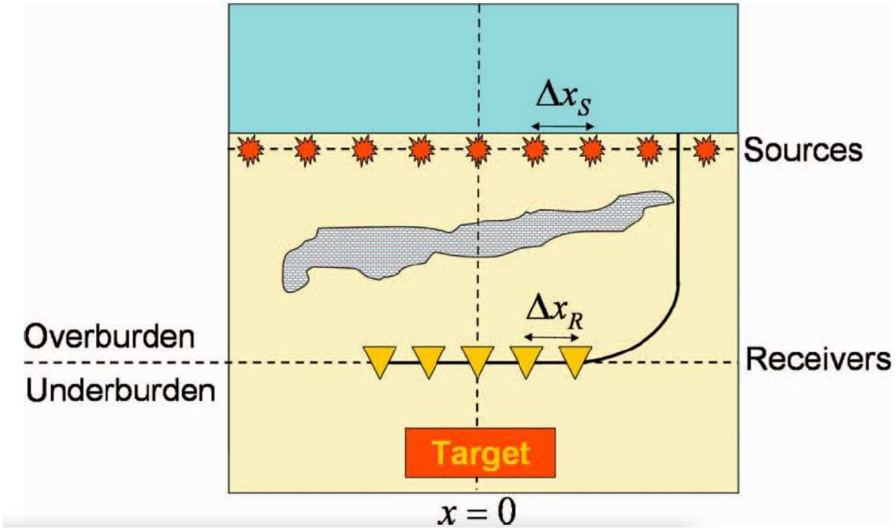
where $X_{\rm c}$ is the decoherence length (distance between sensors beyond which the signals are not correlated) and $\Omega_{\rm c}$ is the decoherence frequency (frequency gap beyond which the frequency components of the recorded signals are not correlated).

- Optimal values $\Omega_{\rm d} = \Omega_{\rm c}$ and $X_{\rm d} = X_{\rm c}$. They can be determined by

- a statistical analysis of the data.
- an adaptive procedure minimizing a suitable norm of the image.
- a good a priori choice !

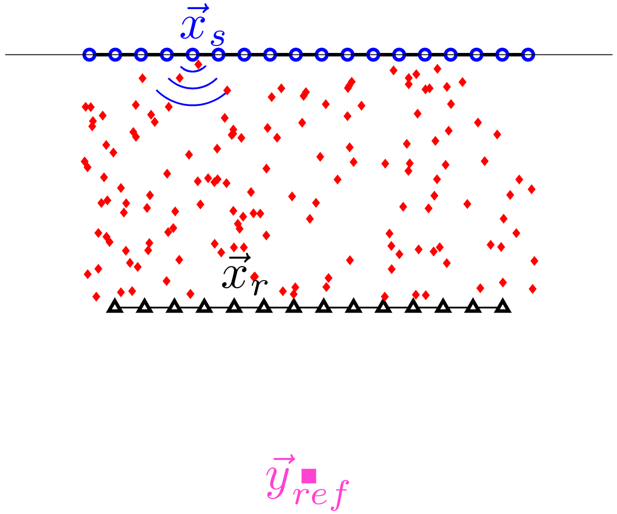
[1] L. Borcea, J. Garnier, G. Papanicolaou, and C. Tsogka, *Inverse Problems* **27**, 085004 (2011).

Imaging below an "overburden"



From van der Neut and Bakulin (2009)

Imaging below an overburden

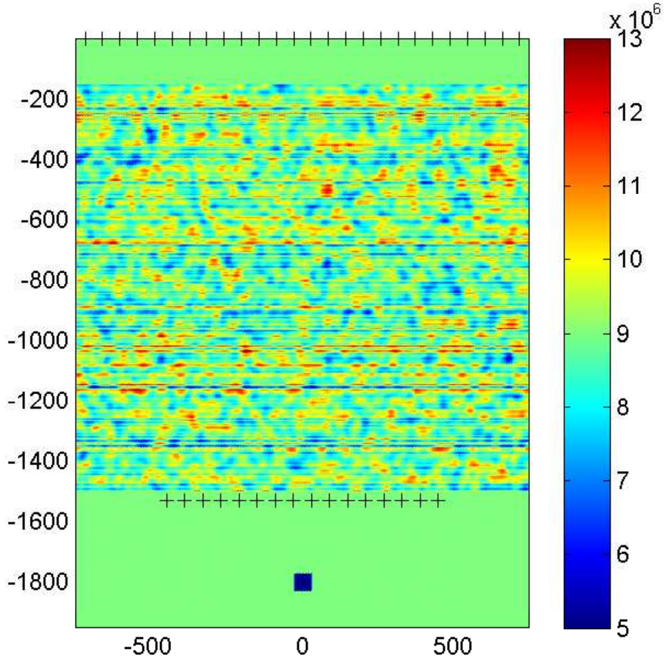


Array imaging of a reflector at $\vec{\boldsymbol{y}}_{\mathrm{ref}}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver located below the scattering medium. Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \dots, N_{\mathrm{r}}, s = 1, \dots, N_{\mathrm{s}}\}$.

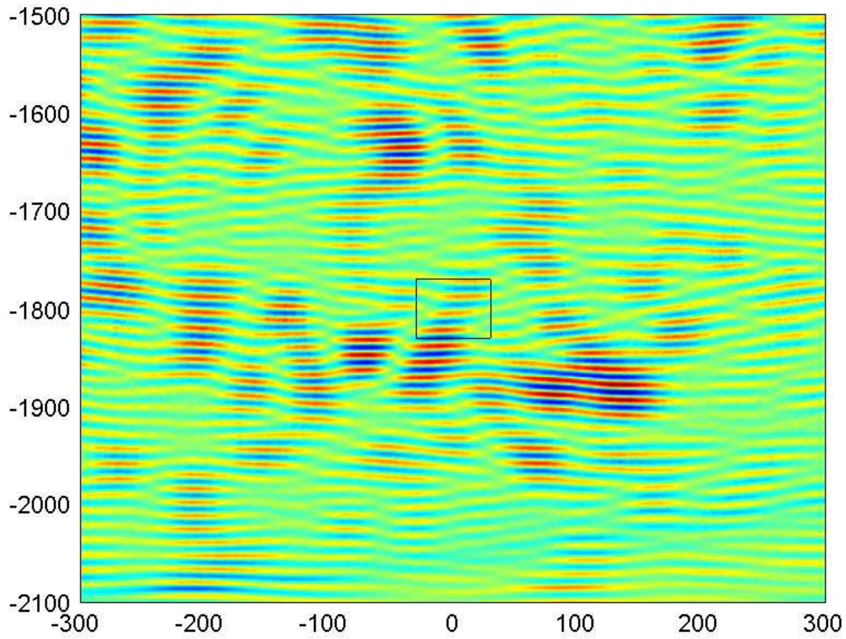
If the overburden is scattering, then Kirchhoff Migration does not work:

$$\mathcal{I}_{\mathrm{KM}}(\vec{\boldsymbol{y}}^S) = \sum_{r=1}^{N_{\mathrm{r}}} \sum_{s=1}^{N_{\mathrm{s}}} u\Big(\frac{|\vec{\boldsymbol{x}}_s - \vec{\boldsymbol{y}}^S|}{c_0} + \frac{|\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}_r|}{c_0}, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s \Big)$$

Numerical simulations



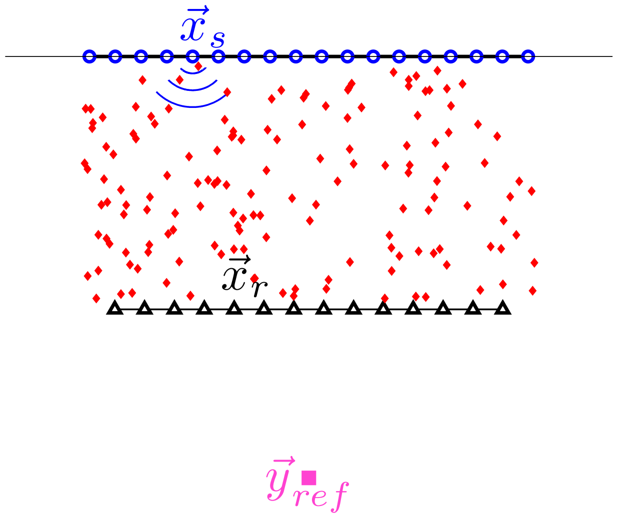
Computational setup



Kirchhoff Migration

(simulations carried out by Chrysoula Tsogka)

Imaging below an overburden



$\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver. Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \ldots, N_{\rm r}, s = 1, \ldots, N_{\rm s}\}$.

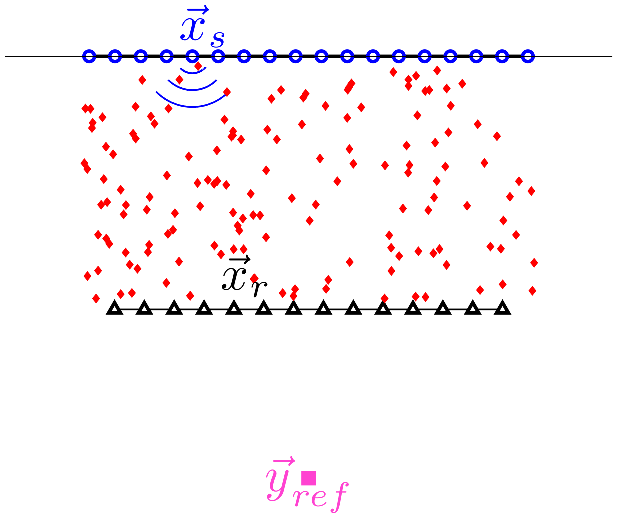
Image with migration of the special cross correlation matrix:

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \sum_{r,r'=1}^{N_{\rm r}} \mathcal{C}\Big(\frac{|\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{y}}^S|}{c_0} + \frac{|\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}_{r'}|}{c_0}, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'} \Big),$$

with

$$\mathcal{C}(\tau, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}) = \sum_{s=1}^{N_{\rm s}} \int u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s) u(t+\tau, \vec{\boldsymbol{x}}_{r'}; \vec{\boldsymbol{x}}_s) dt \ , \qquad r, r' = 1, \ldots, N_{\rm r}$$

Imaging below an overburden



$\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver. Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \dots, N_\mathrm{r}, s = 1, \dots, N_\mathrm{s}\}$.

Image with migration of the special cross correlation matrix:

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \sum_{r,r'=1}^{N_\mathrm{r}} \mathcal{C}\Big(\frac{|\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{y}}^S|}{c_0} + \frac{|\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}_{r'}|}{c_0}, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}\Big),$$

with

$$\mathcal{C}(\tau, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}) = \sum_{s=1}^{N_\mathrm{s}} \int u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s) u(t+\tau, \vec{\boldsymbol{x}}_{r'}; \vec{\boldsymbol{x}}_s) dt \ , \qquad r, r' = 1, \dots, N_\mathrm{r}$$

It is a special CINT function:

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \frac{1}{2\pi} \sum_{s=1}^{N_\mathrm{s}} \sum_{r,r'=1}^{N_\mathrm{r}} \int d\omega \hat{u}(\omega, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s) \overline{\hat{u}(\omega, \vec{\boldsymbol{x}}_{r'}; \vec{\boldsymbol{x}}_s)} \exp\Big\{ i\omega \Big[\frac{|\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{y}}^S|}{c_0} + \frac{|\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}_{r'}|}{c_0} \Big] \Big\}$$

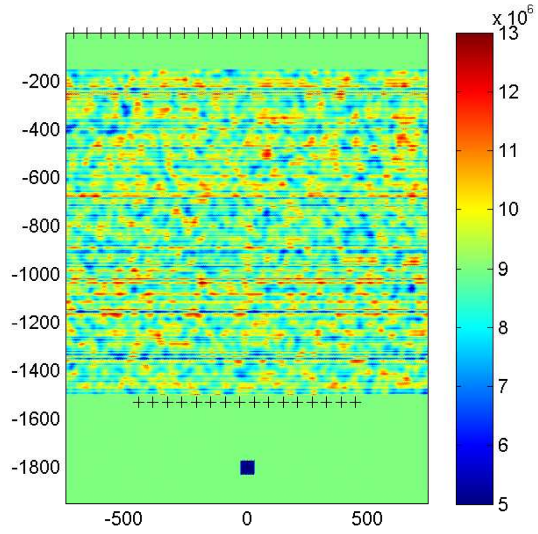
Remark: General CINT function:

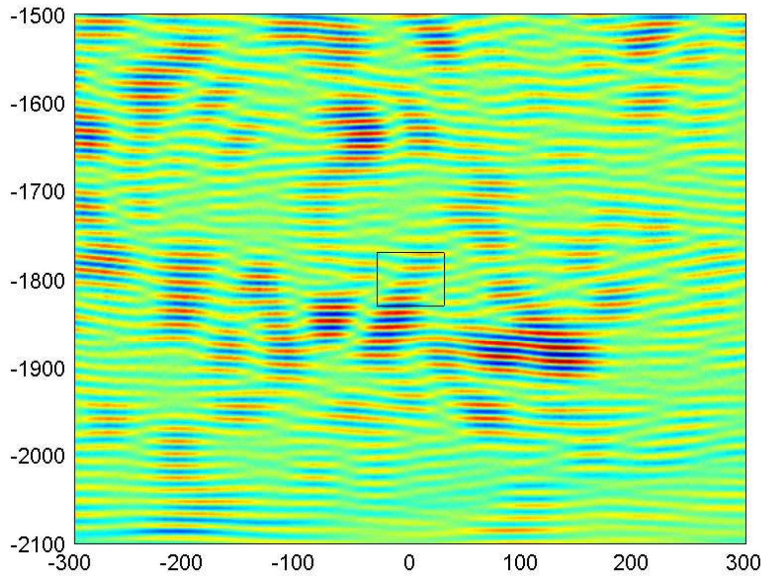
$$\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S) = \sum_{\substack{s,s'=1\\ |\vec{\boldsymbol{x}}_s-\vec{\boldsymbol{x}}_{s'}|\leq X_{\rm d}}}^{N_{\rm s}} \sum_{\substack{r,r'=1\\ |\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{x}}_{r'}|\leq X'_{\rm d}}}^{N_{\rm r}} \iint_{|\omega-\omega'|\leq\Omega_{\rm d}} d\omega d\omega' \, \hat{u}(\omega,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)\overline{\hat{u}(\omega',\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{x}}_{s'})}$$

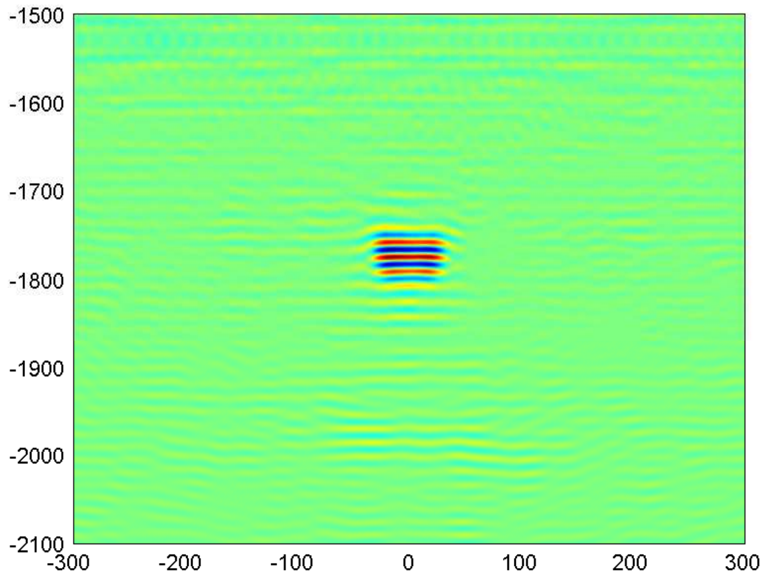
$$\times \exp\Big\{ -i\omega\Big[\frac{|\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{y}}^S|}{c_0}+\frac{|\vec{\boldsymbol{x}}_s-\vec{\boldsymbol{y}}^S|}{c_0}\Big]+i\omega'\Big[\frac{|\vec{\boldsymbol{x}}_{r'}-\vec{\boldsymbol{y}}^S|}{c_0}+\frac{|\vec{\boldsymbol{x}}_{s'}-\vec{\boldsymbol{y}}^S|}{c_0}\Big]\Big\}$$

- If $X_{\rm d} = X'_{\rm d} = \Omega_{\rm d} = \infty$, then $\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S) = \big|\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S)\big|^2$.
- If $X_{\rm d} = 0$, $X'_{\rm d} = \infty$, $\Omega_{\rm d} = 0$, then $\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S)$ is the special CINT.

Numerical simulations







Kirchhoff Migration

Cross Correlation Migration

Analysis in randomly scattering media

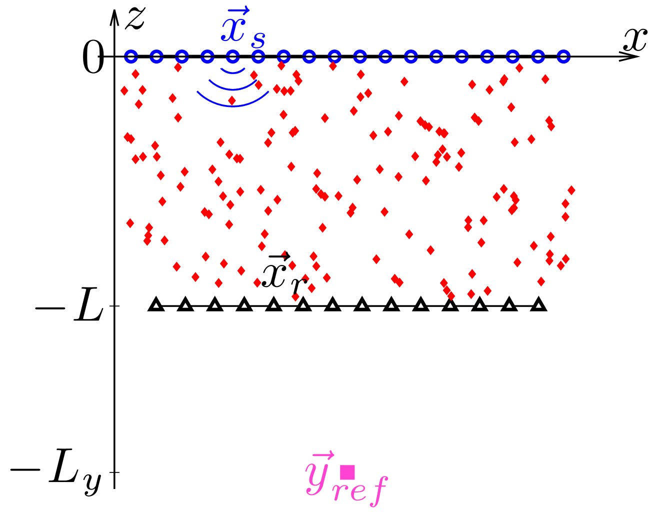
• Does the cross correlation imaging function give good images in scattering media ?
↪ It is possible to analyze the resolution and stability of the imaging function in randomly scattering media.

• General results:
Imaging function is stable provided the bandwidth is large enough and/or the source array is large enough.

• Detailed results:
If there are sources everywhere at the surface: scattering plays no role.
If the source distribution is spatially limited: scattering is important.
- in the random paraxial regime, scattering helps (it enhances the angular diversity of the illumination).
- in the randomly layered regime, scattering does not help (it reduces the angular diversity of the illumination).

[1] J. Garnier and G. Papanicolaou, *Inverse Problems* **28** 075002 (2012).

Imaging below an overburden: analysis in the paraxial regime

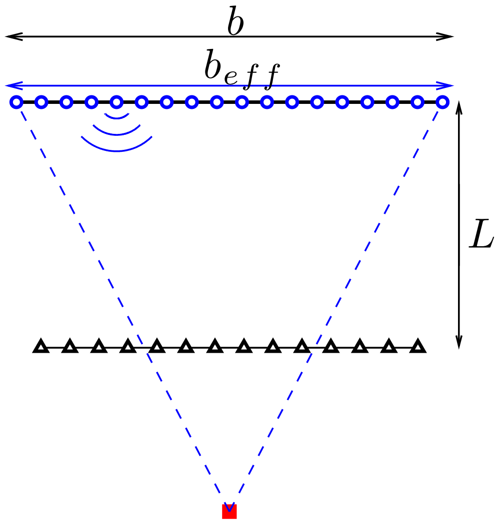


- Consider the regime "$\lambda \ll l_c \ll L$".

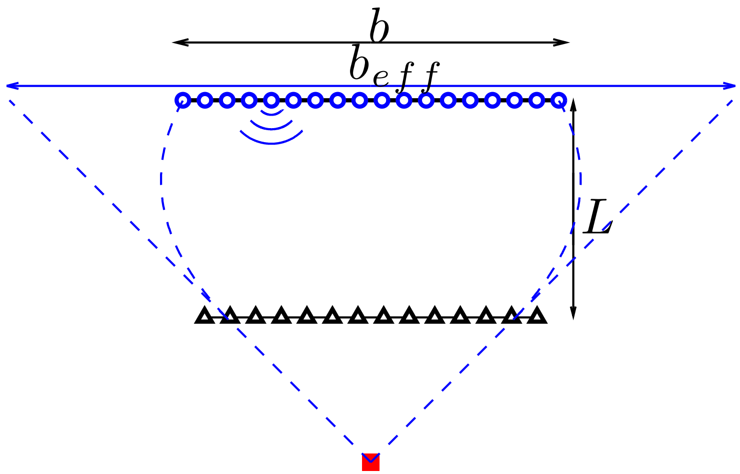
- Assume that:

- the source aperture is b and the receiver aperture is a.
- there is a point reflector at $\vec{\boldsymbol{y}} = (\boldsymbol{y}, -L_y)$.
- the covariance function $\gamma(\boldsymbol{x}) = \int \mathbb{E}[\mu(\boldsymbol{0}, 0)\mu(\boldsymbol{x}, z)]dz$ can be expanded as $\gamma(\boldsymbol{x}) = \gamma(\boldsymbol{0}) - \bar{\gamma}_2|\boldsymbol{x}|^2 + o(|\boldsymbol{x}|^2)$ for small $|\boldsymbol{x}|$.
- scattering is strong: $\frac{\gamma(\boldsymbol{0})\omega_0^2 L}{c_0^2} > 1$ ($\rightarrow$ mean wave is damped).

Imaging below an overburden: analysis in the paraxial regime



Homogeneous medium



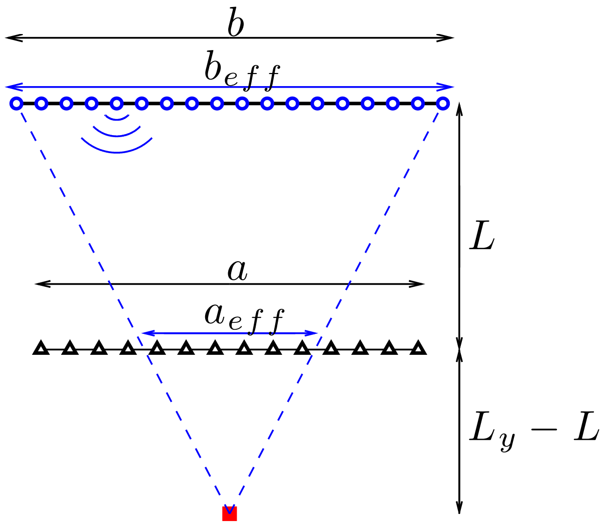
Random medium

Effective source aperture:

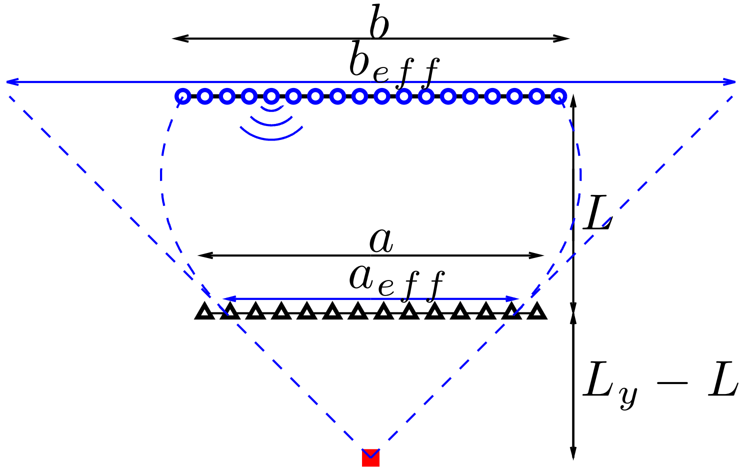
$$b_{\text{eff}} = b$$

$$b_{\text{eff}} = \Big(b^2 + \frac{\bar{\gamma}_2 L^3}{3} \Big)^{1/2}$$

Imaging below an overburden: analysis in the paraxial regime



Homogeneous medium



Random medium

Effective source aperture:

$$b_{\text{eff}} = b \qquad\qquad b_{\text{eff}} = \Big(b^2 + \frac{\bar{\gamma}_2 L^3}{3}\Big)^{1/2}$$

Effective receiver aperture:

$$a_{\text{eff}} = b\frac{L_y - L}{L_y} \qquad\qquad a_{\text{eff}} = b_{\text{eff}}\frac{L_y - L}{L_y}$$

Imaging below an overburden: analysis in the paraxial regime

- The imaging function for the search point $\vec{\boldsymbol{y}}^S$ is

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \frac{1}{N_{\rm r}^2} \sum_{r,r'=1}^{N_{\rm r}} \mathcal{C}\big(\mathcal{T}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_{r'}), \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}\big)$$

- The imaging function is statistically stable ($\lambda_0 \ll b \ll L$).

- The lateral resolution is $\dfrac{\lambda_0(L_y - L)}{a_{\rm eff}}$. The range resolution is $\dfrac{c_0}{B}$.

Here: λ_0 is the carrier wavelength, B is the bandwidth.

- Since $a_{\rm eff}\ |_{\rm rand} > a_{\rm eff}\ |_{\rm homo}$, this shows that scattering helps.

\- physical reason: scattering enhances the angular diversity of the illumination.

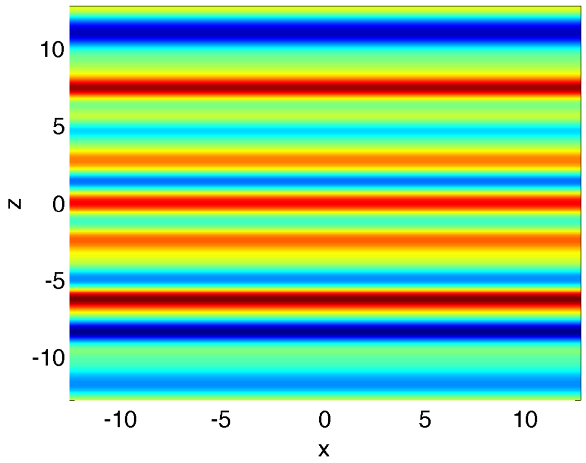
Randomly layered medium

- Random medium model ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$):

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_0^2}\big(1 + \mu(z)\big)$$

c_0 is a reference speed,

$\mu(z)$ is a zero-mean random process.



- Consider the time-harmonic form of the scalar wave equation ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$)

$$(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}\big(1 + \mu(z)\big)\hat{u} = 0$$

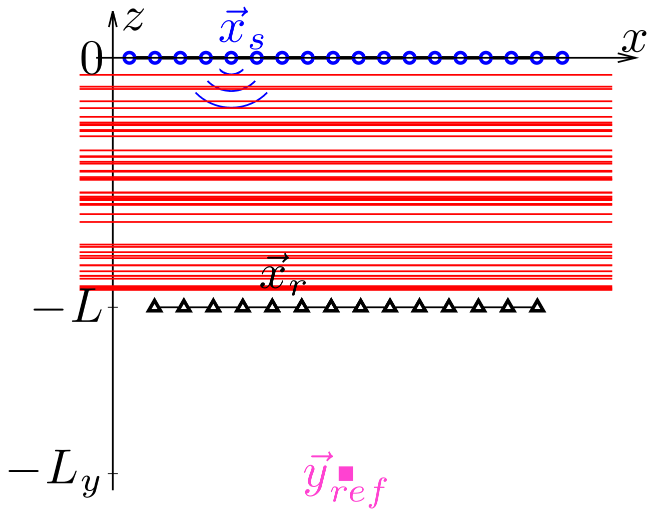
Consider the scaled regime "$l_c \ll \lambda \ll L$":

$$\omega \to \frac{\omega}{\varepsilon}, \qquad \mu(z) \to \mu\Big(\frac{z}{\varepsilon^2}\Big)$$

The moments of the random Green's function are known in the limit $\varepsilon \to 0$ [1].
$\hookrightarrow$ exponential decay of the mean field; exponential decay of the mean intensity (localization regime).

[1] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, *Wave propagation* ..., Springer, 2007.

Imaging below an overburden: analysis in the layered regime

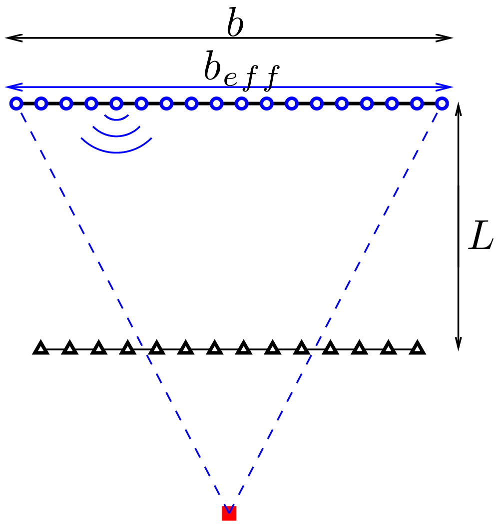


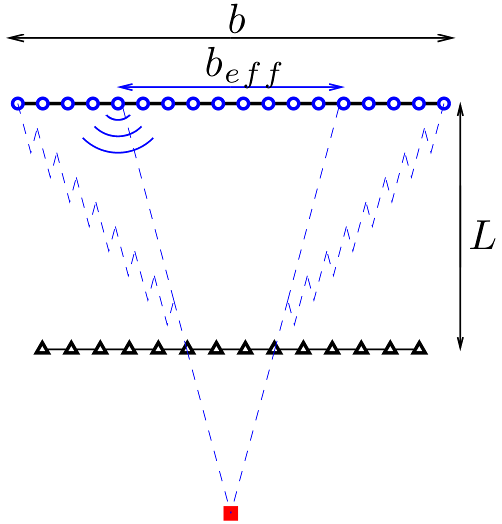
- Consider the regime "$l_c \ll \lambda \ll L$".
- Assume that:

- the source aperture is b and the receiver aperture is a.
- there is a point reflector at $\vec{\boldsymbol{y}}_{\rm ref} = (\boldsymbol{y}, -L_y)$.
- the localization length $L_{\rm loc}$ is smaller than L (strong scattering, mean wave is damped):

$$L_{\rm loc} = \frac{4c_0^2}{\gamma\omega_0^2}, \qquad \gamma = \int_{-\infty}^{\infty} \mathbb{E}[\mu(0)\mu(z)]dz$$

Imaging below an overburden: analysis in the layered regime





Homogeneous medium

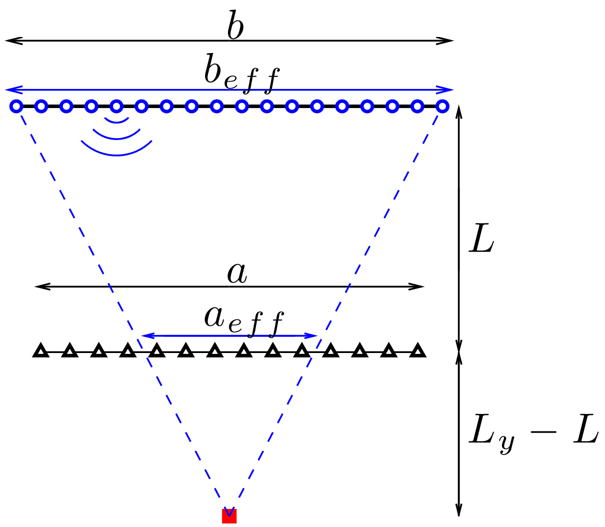
Randomly layered medium

Effective source aperture:

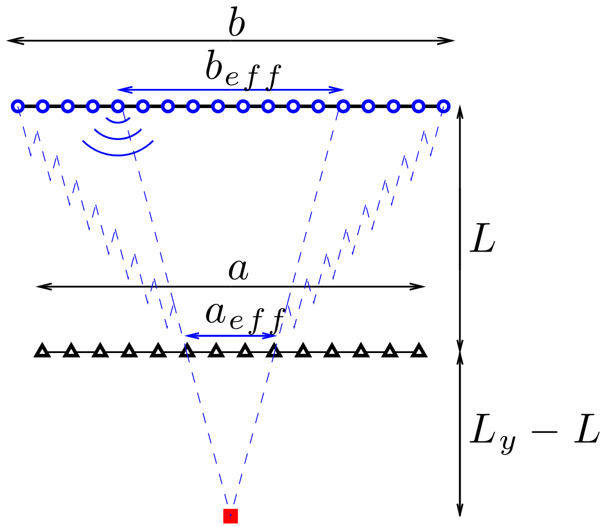
$b_{\rm eff} = b$

$b_{\rm eff}^2 = 4 L_{\rm loc} L \ \ (\ll b^2)$

Imaging below an overburden: analysis in the layered regime



Homogeneous medium



Randomly layered medium

Effective source aperture:

$$b_{\text{eff}} = b \qquad\qquad b_{\text{eff}}^2 = 4L_{\text{loc}}L \ \ (\ll b^2)$$

Effective receiver aperture:

$$a_{\text{eff}} = b\frac{L_y - L}{L_y} \qquad\qquad a_{\text{eff}} = b_{\text{eff}}\frac{L_y - L}{L_y}$$

Imaging below an overburden: analysis in the layered regime

- The imaging function for the search point $\vec{\boldsymbol{y}}^S$ is

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \frac{1}{N_{\rm r}^2} \sum_{r,r'=1}^{N_{\rm r}} \mathcal{C}\big(\mathcal{T}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_{r'}), \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}\big)$$

- The imaging function is statistically stable ($\lambda_0 \ll b, L$).

- The lateral resolution is $\dfrac{\lambda_0(L_y - L)}{a_{\rm eff}}$. The range resolution is $\dfrac{c_0}{B}\big(1 + \dfrac{B^2 L}{4\omega_0^2 L_{\rm loc}}\big)^{1/2}$.

- Since $a_{\rm eff}\mid_{\rm rand} < a_{\rm eff}\mid_{\rm homo}$, this shows that scattering does not help.

- physical reason: scattering reduces the angular and frequency diversity of the illumination.

Further results

• Use of other imaging functions based on cross correlations (or Wigner distribution functions).

• Use of ambient noise sources.
One can apply correlation-based imaging techniques to signals emitted by ambient noise sources.
↪ Useful for applications in seismology (travel time tomography, volcano monitoring, oil reservoir monitoring).

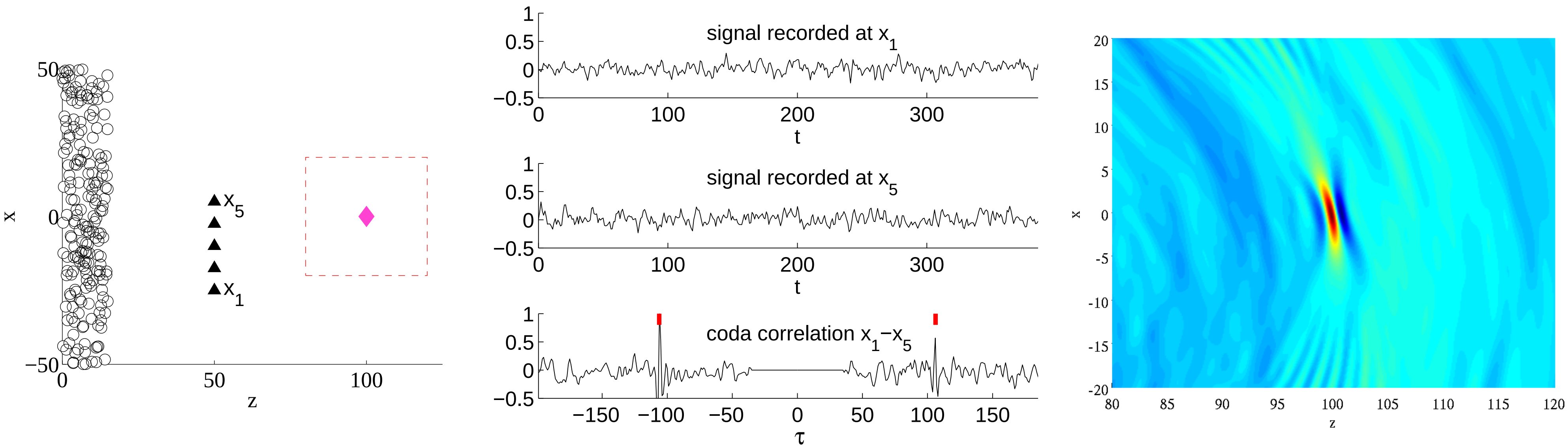
• Use of higher-order correlations.
One can apply imaging techniques based on special fourth-order cross correlations.
↪ Useful when the statistics of the wave fluctuations is not Gaussian.

Passive sensor imaging of a reflector

- Ambient noise sources ($\circ$) emit stationary random signals.
- The signals $(u(t,\vec{\boldsymbol{x}}_r))_{r=1,\ldots,N_\mathrm{r}}$ are recorded by the receivers $(\vec{\boldsymbol{x}}_r)_{r=1,\ldots,N_\mathrm{r}}$ ($\blacktriangle$).
- The cross correlation matrix is computed and migrated:

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \sum_{r,r'=1}^{N_\mathrm{r}} \mathcal{C}_T\big(\mathcal{T}(\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}}^S),\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'}\big)$$

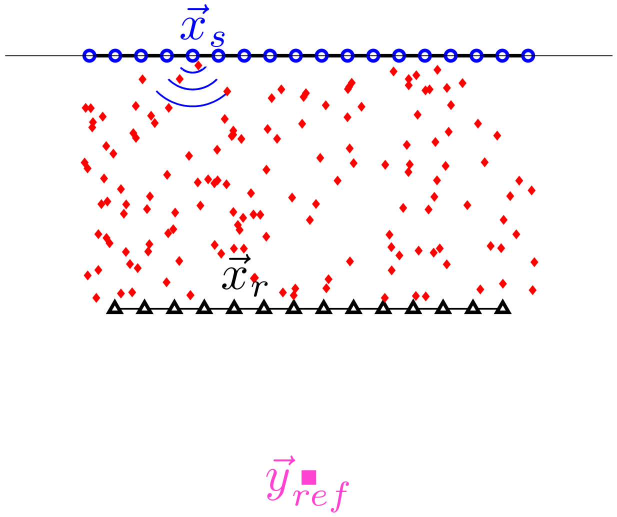
$$\text{with}\quad \mathcal{C}_T(\tau,\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'}) = \frac{1}{T}\int_0^T u(t+\tau,\vec{\boldsymbol{x}}_{r'})u(t,\vec{\boldsymbol{x}}_r)dt$$



Provided the ambient noise illumination is long (in time) and diversified (in angle and frequency): good stability [1].

[1] J. Garnier and G. Papanicolaou, *SIAM J. Imaging Sciences* **2**, 396 (2009).

Conclusions



- In scattering media one should migrate *well chosen* cross correlations of data, not data themselves.
- Method can be applied with ambient noise sources instead of controlled sources.
- Scattering can be mitigated, and even sometimes can help ! Already noticed for time-reversal experiments, but far from clear in imaging problems.