Solutions of the Helmholtz equation in a periodic waveguide : asymptotic behaviour and radiation condition.

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POEMS (UMR 7231 CNRS/ENSTA/INRIA)

This course is mainly based on a joint work with *Sonia Fliss*

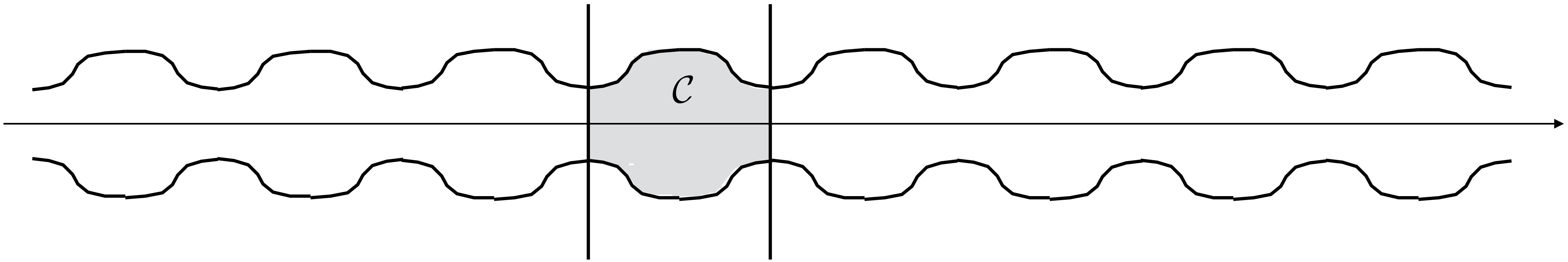
A periodic waveguide : the geometry

By definition, this is a domain $\Omega \subset \mathbb{R}^{d+1} = \{\, x = (x_1, x_T),\; x_1 \in V,\; x_T \in \mathbb{R}^d \}$

which is connected, bounded in $x_T = (x_2, \cdots, x_{d+1})$: $\Omega \subset \{\, (x_1, x_T),\; |x_T| < R \}$

and periodic (with period 1 for simplicity) in the longitudinal x_1 variable

$$\boxed{(x_1, x_T) \in \Omega \quad \longrightarrow \quad (x_1 + 1, x_T) \in \Omega}$$



Example $(\, d = 1\,)$: $\Omega = \{\, (x_1, x_2),\; f_-(x_1) < x_2 < f_+(x_1) \}$ $\quad (f_-, f_+)$ periodic

Unit periodicity cell $\mathcal{C} = \{ (x_1, x_T) \in \Omega \;/\; 0 < x_1 < 1 \}$ $\qquad \Omega = \bigcup_{n \in \mathbb{Z}} \, [\, \mathcal{C} + (n, 0)]$

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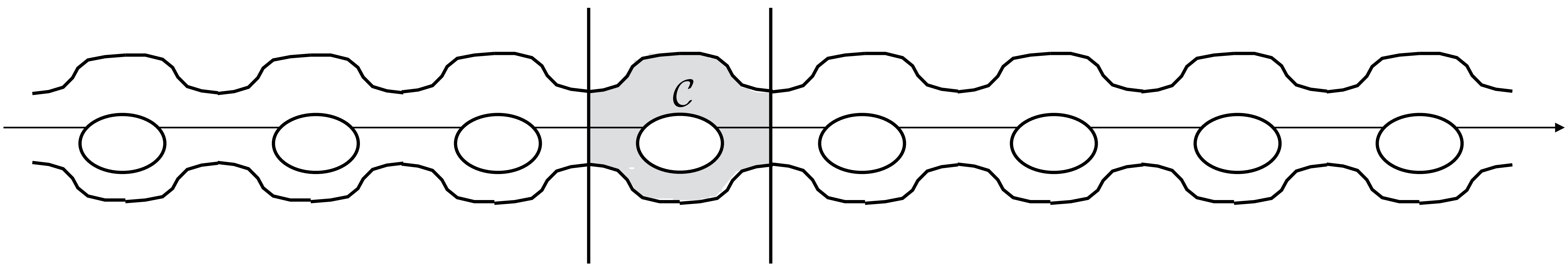

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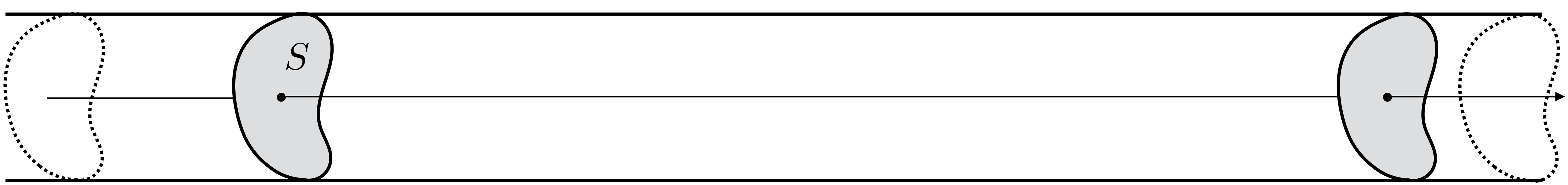
A particular case : the perfect waveguide $\Omega = \mathbb{R} \times S$



A periodic waveguide : governing equations

By analogy with electromagnetism, we assume that the material properties of the propagation medium are reduced to a periodic index of refraction

$$0 < n_- < n(x_1, x_T) < n_+ < +\infty \qquad\qquad n(x_1 + 1, x_T) = n(x_1, x_T)$$

We assume that the unknown $U(x,t) : \Omega \times \mathbb{R} \longrightarrow \mathbb{C}$ satisfies the scalar wave equation

$$n^2\, \partial_t^2 U - \Delta U = F(x,t)$$

where $F(x,t)$ is the source term. Assuming that this source term is time harmonic

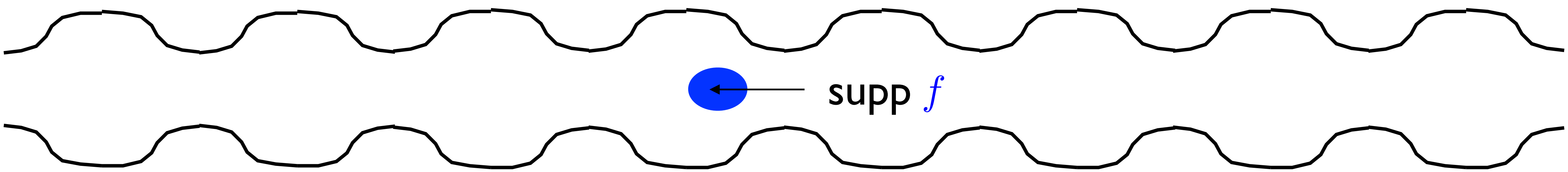
$F(x,t) = f(x)\, e^{-i\omega t}$ $\quad f(x) \in L^2(\Omega)$ (compactly supported) $\quad \omega > 0$ given frequency

we look for a time harmonic solution $U(x,t) = u(x)\, e^{-i\omega t}$ which leads to

$$-\Delta u - n^2\,\omega^2 u = f \qquad\qquad \text{Helmholtz equation}$$

Objective of the course

$$(\mathcal{P}) \qquad -\Delta u - n^2\omega^2 u = f \quad \text{in } \Omega\,, \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega\,.$$



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1. Define and construct properly the good solution of $(\mathcal{P})$
2. Describe the properties of this solution, in particular its behaviour at infinity
3. Find radiations condition at infinity that characterize this solution

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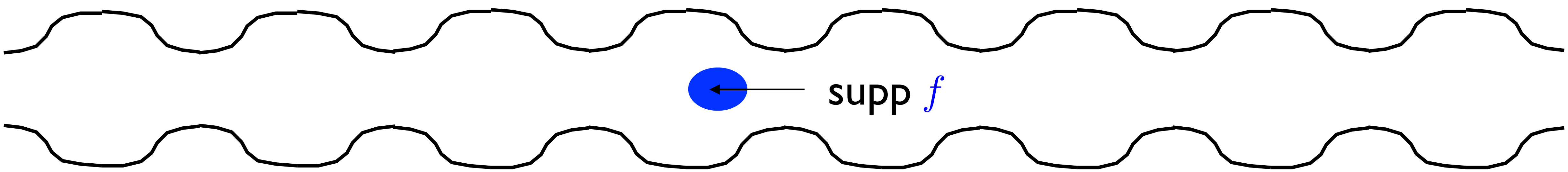

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The underlying selfajoint operator

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In the Hilbert space $L^2(\Omega, n^2\,dx)$, with scalar product $(u,v)_{n^2} := \int_\Omega u\,\overline{v}\,n^2\,dx$ we consider the unbounded operator defined by

$$D(A) = \{v \in H^1(\Omega) \;/\; \Delta v \in L^2(\Omega), \partial_\nu v = 0 \text{ on } \partial\Omega\} \qquad A\,v = -\,n^{-2}\,\Delta v$$

Even though it is not necessary, for technical simplicity, we shall assume that $\partial\Omega$ is smooth enough in order that

$$D(A) = \{v \in H^2(\Omega) \;/\; \partial_\nu v = 0 \text{ on } \partial\Omega\}$$

Theorem : A is a positive selfadjoint operator. Thus $\sigma(A) \subset \mathbb{R}^+$.

The Helmholtz equation writes formally $A\,u - \omega^2\,u = f/n^2$

The existence of a solution in $D(A)$ can only occur when : $\omega^2 \notin \sigma(A)$

When $\omega^2 \in \sigma(A)$, we need to look for a solution in another framework.

The limiting absorption procedure

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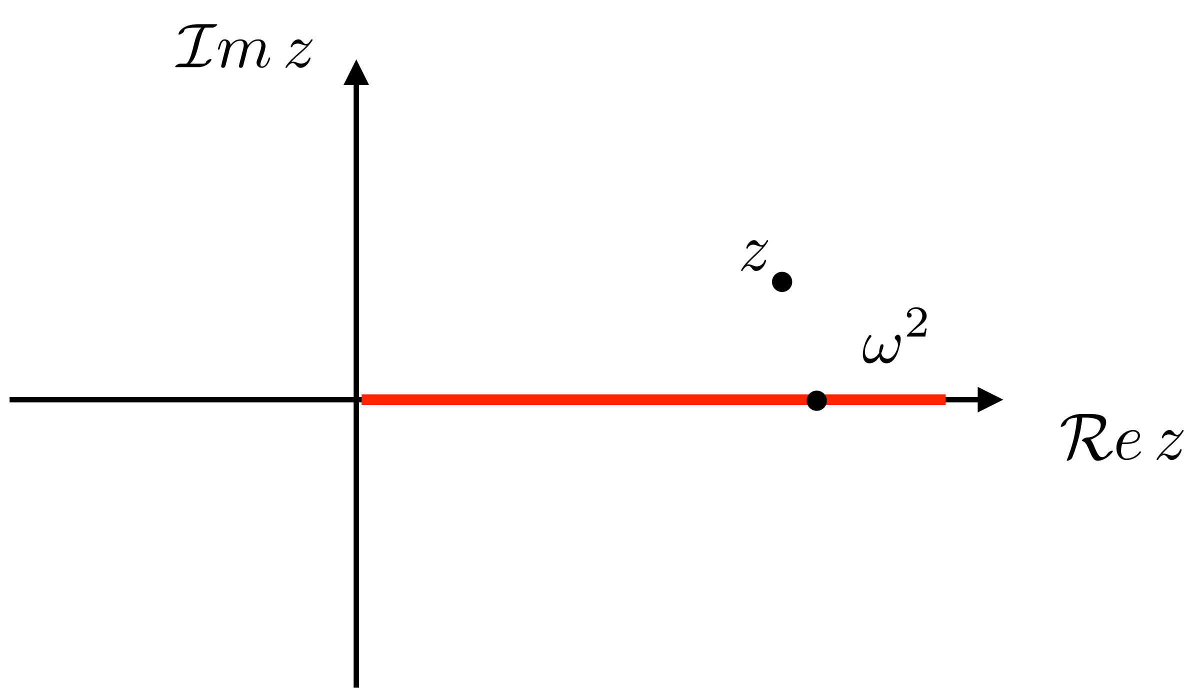
Formally $\quad A\,u - \omega^2\,u = f/n^2 \quad \Longleftrightarrow \quad u = (A-\omega^2)^{-1}\,g, \quad g := f/n^2$

The above formula makes sense in we replace ω^2 by $z \notin \mathbb{R}^+$ which suggests to look at the existence of the following limit

$$\lim_{z \to \omega^2,\, z \notin \mathbb{R}} (A - z)^{-1} g$$



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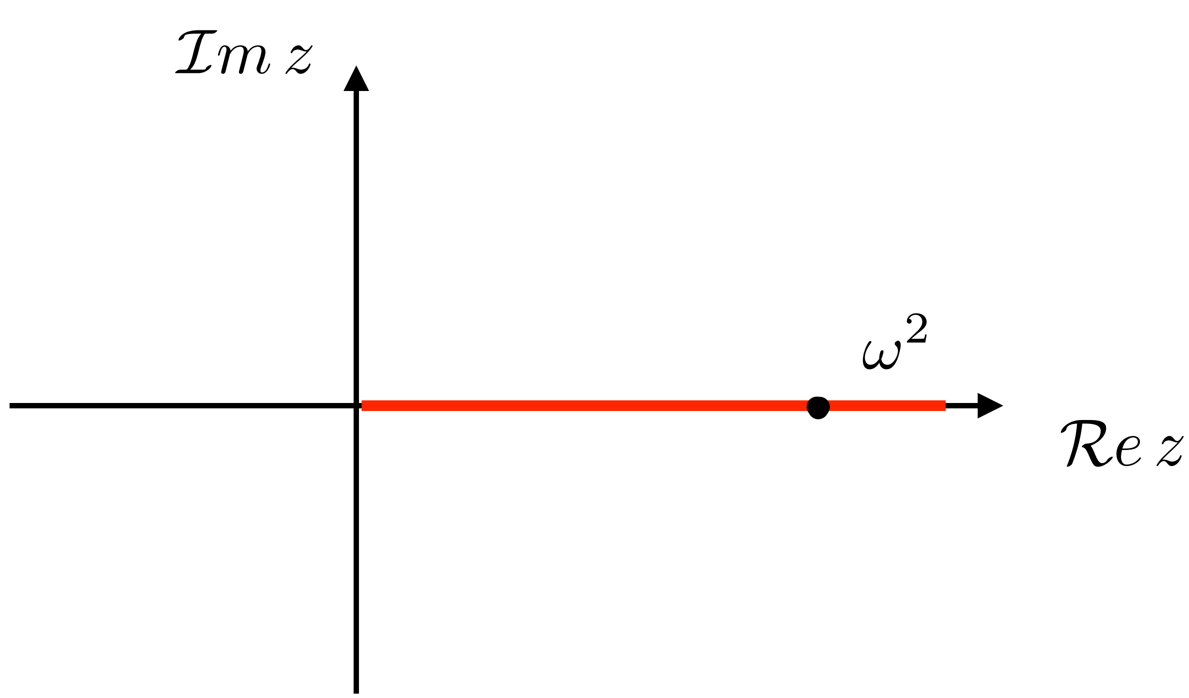
This is not sufficient : two different limits may exist depending on the sign of $\mathcal{I}m\, z$



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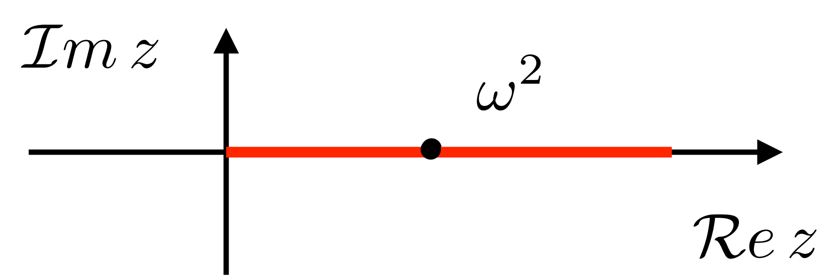
The physically relevant choice is to take $z = (\omega^2 + i\, \varepsilon \omega)\ , \varepsilon > 0$ which amounts to adding a small absorption term to the time dependent wave equation

$$n^2\, (\partial_t^2 U^\varepsilon + \varepsilon\, \partial_t U^\varepsilon) - \Delta U^\varepsilon = 0 \qquad \varepsilon > 0$$

and to look at the limit when $\varepsilon \to 0$



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$$n^2 \left(\partial_t^2 U^\varepsilon + \varepsilon\, \partial_t U^\varepsilon\right) - \Delta U^\varepsilon = 0 \qquad \varepsilon > 0$$

This leads the Helmholtz equation with absorption

$(\mathcal{P}_\varepsilon)$ $\quad -\Delta u^\varepsilon - n^2\, (\omega^2 + i\varepsilon\omega)\, u^\varepsilon = f$ in Ω $\quad \partial_\nu u^\varepsilon = 0$ on $\partial\Omega$ $\quad u^\varepsilon \in H^2(\Omega)$

Objective of the course

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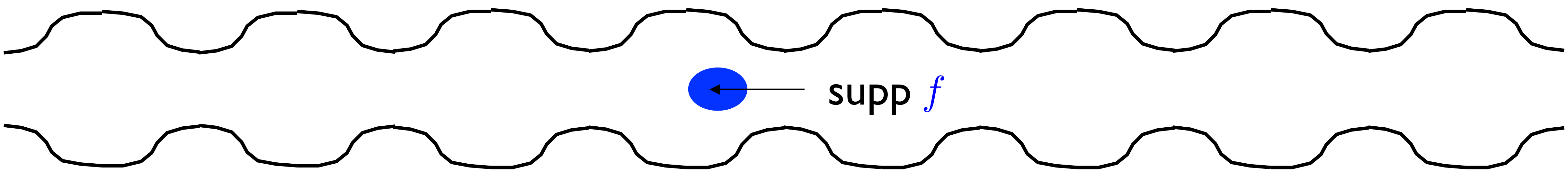

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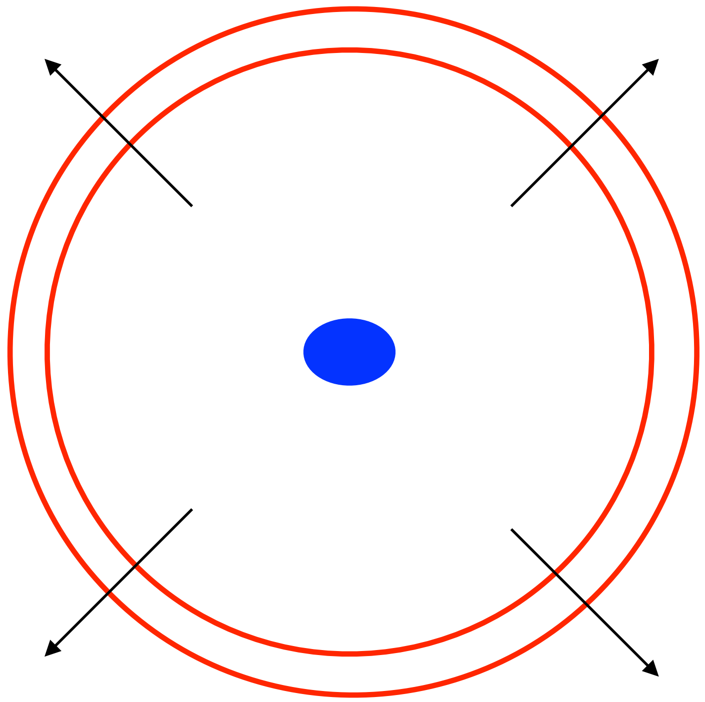
Homogeneous free space



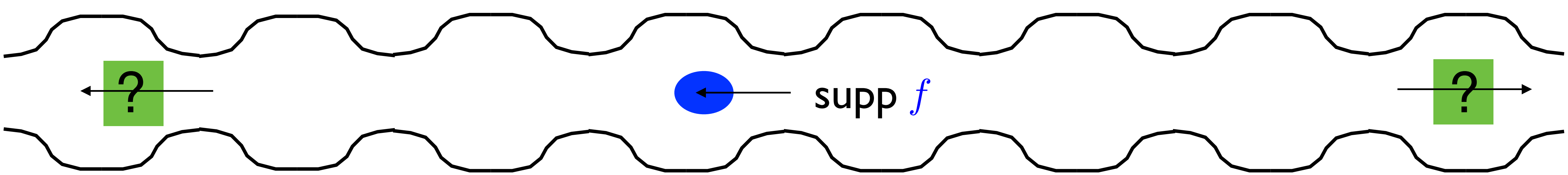
Data for the inverse problem

$$u(r\Theta) \underset{r \to +\infty}{\sim} A(\Theta) \frac{e^{i\omega r}}{r^{\frac{d-1}{2}}}$$

$$\partial_r u + i\,\omega u = O(r^{-2})$$

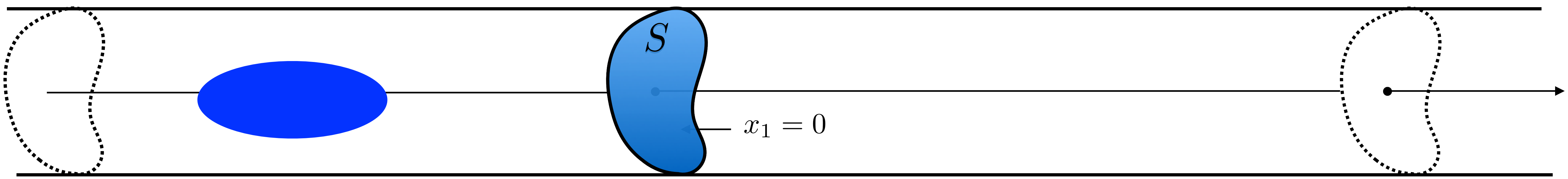
Sommerfeld condition

Waveguide



Behaviour at infinity and radiation conditions

The case of the perfect waveguide



Here, we assume that index of refraction only depends on the transverse variable

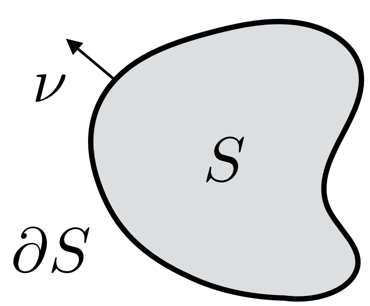
$$n(x_1, x_T) = n(x_T)$$

$$-\Delta_T u^\varepsilon - \partial_{x_1}^2 u^\varepsilon - n(x_T)^2 \left(\omega^2 + i\varepsilon\omega\right) u^\varepsilon = f$$

We use separation of variables by introducing the eigenfuntions of the transverse operator

$$-\Delta_T \theta_n = \lambda_n \, n^2 \, \theta_n \quad \text{in } S$$

$$\partial_\nu \theta_n = 0 \quad \text{on } \partial S$$



$$\lambda_n > 0, \quad \lambda_n \longrightarrow +\infty$$

We look for the solution in the form $\quad u^\varepsilon(x_1, x_T) = \displaystyle\sum_{n=0}^{+\infty} u_n^\varepsilon(x_1)\, \theta_n(x_T)$

$$\Longrightarrow \qquad -\left(u_n^\varepsilon\right)'' + \left(\lambda_n - (\omega^2 + i\,\varepsilon\,\omega)\right) u_n^\varepsilon = 0, \quad x_1 > 0$$

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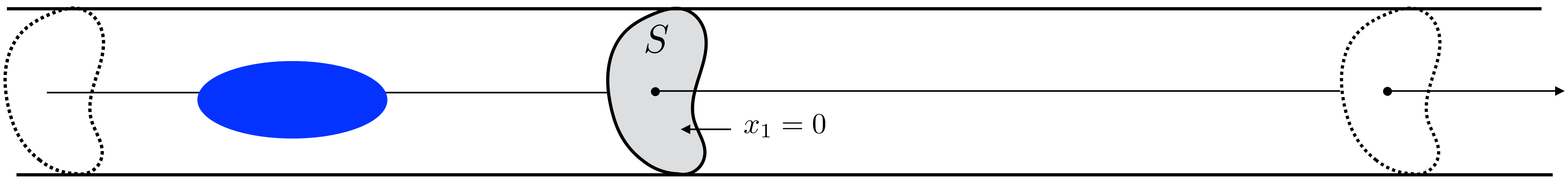

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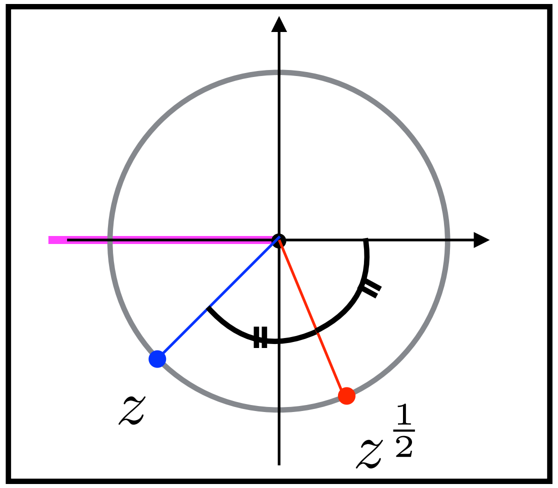
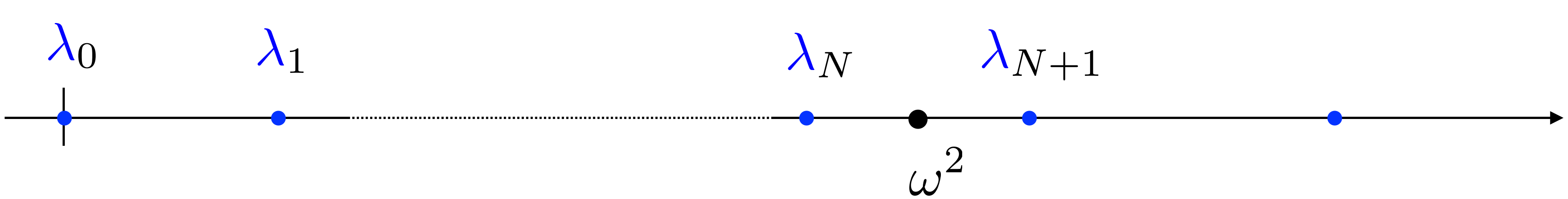
$$-\left(u_n^\varepsilon\right)'' + \left(\lambda_n - (\omega^2 + i\,\varepsilon\,\omega)\right) u_n^\varepsilon = 0, \quad x_1 > 0$$

Introducing $\zeta_n^\varepsilon := \left(\lambda_n - (\omega^2 + i\,\varepsilon\,\omega)\right)^{\frac{1}{2}}$ with $\mathcal{R}e\, z^{\frac{1}{2}} > 0$



Since we look for $u^\varepsilon \in L^2(\Omega)$, $\quad u^\varepsilon(x_1, x_T) = \sum_{n=0}^{+\infty} u_n^\varepsilon(0)\, \theta_n(x_T)\, e^{-\zeta_n^\varepsilon x_1}$



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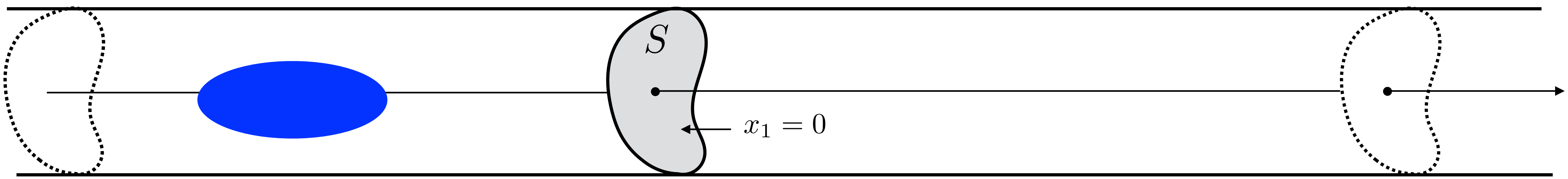
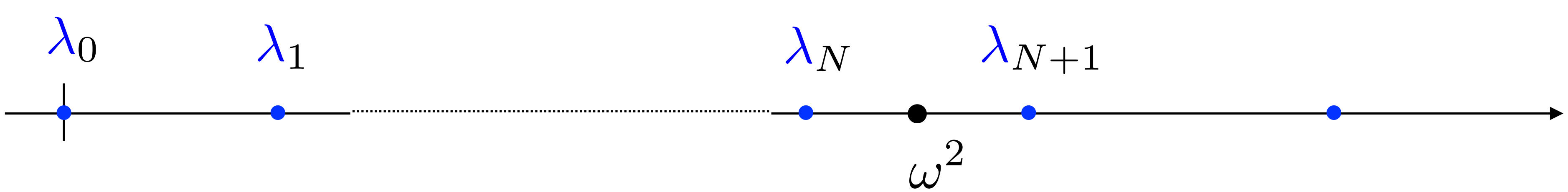


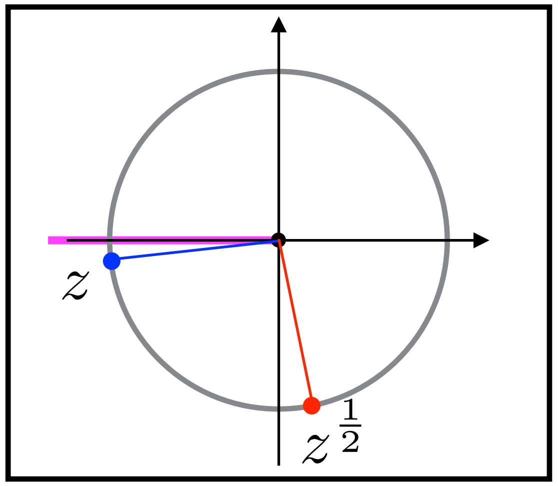
$$\zeta_n^\varepsilon := \left(\lambda_n - (\omega^2 + i\,\varepsilon\,\omega)\right)^{\frac{1}{2}} \qquad \mathcal{R}e\, z^{\frac{1}{2}} > 0 \qquad u^\varepsilon(x_1, x_T) = \sum_{n=0}^{+\infty} u_n^\varepsilon(0)\,\theta_n(x_T)\,e^{-\zeta_n^\varepsilon\, x_1}$$



Passage to the limit when $\varepsilon \to 0$

$$n > N \qquad \zeta_n^\varepsilon \longrightarrow \sqrt{\lambda_n - \omega^2}$$

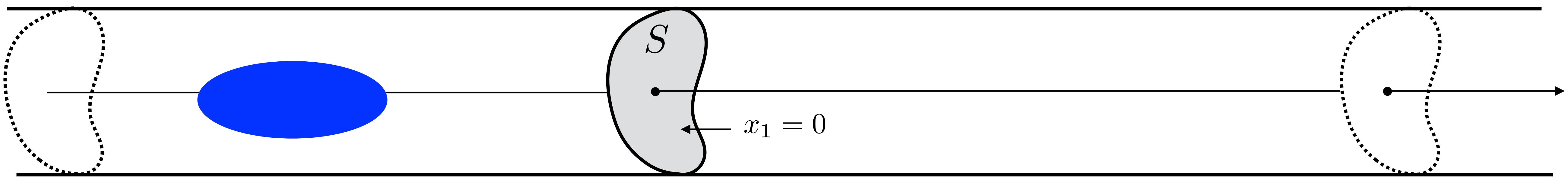
$$n \leq N \qquad \zeta_n^\varepsilon \longrightarrow -\,i\,\sqrt{\omega^2 - \lambda_n}$$



$$\Longrightarrow \quad u(x_1, x_T) = \sum_{n=0}^{N} u_n(0)\,\theta_n(x_T)\,e^{i\,\sqrt{\omega^2 - \lambda_n}\,x_1} + \sum_{n=N+1}^{+\infty} u_n(0)\,\theta_n(x_T)\,e^{-\sqrt{\lambda_n - \omega^2}\,x_1}$$

propagative modes evanescent modes

The case of the perfect waveguide



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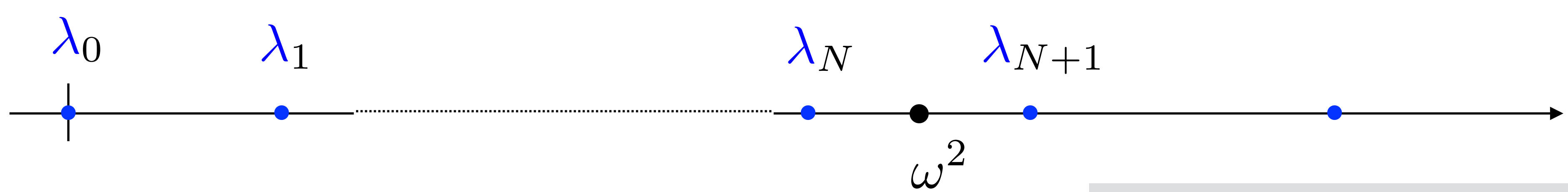

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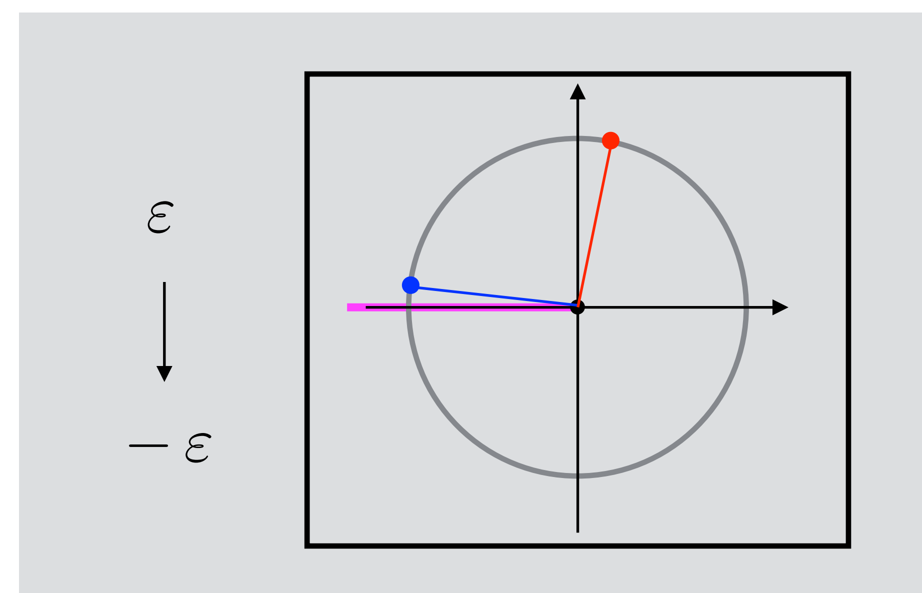

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propagative modes — evanescent modes

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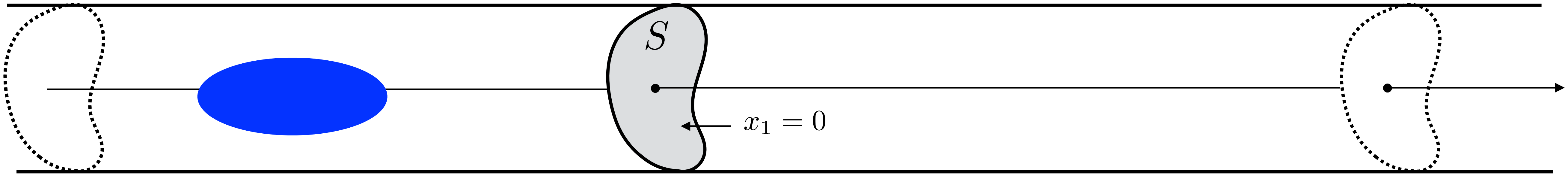

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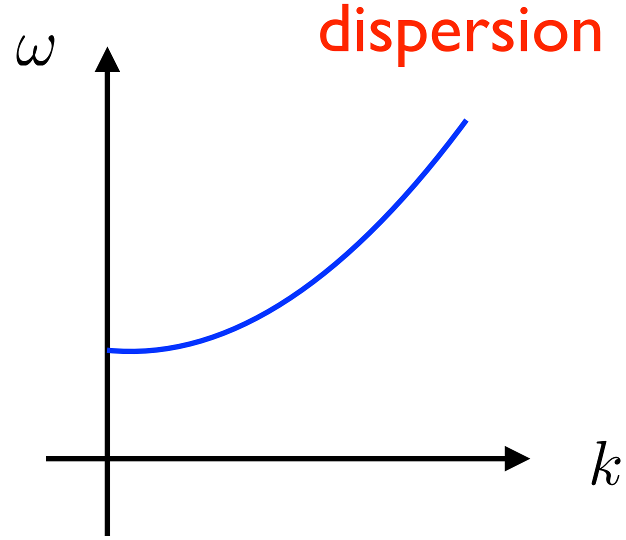
propagative modes   evanescent modes

The propagative modes are outgoing : they propagate in the direction $x_1 > 0$

$$e^{i\sqrt{\omega^2-\lambda_n}\,x_1}\; e^{-i\omega t} = \exp i\,(k\,x_1 - \omega\,t) \qquad k = \sqrt{\omega^2 - \lambda_n} \quad \Longleftrightarrow \quad \omega = \sqrt{k^2 + \lambda_n}$$

dispersion relation

dispersion curve



$\boxed{\text{group velocity : } \quad \partial_k \omega > 0}$

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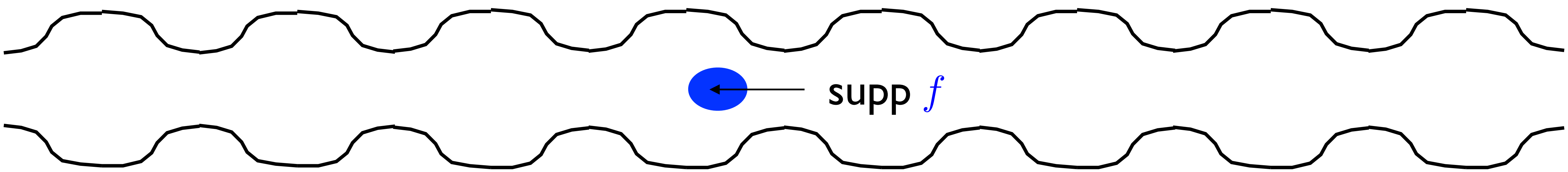

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In periodic waveguides, the result is more complex and the analysis very technical.

Construction of the outgoing solution

Consider the (unique) solution of the Helmholtz equation with absorption

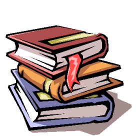
$$(\mathcal{P}_\varepsilon) \quad -\Delta u^\varepsilon - n^2\,(\omega^2 + i\varepsilon\omega)\,u^\varepsilon = f \quad \text{in } \Omega \quad \partial_\nu u^\varepsilon = 0 \ \text{ on } \partial\Omega \quad u^\varepsilon \in H^2(\Omega)$$

Find an appropriate representation (decomposition) of this solution

Technical tool : the Floquet-Bloch transform

Use this representation to pass to the limit when $\varepsilon \to 0$

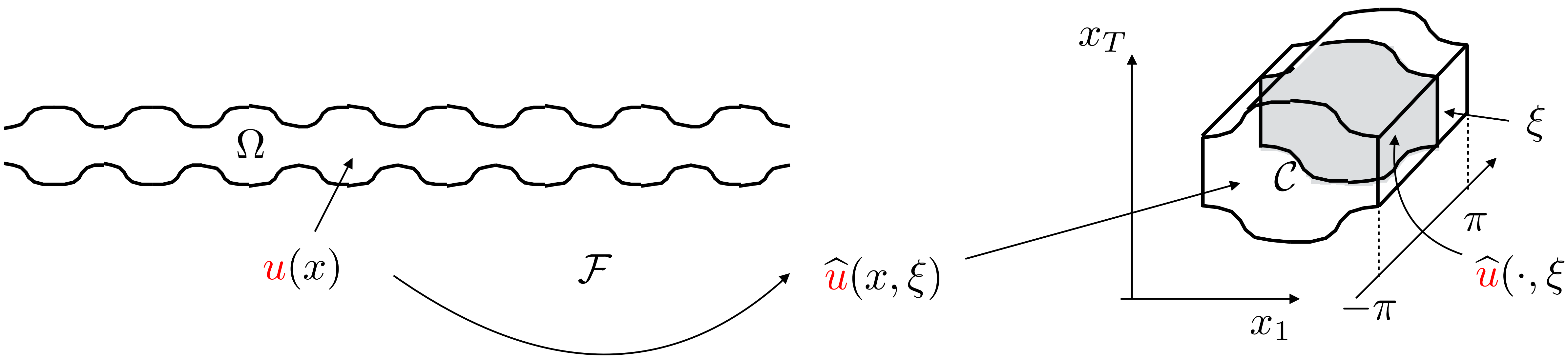
Technical tool : the Plemelj-Privalov theorem

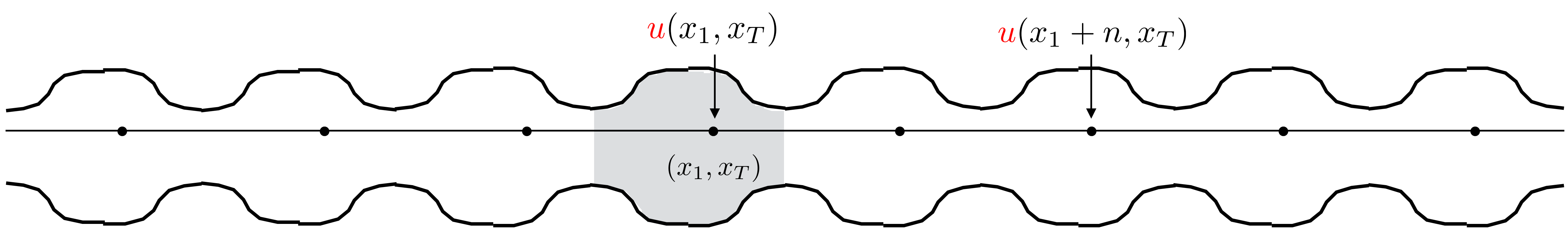


P. Kuchment. Floquet theory for partial differential equations, *Operator theory : Advances and Applications*, *Birkhäuser Verlag*, (1993)

The Floquet-Bloch tranform

This is a unitary fransform between $L^2(\Omega)$ and $L^2\big(\mathcal{C}\times]-\pi,\pi[\big)$ that can be seen as an adequate version of the Fourier transform in the x_1 variable.

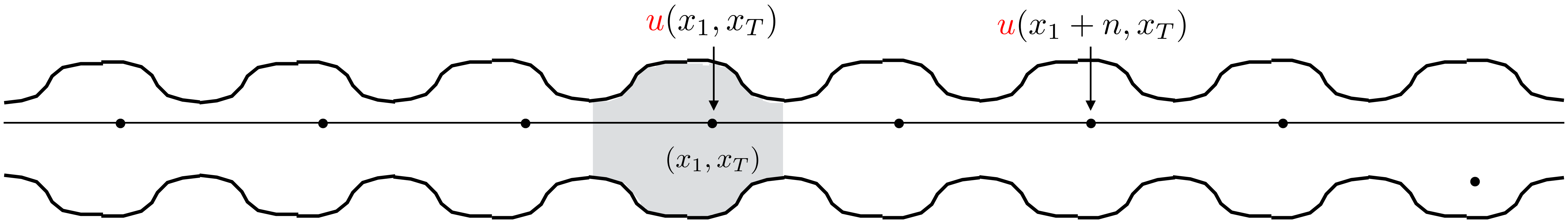




Given $u \in D(\Omega)$, one constructs $\widehat{u}(x,\xi)$ as the sum of the Fourier series associated with the sequence $\left\{\, u(x_1+n,\, x_T), n\in\mathbb{Z}\,\right\}$

$$\boxed{\widehat{u}(x_1,\, x_T,\xi) = (2\pi)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} u(x_1+n,\, x_T)\, e^{-in\xi}}$$

The Floquet-Bloch tranform



Given $u \in D(\Omega)$, one constructs $\widehat{u}(x, \xi)$ as the sum oh the Fourier series associated with the sequence $\{\, u(x_1 + n,\, x_T), n \in \mathbb{Z}\,\}$

$$\boxed{\widehat{u}(x_1,\, x_T, \xi) = (2\pi)^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}} u(x_1 + n,\, x_T)\, e^{-in\xi}}$$

Since $\{(2\pi)^{-\frac{1}{2}}\, e^{-in\xi},\, n \in \mathbb{Z}\}$ form an hilbertian basis of $L^2(-\pi, \pi)$

$$\int_{-\pi}^{\pi} \left|\widehat{u}(x_1,\, x_T, \xi)\right|^2 d\xi = \sum_{n \in \mathbb{Z}} \left|u(x_1 + n,\, x_T)\right|^2$$

and after integration over $\mathcal{C}$

FB Plancherel theorem

$$\int_{\mathcal{C}} \int_{-\pi}^{\pi} \left|\widehat{u}(x_1,\, x_T, \xi)\right|^2 d\xi\; dx = \sum_{n \in \mathbb{Z}} \int_{\mathcal{C}} \left|u(x_1 + n,\, x_T)\right|^2 dx \equiv \int_{\Omega} \left|u(x)\right|^2 dx$$

The Floquet-Bloch tranform

$$\boxed{\widehat{u}(x_1,\, x_T, \xi) = (2\pi)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} u(x_1+n,\, x_T)\, e^{-in\xi}}$$

We have thus shown that $\forall\, u \in \mathcal{D}(\Omega), \quad \|\widehat{u}\|_{L^2\left(\mathcal{C}\times]-\pi,\pi\,[\,\right)} = \|u\|_{L^2(\Omega)}$

As a consequence, the map $\mathcal{F} : u \longrightarrow \widehat{u}$ extends continuously into an isometry from $L^2(\Omega)$ into $L^2\big(\mathcal{C}\times]-\pi,\pi[\,\big)$ also defined by

$$\boxed{\widehat{u}(\cdot,\xi) = \lim_{N\longrightarrow 0} (2\pi)^{-\frac{1}{2}} \sum_{|n|\leq N} u(\cdot+n\,e_1)\, e^{-in\xi} \qquad \text{in } L^2(\mathcal{C})}$$

The transformation $\mathcal{F} : u \longrightarrow \widehat{u}$ is one to one with the reconstruction formula :

$$\boxed{\forall\, x\in\mathcal{C}, \quad u(x_1+n,\, x_T) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \widehat{u}(x_1,\, x_T,\xi)\, e^{in\xi}\, d\xi}$$

Remark : the same proof as in the previous slide shows that the isometry result remains valid with weighted L^2 spaces provided that the weight is a periodic function.

$$\int_{\mathcal{C}} \int_{-\pi}^{\pi} \big|\widehat{u}(x_1,\, x_T,\xi)\big|^2\, n(x)^2\, d\xi\, dx = \int_{\Omega} \big|u(x)\big|^2 n(x)^2\, dx$$

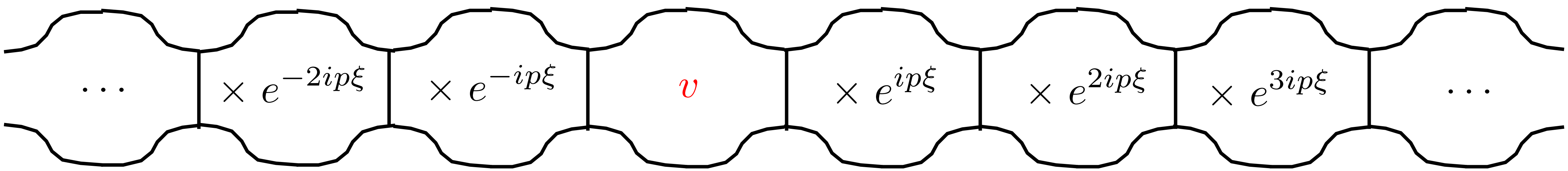
Quasiperiodic functions

By definition, a function $v : \Omega \to \mathbb{C}$ is said to be $\xi -$ quasiperiodic if and only if

$$v(x_1 + 1, x_T) = e^{i\xi}\, v(x_1, x_T)$$

Given $v : \mathcal{C} \to \mathbb{C}$, one defines its $\xi -$ quasiperiodic extension to Ω, $E_\xi v$, by

$$E_\xi v(x_1 + n, x_T) = e^{in\xi}\, v(x_1 + n, x_T) \qquad \forall\, n \in \mathbb{Z}$$



Link with the Floquet-Bloch transform

$$(1) \qquad \boxed{\widehat{u}(x_1,\, x_T, \xi) = (2\pi)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} u(x_1 + n,\, x_T)\, e^{-in\xi}}$$

Extending formula (1) to any $(x_1, x_T) \in \Omega$, $\widehat{u}(\cdot, \xi)$ is $\xi -$quasiperiodic

The Floquet-Bloch tranform : properties

$$\boxed{\widehat{u}(x_1,\, x_T, \xi) = (2\pi)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} u(x_1 + n,\, x_T)\, e^{-in\xi}}\, .$$

1. The Floquet-Bloch transform commutes with differential operators

$$\forall\, u \in H^1(\Omega),\; \forall\, \xi \in (-\pi, \pi), \quad \mathcal{F}\left(\frac{\partial u}{\partial x_i}\right)(\cdot;\xi) = \frac{\partial}{\partial x_i}\big(\mathcal{F}u(\cdot;\xi)\big)$$

2. The Floquet-Bloch transform commutes with multiplication with periodic functions

$$\forall\, u \in L^2(\Omega),\; \forall\, \xi \in (-\pi, \pi), \quad \mathcal{F}\left(n^2 u\right)(\cdot;\xi) = n^2\big(\mathcal{F}u(\cdot;\xi)\big)$$

3. The Floquet-Bloch transform diagonalizes the translations

$$\tau_n u(x_1, x_T) := u(x_1 + n, x_T), \quad n \in \mathbb{Z}$$

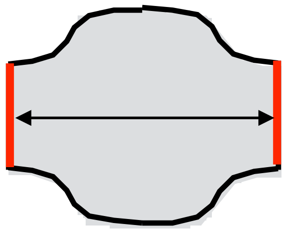
$$\forall\, u \in L^2(\Omega),\; \forall\, \xi \in (-\pi, \pi), \quad \mathcal{F}\left(\tau_n u\right)(\cdot;\xi) = e^{in\xi}\,\big(\mathcal{F}u(\cdot;\xi)\big)$$

The Floquet-Bloch tranform : properties

$$\widehat{u}(x_1,\, x_T, \xi) = (2\pi)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} u(x_1+n,\, x_T)\, e^{-in\xi}$$

4. The Floquet-Bloch transform in Sobolev spaces

For any $s \geq 0$ and $\xi \in \,]-\pi,\pi\,[$, we define



$$H^s_\xi(\mathcal{C}) := \left\{u \in H^s(\mathcal{C}) \;/\; E_\xi u \in H^s_{loc}(\Omega)\right\}$$

$$H^1_\xi(\mathcal{C}) := \left\{u \in H^1(\mathcal{C}) \;/\; u(1,x_T) = e^{i\xi}\; u(0,x_T)\right\}$$

$$H^s_\xi(\mathcal{C}) := \left\{u \in H^s(\mathcal{C}) \;/\; u(1,x_T) = e^{i\xi}\; u(0,x_T)\right\} \quad 1/2 < s < 3/2$$

$$H^2_\xi(\mathcal{C}) := \left\{u \in H^2(\mathcal{C}) \cap H^1_\xi(\mathcal{C}) \;/\; \partial_{x_1} u(1,x_T) = e^{i\xi}\; \partial_{x_1} u(0,x_T)\right\}$$

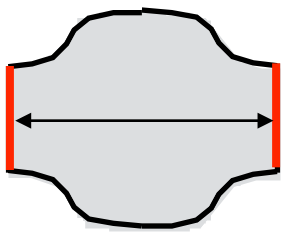
$$H^s_\xi(\mathcal{C}) := \left\{u \in H^s(\mathcal{C}) \cap H^1_\xi(\mathcal{C}) \;/\; \partial_{x_1} u(1,x_T) = e^{i\xi}\; \partial_{x_1} u(0,x_T)\right\} \quad 3/2 < s < 5/2$$

The Floquet-Bloch tranform : properties

$$\boxed{\widehat{u}(x_1,\, x_T, \xi) = (2\pi)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} u(x_1+n,\, x_T)\, e^{-in\xi}}$$

4. The Floquet-Bloch transform in Sobolev spaces

For any $s \geq 0$ and $\xi \in \,]-\pi,\pi[$, we define



$$H^s_\xi(\mathcal{C}) := \left\{ u \in H^s(\mathcal{C}) \;/\; E_\xi u \in H^s_{loc}(\Omega) \right\}$$

$$H^1_\xi(\mathcal{C}) := \left\{ u \in H^1(\mathcal{C}) \;/\; u(1,x_T) = e^{i\xi}\, u(0,x_T) \right\}$$

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$$H^s_\xi(\mathcal{C}) := \left\{ u \in H^s(\mathcal{C}) \cap H^1_\xi(\mathcal{C}) \;/\; \partial_{x_1} u(1,x_T) = e^{i\xi}\, \partial_{x_1} u(0,x_T) \right\} \quad 3/2 < s < 5/2$$

$\xi -$ quasi-periodic boundary conditions

The Floquet-Bloch tranform : properties

$$\boxed{\widehat{u}(x_1,\, x_T, \xi) = (2\pi)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} u(x_1+n,\, x_T)\, e^{-in\xi}}.$$

4. The Floquet-Bloch transform in Sobolev spaces

For any $s \geq 0$ and $\xi \in \,]-\pi,\pi[$, we define

$$H^s_\xi(\mathcal{C}) := \left\{ u \in H^s(\mathcal{C}) \;/\; E_\xi u \in H^s_{loc}(\Omega) \right\}$$

Accordingly, we define

$$H^s_{qp}\big(\mathcal{C}\times]-\pi,\pi[\big) := \left\{ u \in L^2\big(-\pi,\pi; H^s(\mathcal{C})\big) \;/\; \text{a. e. } \xi \in \,]-\pi,\pi[\,, u(\cdot,\xi) \in H^s_\xi(\mathcal{C}) \right\}$$

Theorem : The Floquet-Bloch transform $\mathcal{F}$ defines an isomorphism between

$H^s(\Omega)$ and $H^s_{qp}\big(\mathcal{C}\times]-\pi,\pi[\big)$

The Floquet-Bloch tranform : properties

$$\boxed{\widehat{u}(x_1,\, x_T, \xi) = (2\pi)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} u(x_1+n,\, x_T)\, e^{-in\xi}}$$

5. Decay properties in x_1 / Sobolev regularity in ξ

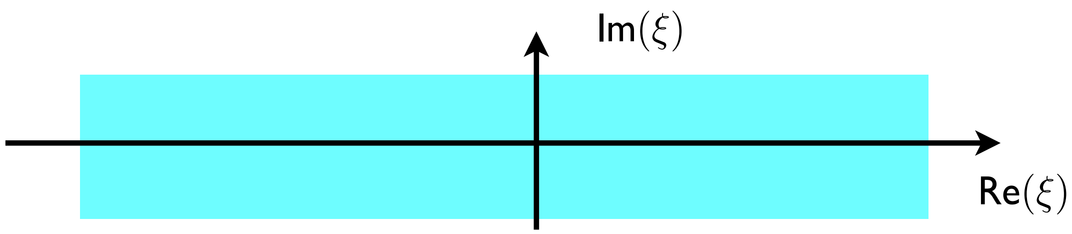
$$L^2_s(\Omega) := \left\{ u \in L^2(\Omega) \;/\; (1+x_1^2)^{\frac{s}{2}} u \in L^2(\Omega) \right\} \qquad s > 0$$

$$u \in L^2_s(\Omega) \quad \Longrightarrow \quad \widehat{u} := \mathcal{F}u \in H^s\big(-\pi, \pi; L^2(\mathcal{C})\big)$$

6. Analyticity properties of Floquet-Bloch transforms

Assume that, for some $\alpha > 0$, $e^{\alpha\sqrt{1+x_1^2}}\, u \in H^s(\Omega)$, then the function $\xi \mapsto \widehat{u}(\cdot, \xi)$ can be extended to complex values of ξ in the strip

$$B_\alpha = \{\, \xi \;/\; |\mathcal{I}m\, \xi| < \alpha \,\}$$



as an analytic function from B_α with values in $H^s(\mathcal{C})$. Moreover $\widehat{u}(\cdot, \xi)$ is 2π periodic in B_α.

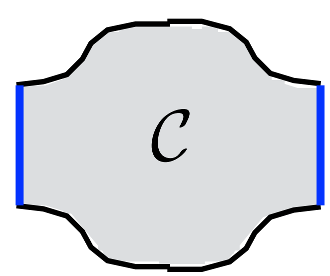
In particular, if u is compactly supported, $\xi \mapsto \widehat{u}(\cdot, \xi)$ is an entire function in $\mathbb{C}$.

Computation of the solution with absorption

$$(\mathcal{P}_\varepsilon) \quad -\Delta u^\varepsilon - n^2\,(\omega^2 + i\varepsilon\omega)\,u^\varepsilon = f \quad \text{in } \Omega \quad \partial_\nu u^\varepsilon = 0 \text{ on } \partial\Omega \qquad \boxed{u^\varepsilon \in H^2(\Omega)}$$

Let us denote $\widehat{u}^\varepsilon(x,\xi)$ the FB-transform of $u^\varepsilon(x)$ and applying $\mathcal{F}$ to $(\mathcal{P}_\varepsilon)$, we deduce that, for each $\xi \in \,]-\pi,\pi[$, $\widehat{u}^\varepsilon(\cdot,\xi)$ satisfies

$$\boxed{\begin{array}{l} -\Delta\widehat{u}^\varepsilon(\cdot,\xi) - n^2\,(\omega^2 + i\varepsilon\omega)\,\widehat{u}^\varepsilon(\cdot,\xi) = \widehat{f}(\cdot,\xi) \qquad \text{in } \mathcal{C} \\ \partial_\nu\widehat{u}^\varepsilon(\cdot,\xi) = 0 \qquad \text{on} \qquad \partial\Omega \cap \partial\mathcal{C} \\ \widehat{u}^\varepsilon(1,x_T) = e^{i\xi}\,\widehat{u}^\varepsilon(0,x_T), \quad \partial_{x_1}\widehat{u}^\varepsilon(1,x_T) = e^{i\xi}\,\partial_{x_1}\widehat{u}^\varepsilon(0,x_T) \quad \Longleftrightarrow \quad \widehat{u}^\varepsilon(\cdot,\xi) \in H^2_\xi(\Omega) \end{array}}$$



Boundary value problem in $\mathcal{C}$, in which ξ plays the role of a parameter

The reduced cell operators

$$-\Delta \widehat{u}^\varepsilon(\cdot,\xi) - n^2\,(\omega^2 + i\varepsilon\omega)\,\widehat{u}^\varepsilon(\cdot,\xi) = \widehat{f}(\cdot,\xi) \qquad \text{in } \mathcal{C}$$

$$\partial_\nu \widehat{u}^\varepsilon(\cdot,\xi) = 0 \qquad \text{on} \qquad \partial\Omega \cap \partial\mathcal{C}$$

$$\widehat{u}^\varepsilon(1,x_T) = e^{i\xi}\,\widehat{u}^\varepsilon(0,x_T), \quad \partial_{x_1}\widehat{u}^\varepsilon(1,x_T) = e^{i\xi}\,\partial_{x_1}\widehat{u}^\varepsilon(0,x_T)$$

For $\xi \in \mathbb{C}$, let $A(\xi)$ be the unbounded operator in the Hilbert space $L^2(\mathcal{C}, n^2\,dx)$ such that (note that $A(\xi + 2\pi) = A(\xi)$)

$$D\big(A(\xi)\big) = \big\{v \in H^2_\xi(\mathcal{C}) \;/\; \partial_\nu v = 0 \text{ on } \partial\mathcal{C} \cap \partial\Omega\big\} \qquad A(\xi)\,v = -\,n^{-2}\,\Delta v$$

Theorem : $A(\xi)$ has a compact resolvent and is positive selfadjoint in $L^2(\mathcal{C}, n^2\,dx)$ for real values of ξ .

$$\mathcal{C} \text{ bounded} \implies H^2_\xi(\mathcal{C}) \subset L^2(\mathcal{C}) \text{ compact}$$

The reduced cell operators

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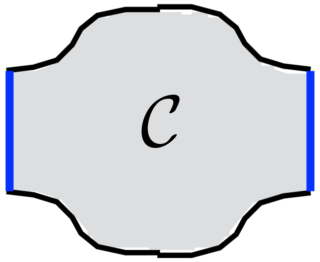
Theorem : $A(\xi)$ has a compact resolvent and is positive selfadjoint in $L^2(\mathcal{C}, n^2\, dx)$ for real values of ξ .

Proof of the hermitian positive nature of $A(\xi)$: $(\xi \in \mathbb{R})$

$$\big(A(\xi) u, v\big)_{n^2} = -\int_{\mathcal{C}} \Delta u\, \overline{v}\; dx$$

Green's formula



$$= \int_{\mathcal{C}} \nabla u\, \nabla \overline{v}\; dx + \int \partial_{x_1} u(1, x_T)\, \overline{v}(1, x_T)\; dx_T - \int \partial_{x_1} u(0, x_T)\, \overline{v}(0, x_T)\; dx_T$$

Using quasi-periodicity conditions

$$\partial_{x_1} u(1, x_T)\, \overline{v}(1, x_T) = e^{i\xi}\, \partial_{x_1} u(0, x_T)\, e^{-i\xi}\, \overline{v}(0, x_T) = \partial_{x_1} u(0, x_T)\, \overline{v}(0, x_T)$$

The reduced cell operators

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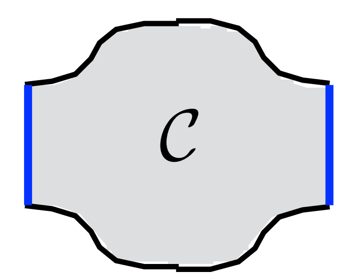

$$= \int_{\mathcal{C}} \nabla u\, \nabla \overline{v}\; dx + \cancel{\int \partial_{x_1} u(1, x_T)\, \overline{v}(1, x_T)\; dx_T} - \cancel{\int \partial_{x_1} u(0, x_T)\, \overline{v}(0, x_T)\; dx_T}$$

Using quasi-periodicity conditions

$$\partial_{x_1} u(1, x_T)\, \overline{v}(1, x_T) = e^{i\xi}\, \partial_{x_1} u(0, x_T)\, e^{-i\xi}\, \overline{v}(0, x_T) = \partial_{x_1} u(0, x_T)\, \overline{v}(0, x_T)$$

$$\Longrightarrow \quad \big(A(\xi)u, v\big)_{n^2} = \int_{\mathcal{C}} \nabla u\, \nabla \overline{v}\; dx$$

The reduced cell operators

For $\xi \in \mathbb{C}$, let $A(\xi)$ be the unbounded operator in the Hilbert space $L^2(\mathcal{C}, n^2\, dx)$ such that (note that $A(\xi + 2\pi) = A(\xi)$)

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Theorem : $A(\xi)$ has a compact resolvent and is positive selfadjoint in for real values of ξ .

Corollary : For $\xi \in \mathbb{R}$, there exists an hilbertian basis $\big\{\varphi_n(\cdot, \xi) \in H^2_\xi(\mathcal{C}), n \in \mathbb{N}\big\}$ of $L^2(\mathcal{C}, n^2\, dx)$ and a non decreasing sequence $\lambda_n(\xi) \geq 0$ such that

$$A(\xi)\, \varphi_n(\cdot, \xi) = \lambda_n(\xi)\, \varphi_n(\cdot, \xi) \qquad \lambda_n(\xi) \longrightarrow +\infty \quad (n \to +\infty)$$

The reduced cell operators

$$\boxed{\begin{array}{ll} -\Delta\psi_n(\cdot,\xi) = \lambda_n(\xi)\, n^2\, \psi_n(\cdot,\xi) & \partial_\nu \psi_n(\cdot,\xi) = 0 \\ \partial_{x_1}\psi_n(\cdot,\xi)|_{\Gamma_1} = \; e^{i\xi}\, \partial_{x_1}\psi_n(\cdot,\xi)|_{\Gamma_0} & \psi_n(\cdot,\xi)|_{\Gamma_1} = \; e^{i\xi}\, \psi_n(\cdot,\xi)|_{\Gamma_0} \end{array}}$$

Theorem : $A(\xi)$ has a compact resolvent and is positive selfadjoint in for real values of ξ .

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$$A(\xi)\,\varphi_n(\cdot,\xi) = \lambda_n(\xi)\;\varphi_n(\cdot,\xi) \qquad \lambda_n(\xi) \longrightarrow +\infty \quad (n \to +\infty)$$

The functions $\varphi_n(\cdot,\xi)$ can be chosen in such a way that

$\xi \to \lambda_n(\xi)$ and $\xi \to \varphi_n(\cdot,\xi) \in H^2(\mathcal{C})$ are Lipschitz continuous

$$\boxed{\begin{array}{ll} \lambda_n(\xi + 2\pi) = \lambda_n(\xi) & \varphi_n(\cdot\,, \xi + 2\pi) = \varphi_n(\cdot,\xi) \\ \lambda_n(-\xi) = \lambda_n(\xi) & \varphi_n(\cdot\,, -\xi) = \overline{\varphi_n(\cdot,\xi)} \end{array}}$$

The reduced cell operators

$$-\Delta\overline{\psi_n(\cdot,\xi)} = \lambda_n(\xi)\, n^2\, \overline{\psi_n(\cdot,\xi)} \qquad \partial_\nu \overline{\psi_n(\cdot,\xi)} = 0$$

$$\partial_{x_1}\overline{\psi_n(\cdot,\xi)}|_{\Gamma_1} = \overline{e^{i\xi}}\, \partial_{x_1}\overline{\psi_n(\cdot,\xi)}|_{\Gamma_0} \qquad \overline{\psi_n(\cdot,\xi)}|_{\Gamma_1} = \overline{e^{i\xi}}\, \overline{\psi_n(\cdot,\xi)}|_{\Gamma_0}$$

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$$\lambda_n(\xi + 2\pi) = \lambda_n(\xi) \qquad \varphi_n(\cdot\,,\xi+2\pi) = \varphi_n(\cdot,\xi)$$

$$\lambda_n(-\xi) = \lambda_n(\xi) \qquad \varphi_n(\cdot\,,-\xi) = \overline{\varphi_n(\cdot,\xi)}$$

The reduced cell operators

$$-\Delta\overline{\psi_n(\cdot,\xi)} = \lambda_n(\xi)\, n^2\, \overline{\psi_n(\cdot,\xi)} \qquad \partial_\nu \overline{\psi_n(\cdot,\xi)} = 0$$

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Theorem : $A(\xi)$ has a compact resolvent and is positive selfadjoint in for real values of ξ .

Corollary : For $\xi \in \mathbb{R}$, there exists an hilbertian basis $\{\varphi_n(\cdot,\xi) \in H^2_\xi(\mathcal{C}), n \in \mathbb{N}\}$ of $L^2(\mathcal{C}, n^2\,dx)$ and a non decreasing sequence $\lambda_n(\xi) \geq 0$ such that

$$A(\xi)\,\varphi_n(\cdot,\xi) = \lambda_n(\xi)\,\varphi_n(\cdot,\xi) \qquad \lambda_n(\xi) \longrightarrow +\infty \quad (n \to +\infty)$$

The functions $\varphi_n(\cdot,\xi)$ can be chosen in such a way that

$\xi \to \lambda_n(\xi)$ and $\xi \to \varphi_n(\cdot,\xi) \in H^2(\mathcal{C})$ are Lipschitz continuous

$$\lambda_n(\xi + 2\pi) = \lambda_n(\xi) \qquad \varphi_n(\cdot\,,\xi + 2\pi) = \varphi_n(\cdot,\xi)$$

$$\lambda_n(-\xi) = \lambda_n(\xi) \qquad \varphi_n(\cdot\,,-\xi) = \overline{\varphi_n(\cdot,\xi)}$$

Analytic families of unbounded operators

For our purpose, we shall use the theory of (possibly unbounded) operators depending analytically of one scalar complex parameter (denoted ξ here)

$$A(\xi) \quad : D\big(A(\xi)\big) \subset H \quad \longrightarrow H \qquad\qquad H \; : \text{ Hilbert space}$$

For bounded operators, one says that $A(\xi)$ is bounded analytic if

$$\boxed{\xi \mapsto A(\xi) \quad \text{is analytic from} \quad \mathbb{C} \quad \text{into} \quad \mathcal{L}(H)}$$

In the case where $D\big(A(\xi)\big) = D$ (independent of ξ), one says that $A(\xi)$ is analytic of type (A) if

$$\boxed{\forall\, v \in D, \quad \xi \mapsto A(\xi)v \quad \text{is analytic from} \quad \mathbb{C} \quad \text{into} \quad H}$$

In the case where the domain depends on ξ , $A(\xi)$ is analytic of class (B) if it is analytically equivavent to an analytic family of class (A), i. e.

There exists (S_ξ, S_ξ^{-1}) bounded analytic and $\widetilde{A}(\xi)$ analytic of class (A) such that

$$D\big(A(\xi)\big) = S_\xi \; D \qquad\qquad A(\xi) = S_\xi \; \widetilde{A}(\xi) \; S_\xi^{-1}$$

The reduced cell operators : analyticity properties

For $\xi \in \mathbb{C}$, let $A(\xi)$ be the unbounded operator in the Hilbert space $L^2(\mathcal{C}, n^2\, dx)$ such that (note that $A(\xi + 2\pi) = A(\xi)$)

$$D\big(A(\xi)\big) = \big\{ v \in H^2_\xi(\mathcal{C}) \;/\; \partial_\nu v = 0 \text{ on } \partial\mathcal{C} \cap \partial\Omega \big\} \qquad A(\xi)\, v = -\, n^{-2}\, \Delta v$$

Let us introduce $S_\xi \in \mathcal{L}\big(L^2(\mathcal{C}, n^2\, dx)\big)$ such that $S_\xi v(x) = e^{i\xi x_1}\, v(x)$. Each S_ξ is an isomorphism, $S_\xi^{-1} = S_{-\xi}$, unitary for real ξ.

$$\xi \in \mathbb{C} \longrightarrow S_\xi \in \mathcal{L}\big(L^2(\mathcal{C}, n^2\, dx)\big) \quad \text{is bounded analytic}$$

$$\boxed{S_\xi v \in H^2_\xi(\mathcal{C}) \iff v \in H^2_{per}(\mathcal{C}) \qquad H^2_{per}(\mathcal{C}) = H^2_{\xi=0}(\mathcal{C})}$$

$$D\big(A(\xi)\big) = S_\xi\, D \qquad D = \big\{ v \in H^2_{per}(\mathcal{C}) \;/\; \partial_\nu v = 0 \text{ on } \partial\mathcal{C} \cap \partial\Omega \big\}$$

$$\begin{tikzcd} D\big(A(\xi)\big) \arrow[r, "A(\xi)"] & L^2(\mathcal{C}, n^2\, dx) \arrow[d, "S_\xi^{-1}"] \\ D \arrow[u, "S_\xi"] \arrow[r, "\widetilde{A}(\xi)"] & L^2(\mathcal{C}, n^2\, dx) \end{tikzcd}$$

$$A(\xi) = S_\xi\, \widetilde{A}(\xi)\, S_\xi^{-1}$$

The reduced cell operators : analyticity properties

For $\xi \in \mathbb{C}$, let $A(\xi)$ be the unbounded operator in the Hilbert space $L^2(\mathcal{C}, n^2\, dx)$ such that (note that $A(\xi + 2\pi) = A(\xi)$)

$$D\big(A(\xi)\big) = \{v \in H^2_\xi(\mathcal{C}) \;/\; \partial_\nu v = 0 \text{ on } \partial\mathcal{C} \cap \partial\Omega\} \qquad A(\xi)\, v = -\, n^{-2}\, \Delta v$$

$$\boxed{D\big(A(\xi)\big) = S_\xi\; D \qquad D = \{v \in H^2_{per}(\mathcal{C}) \;/\; \partial_\nu v = 0 \text{ on } \partial\mathcal{C} \cap \partial\Omega\} \qquad S_\xi v(x) = e^{i\xi x_1}\; v(x)}$$

$$A(\xi) = S_\xi\; \widetilde{A}(\xi)\; S_\xi^{-1} \qquad \widetilde{A}(\xi)\, v = -n^{-2}\, \big(\Delta v + 2i\xi\, \partial_{x_1} v - \xi^2 v\big)$$

Since for any $v \in D$, $\xi \longrightarrow \widetilde{A}(\xi)\, v \in L^2(\mathcal{C}, n^2\, dx)$ is analytic, $\widetilde{A}(\xi)$ is analytic of class (A)

Thus, $A(\xi)$ is analytic of class (B).

Since we already know that in addition the operators $A(\xi)$ have a compact resolvent and are selfadjoint for real ξ , we can apply very useful theorems from perturbation theory for linear operators.

T. Kato. Perturbation theory for linear operators.
Springer Verlag, (1994 , reprint od the edition of 1980)

The reduced cell operators : analyticity properties

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There exists a sequence D_n of complex neighborhoods of the real axis and two sequences of analytic functions

$$\mu_n(\xi) : D_n \longrightarrow \mathbb{C} \qquad \psi_n(\cdot, \xi) : D_n \longrightarrow H^2(\mathcal{C})$$

which coincide for real ξ to the eigenvalues and eigenvectors of $A(\xi)$

$$\big\{ \mu_n(\xi), n \in \mathbb{N} \big\} \equiv \big\{ \lambda_n(\xi), n \in \mathbb{N} \big\} \qquad \big\{ \psi_n(\cdot, \xi), n \in \mathbb{N} \big\} \equiv \big\{ \varphi_n(\cdot, \xi), n \in \mathbb{N} \big\}$$

$$A(\xi)\, \psi_n(\cdot, \xi) = \mu_n(\xi)\; \psi_n(\cdot, \xi) \qquad \big(\psi_n(\cdot, \xi), \psi_n(\cdot, \xi) \big)_{n^2} = \delta_{mn}$$

The reduced cell operators : analyticity properties

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still holds for $\xi \in D_n$ no longer holds for $\xi \in D_n$

The reduced cell operators : analyticity properties

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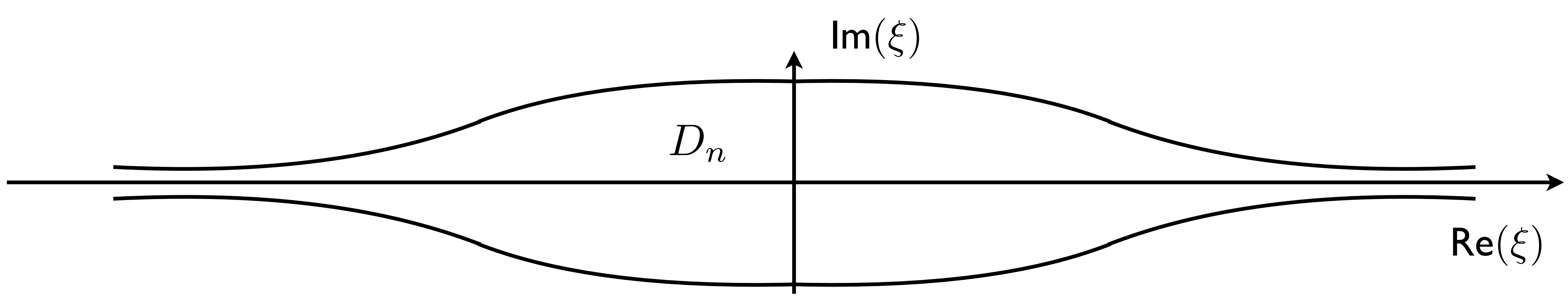
which coincide for real ξ to the eigenvalues and eigenvectors of $A(\xi)$

Remark : without any loss of generality, we can assume that the domains are symmetric with respect to the real axis



Dispersion curves

For $\xi \in \mathbb{C}$, let $A(\xi)$ be the unbounded operator in the Hilbert space $L^2(\mathcal{C}, n^2\,dx)$ such that (note that $A(\xi + 2\pi) = A(\xi)$)

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$$\boxed{A(\xi)\, \psi_n(\cdot,\xi) = \mu_n(\xi)\, \psi_n(\cdot,\xi) \qquad \big(\psi_n(\cdot,\xi), \psi_n(\cdot,\xi)\big)_{n^2} = \delta_{mn}}$$

By definition the (smooth) curves $\xi \longrightarrow \mu_n(\xi)$ are the dispersion curves of the periodic medium

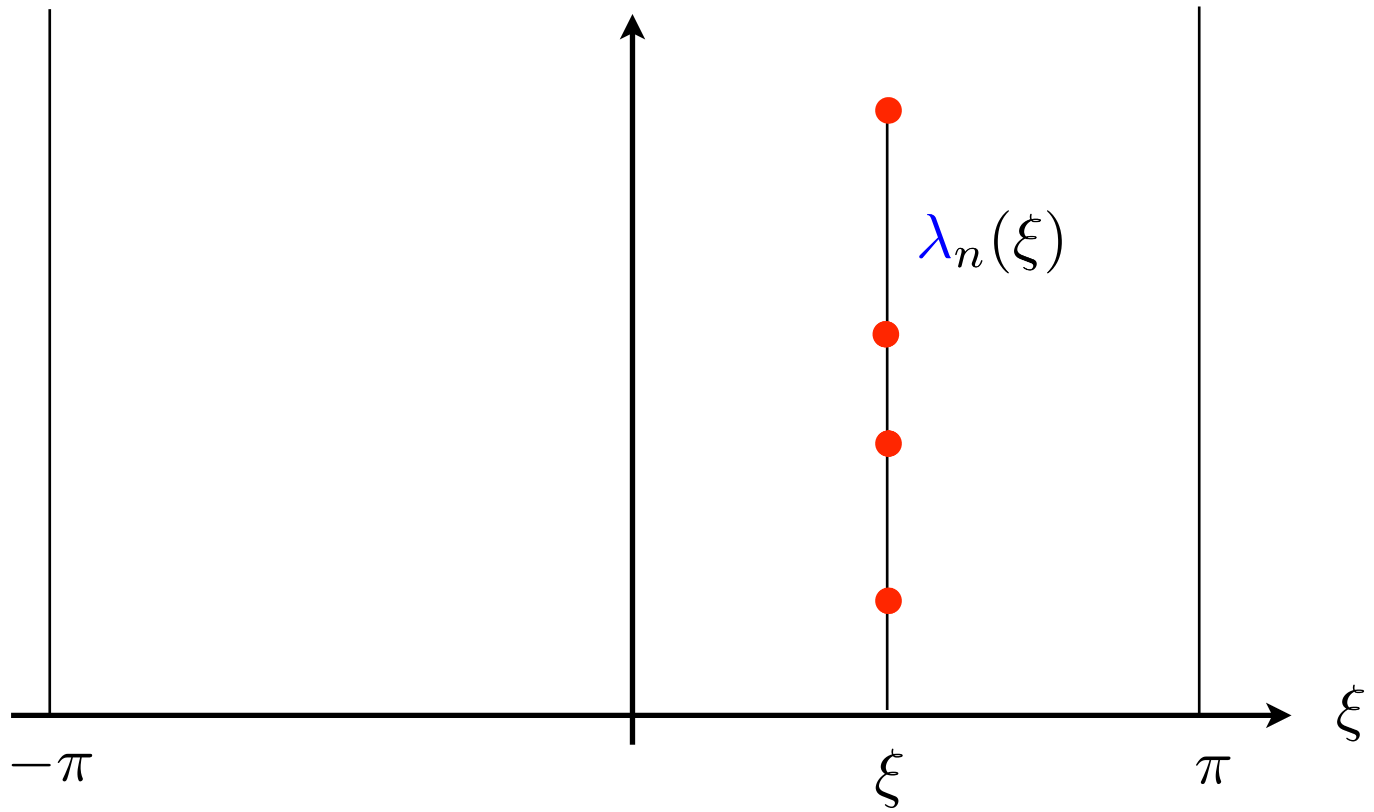
As already seen, for fixed ξ, the set $\{\mu_n(\xi), n \in \mathbb{N}\}$ is simply a rearrangement of the set $\{\lambda_n(\xi), n \in \mathbb{N}\}$. Seen as functions of ξ, the two sets only differ due to crossing points between different dispersion curves.

Using the relationships between the μ_n's and the λ_n's, one can prove that

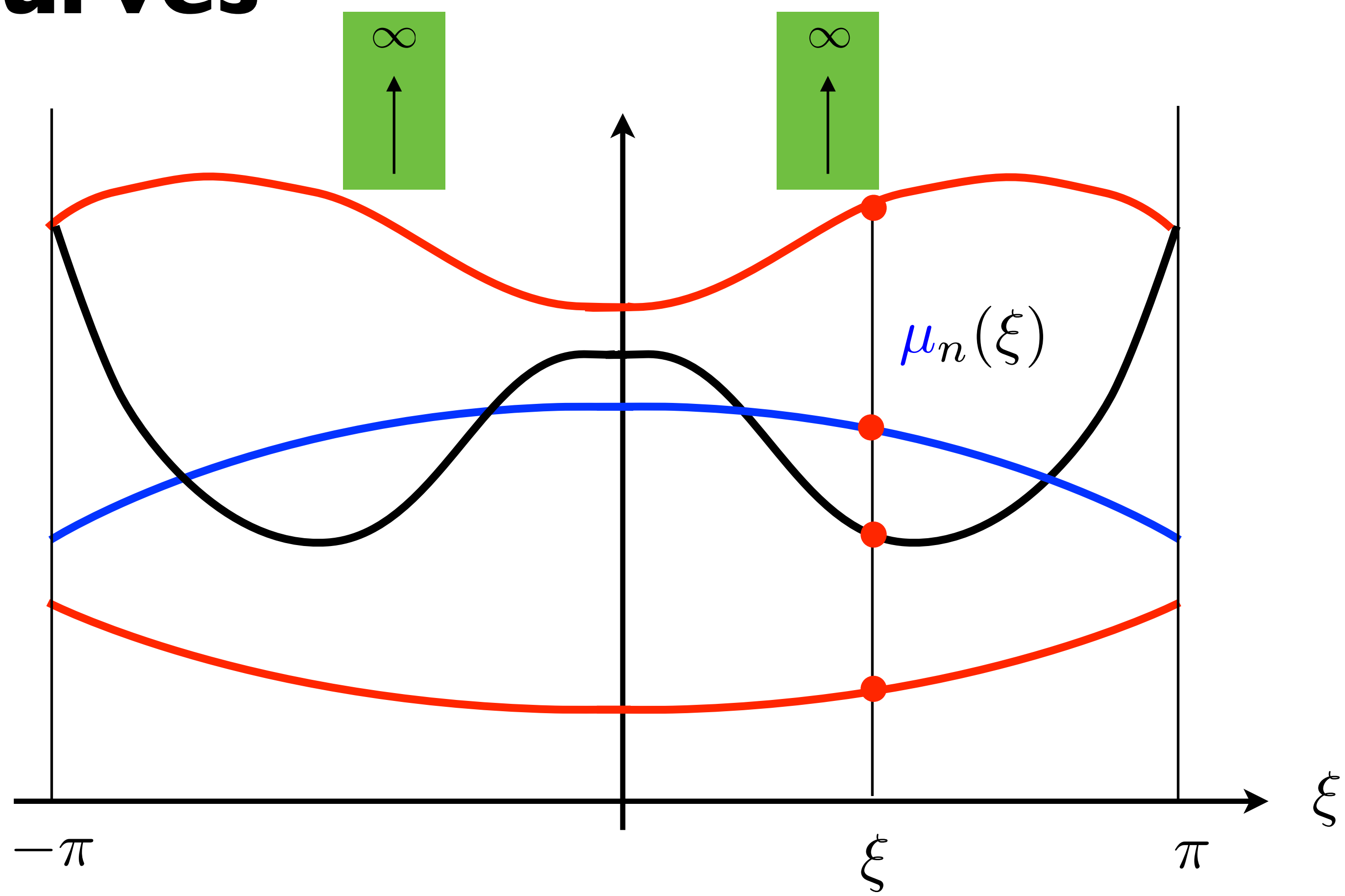
$$\lim_{n \to +\infty}\ \min_{\xi \in [-\pi,\pi]}\ \mu_n(\xi) = +\infty$$

The functions $\mu_n(\xi)$ are not necessarily periodic nor even functions of ξ.

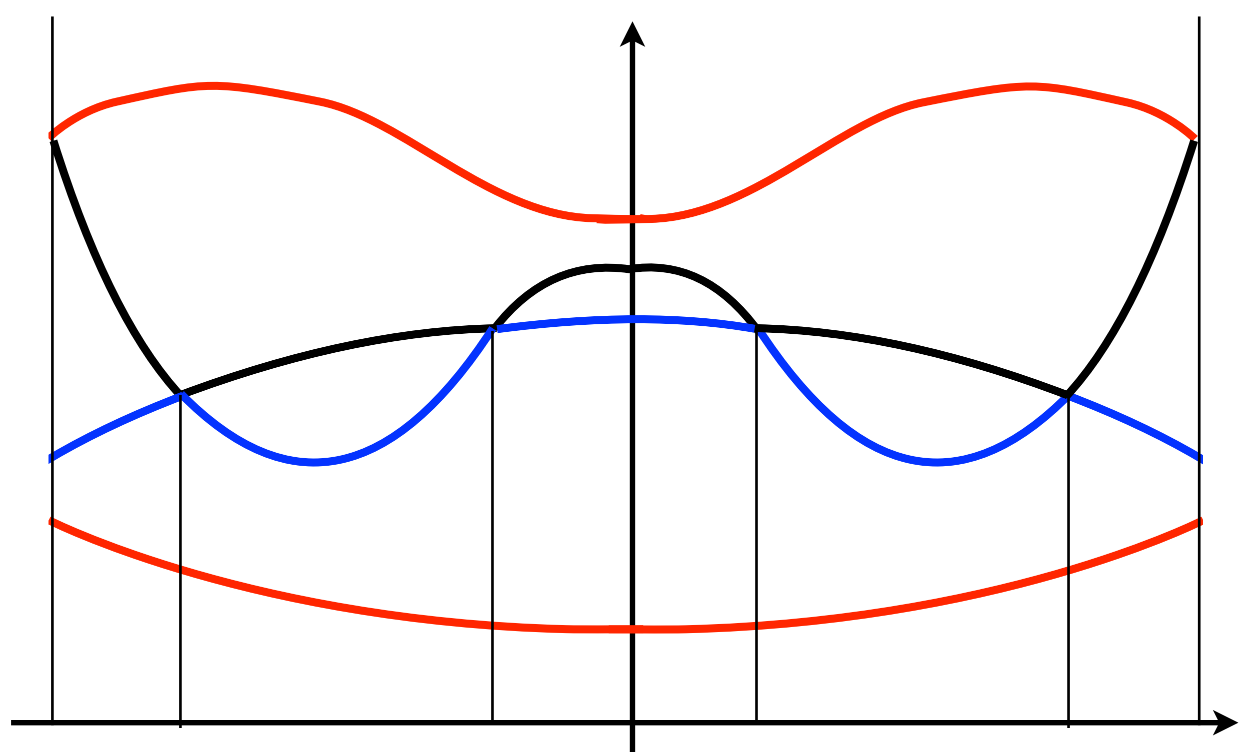
Dispersion curves



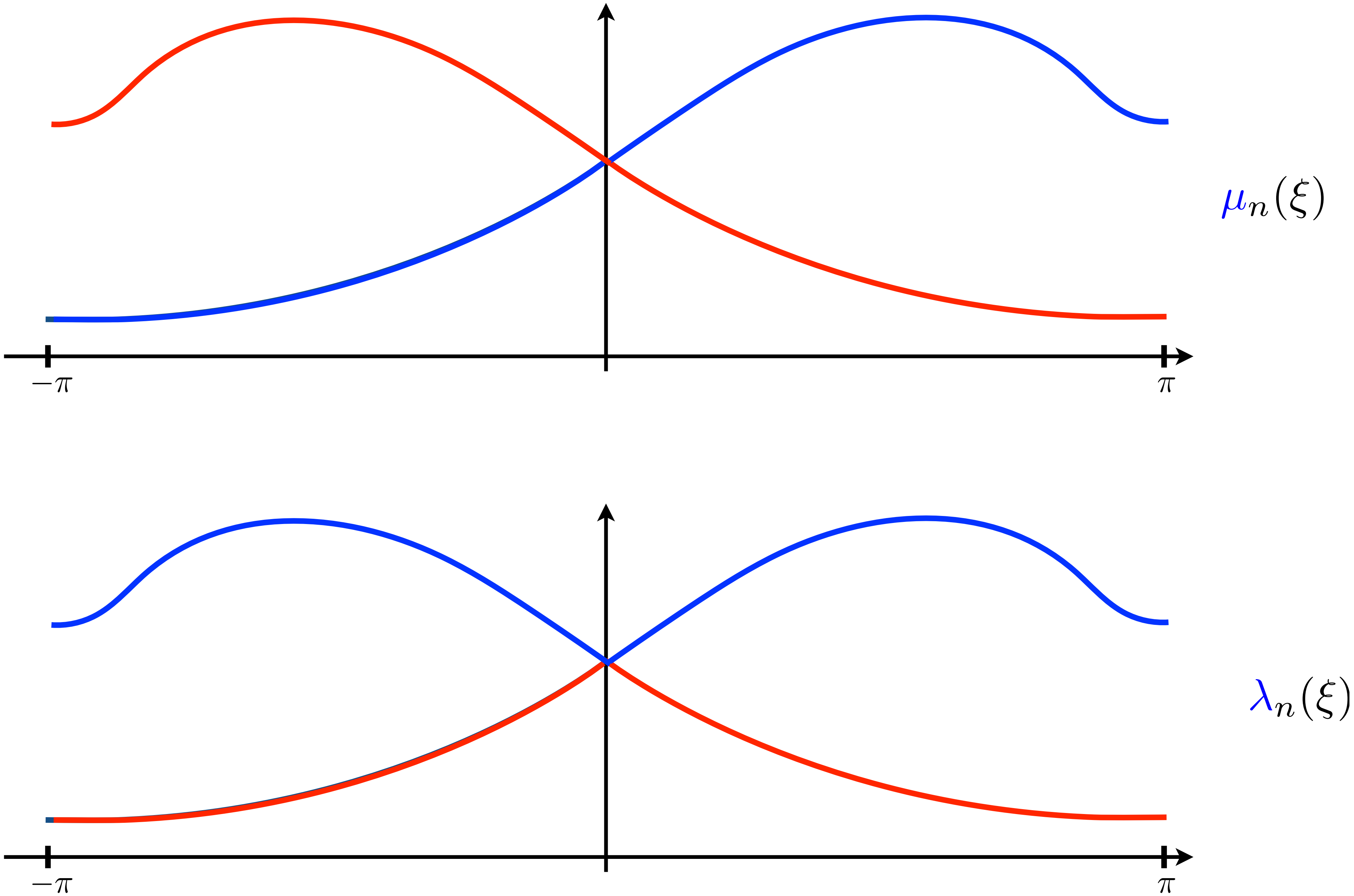
Dispersion curves



For each n , $]-\pi,\pi[$ is decomposed in a finite number of intervals along which λ_n coincides with one function μ_m : λ_n is piecewise analytic .



Dispersion curves



The functions $\mu_n(\xi)$ are not necessarily periodic nor even functions of ξ.

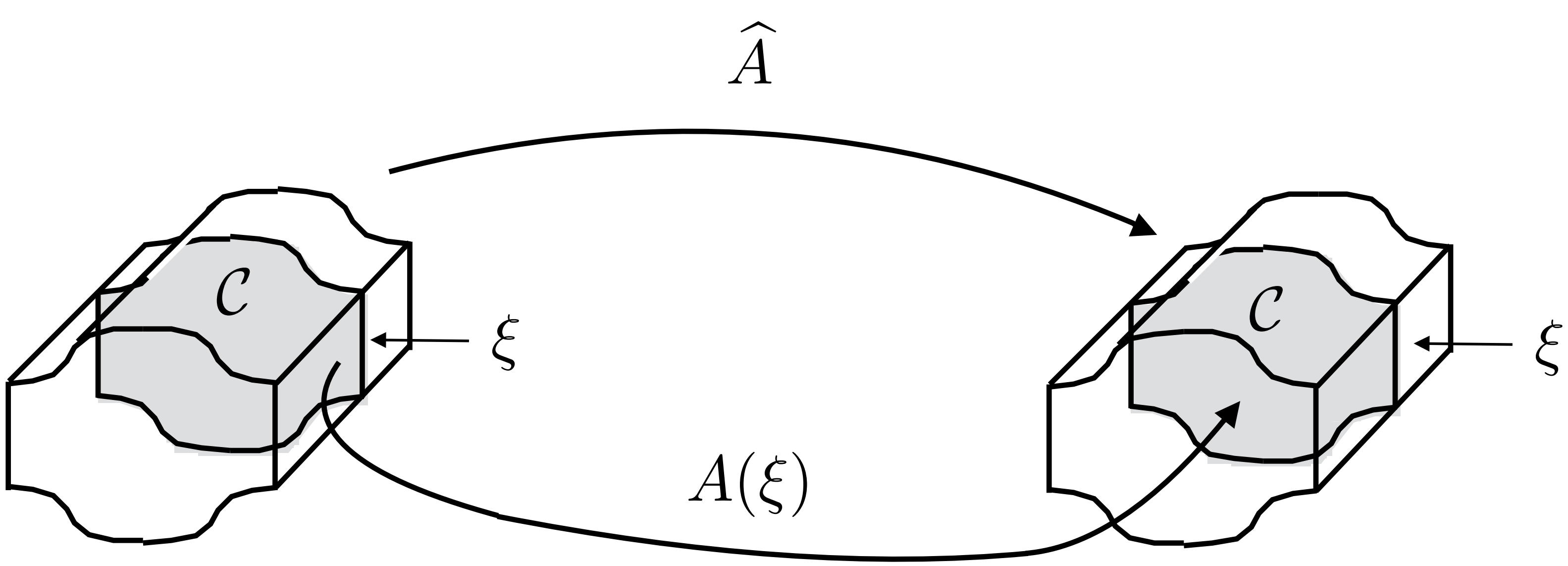
The fibered structure of the operator A

The link between the operator A and the reduced operators $A(\xi)$ is

$$\boxed{u \in D(A) \iff \widehat{u}(\xi) \in D\big(A(\xi)\big), \text{ a. e. } \xi \qquad \widehat{Au}(\xi) = A(\xi)\,\widehat{u}(\xi)}$$

$$\begin{tikzcd}
D(A) \arrow[r, "A"] \arrow[d, "\mathcal{F}"'] & L^2(\Omega) \arrow[d, "\mathcal{F}"] \\
D(\widehat{A}) \arrow[r, "\widehat{A}"] & L^2\big(\mathcal{C}\times]-\pi,\pi[\,\big)
\end{tikzcd}$$

$$\boxed{A = \mathcal{F}^{-1}\,\widehat{A}\,\mathcal{F}}$$



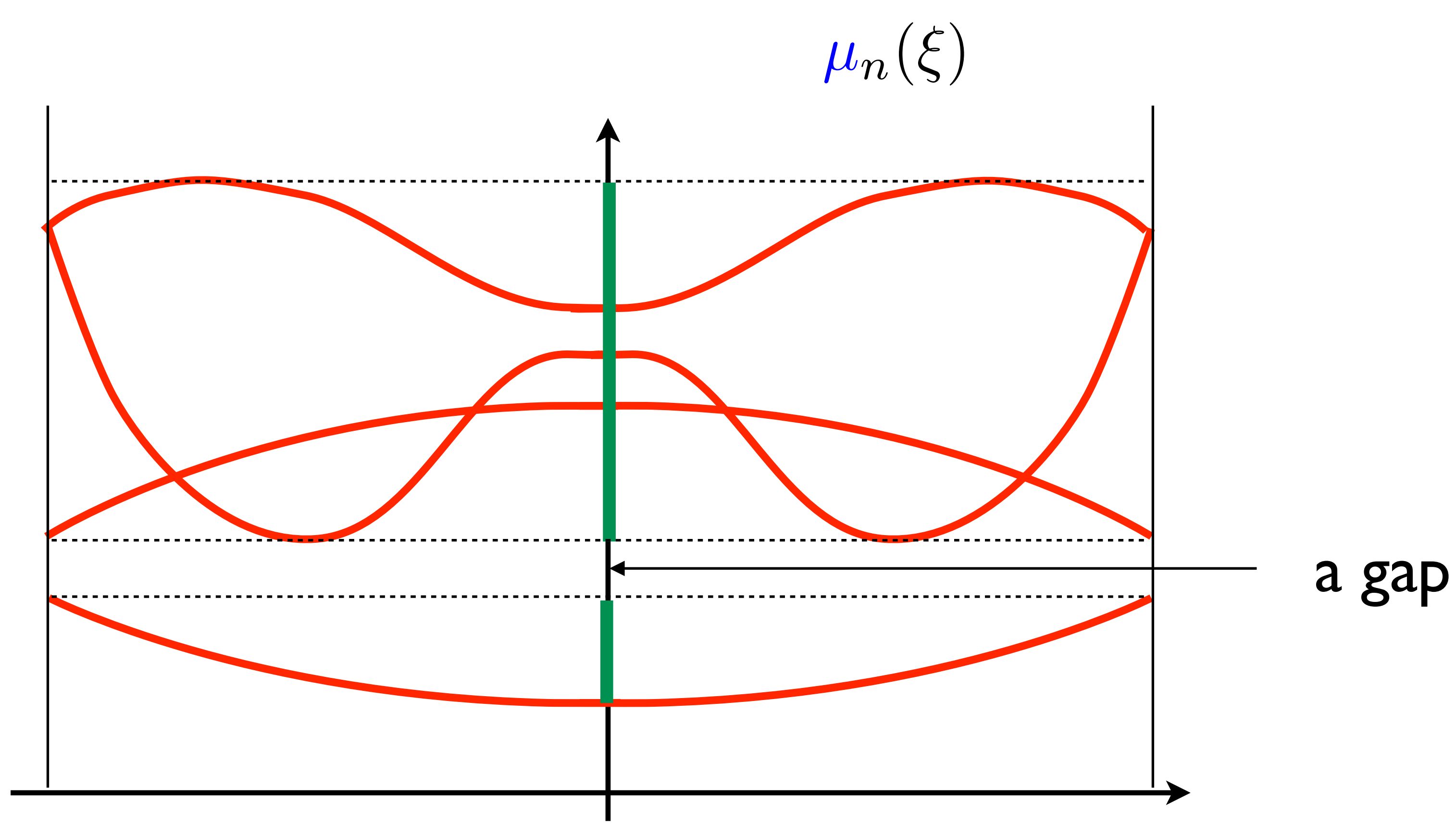
$$\boxed{\widehat{A} = \int^{\oplus} \widehat{A}(\xi)\, d\xi}$$

The spectrum of the operator A

$$A = \mathcal{F}^{-1}\,\widehat{A}\,\mathcal{F} \ \text{ and } \ \widehat{A} = \int^{\oplus} \widehat{A}(\xi)\,d\xi \qquad \Longrightarrow \qquad \sigma(A) = \bigcup_{\xi \in]-\pi,\pi[} \sigma\big(A(\xi)\big)$$

$$\sigma\big(A(\xi)\big) = \{\mu_n(\xi), n \in \mathbb{N}\} \quad \Longrightarrow \quad \sigma\big(A\big) = \bigcup_{n=0}^{+\infty} \mathcal{I}m\,\mu_n \qquad \mathcal{I}m\,\mu_n \text{ : closed, bounded interval}$$

The spectrum of A has a band structure. Gaps may exist.



The spectrum of the operator A

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$$\sigma\big(A(\xi)\big) = \big\{\mu_n(\xi), n \in \mathbb{N}\big\} \quad \Longrightarrow \quad \sigma\big(A\big) = \bigcup_{n=0}^{+\infty} \mathcal{I}m\,\mu_n \qquad \mathcal{I}m\,\mu_n \text{ : closed, bounded interval } \equiv \mu_n\big([-\pi,\pi]\big)$$

The spectrum of A has a band structure. Gaps may exist. For the point spectrum

$$\mu \in \sigma_p(A) \quad \Longleftrightarrow \quad \exists\, n \geq 0 \;/\; \mu_n(\xi) = \mu, \quad \xi \in\,]-\pi,\pi[$$

i. e. , the existence of eigenvalues is linked to the existence of flat dispersion curves

Theorem (Sobolev, Walthoe (2002), Suslina (2002))

When $d = 1$, $\sigma_p(A) = \emptyset$ i. e. the spectrum of A is absolutely continuous.

Conjecture : The above result is true whatever is d .

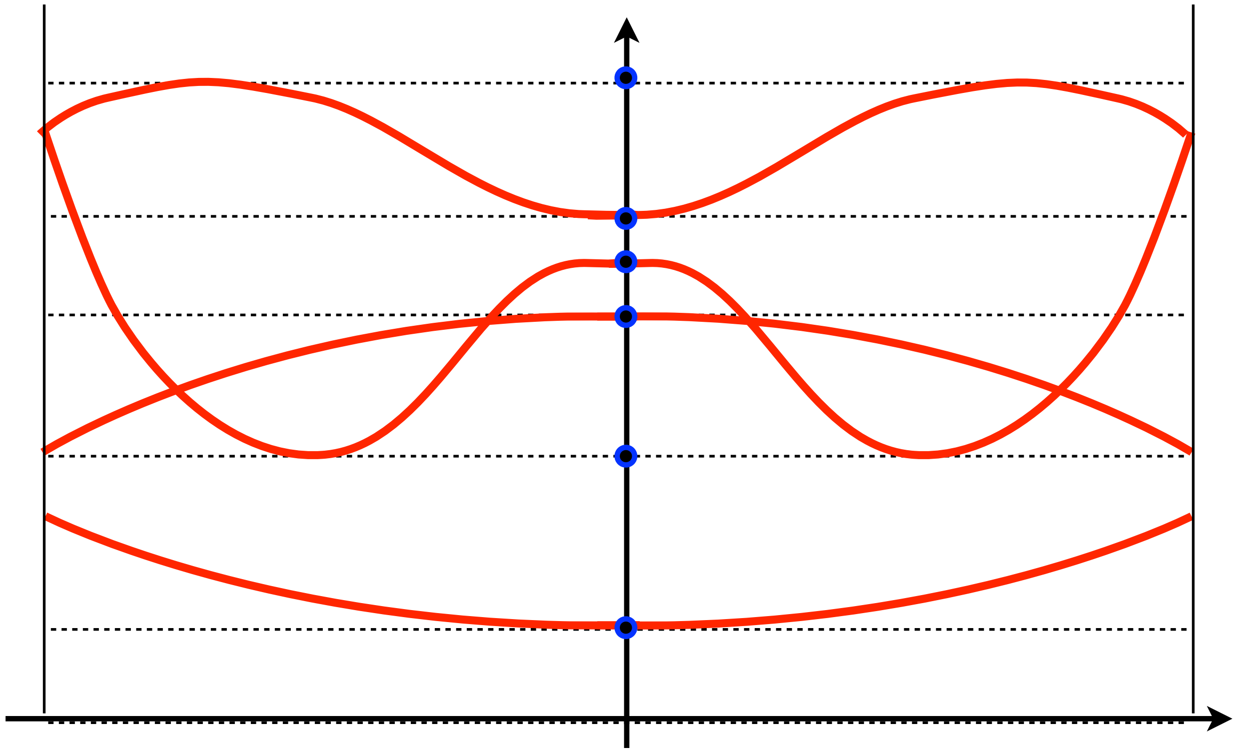
The forbidden frequencies

Definition : A forbidden frequency is a frequency ω such that there exists $n \geq 0$ and $\xi \in [-\pi, \pi]$ satisfying :

$$\mu_n(\xi) = \omega^2 \qquad \text{and} \qquad \mu_n'(\xi) = 0$$

$\sigma_0 := \{\omega^2 \,/\, \omega$ is a forbidden frequency$\}$ set of thresholds of the spectrum

Theorem : σ_0 is a discrete subset of $\mathbb{R}^+$



$\sigma_0 = \{\bullet\}$

The limiting absorption principle will hold only if ω is not a forbidden frequency.

Computation of the solution with absorption

$$(\mathcal{P}_\varepsilon) \quad -\Delta u^\varepsilon - n^2\,(\omega^2 + i\varepsilon\omega)\,u^\varepsilon = f \quad \text{in } \Omega \quad \partial_\nu u^\varepsilon = 0 \text{ on } \partial\Omega \quad u^\varepsilon \in H^2(\Omega)$$

$$\boxed{\begin{array}{l} -\Delta\widehat{u}^\varepsilon(\cdot,\xi) - n^2\,(\omega^2 + i\varepsilon\omega)\,\widehat{u}^\varepsilon(\cdot,\xi) = \widehat{f}(\cdot,\xi) \qquad \text{in } \mathcal{C} \\ \partial_\nu \widehat{u}^\varepsilon(\cdot,\xi) = 0 \qquad \text{on} \qquad \partial\Omega \cap \partial\mathcal{C} \\ \widehat{u}^\varepsilon(1,x_T) = e^{i\xi}\,\widehat{u}^\varepsilon(0,x_T), \quad \partial_{x_1}\widehat{u}^\varepsilon(1,x_T) = e^{i\xi}\,\partial_{x_1}\widehat{u}^\varepsilon(0,x_T) \iff \widehat{u}^\varepsilon(\cdot,\xi) \in H^2_\xi(\Omega) \end{array}}$$

$$D\big(A(\xi)\big) = \big\{v \in H^2_\xi(\mathcal{C}) \;/\; \partial_\nu v = 0 \text{ on } \partial\mathcal{C} \cap \partial\Omega\big\} \qquad A(\xi)\,v = -\,n^{-2}\,\Delta v$$

$$\boxed{A(\xi)\,\widehat{u}^\varepsilon(\cdot,\xi) - (\omega^2 + i\varepsilon\omega)\,\widehat{u}^\varepsilon(\cdot,\xi) = \widehat{f}(\cdot,\xi)/n^2}$$

To exploit the diagonalization of $A(\xi)$, we write $\widehat{u}^\varepsilon(\cdot,\xi) = \sum_{n=0}^{+\infty} \widehat{u}^\varepsilon_n(\xi)\,\psi_n(\cdot,\xi)$ that we substitute into the above equation to obtain

$$\big[\,\mu_n(\xi) - (\omega^2 + i\varepsilon\omega)\,\big]\,\widehat{u}^\varepsilon_n(\xi) = \widehat{f}_n(\xi) \qquad \widehat{f}_n(\xi) := \int_{\mathcal{C}} \widehat{f}(\cdot,\xi)\,\overline{\psi_n(\cdot,\xi)}\,dx$$

Computation of the solution with absorption

$$(\mathcal{P}_\varepsilon) \quad -\Delta u^\varepsilon - n^2\,(\omega^2 + i\varepsilon\omega)\,u^\varepsilon = f \quad \text{in } \Omega \quad \partial_\nu u^\varepsilon = 0 \;\text{ on } \partial\Omega \qquad u^\varepsilon \in H^2(\Omega)$$

$$\boxed{\begin{array}{l} -\Delta \widehat{u}^\varepsilon(\cdot,\xi) - n^2\,(\omega^2 + i\varepsilon\omega)\,\widehat{u}^\varepsilon(\cdot,\xi) = \widehat{f}(\cdot,\xi) \qquad \text{in } \mathcal{C} \\ \partial_\nu \widehat{u}^\varepsilon(\cdot,\xi) = 0 \qquad \text{on} \qquad \partial\Omega \cap \partial\mathcal{C} \\ \widehat{u}^\varepsilon(1,x_T) = e^{i\xi}\,\widehat{u}^\varepsilon(0,x_T), \quad \partial_{x_1}\widehat{u}^\varepsilon(1,x_T) = e^{i\xi}\,\partial_{x_1}\widehat{u}^\varepsilon(0,x_T) \quad \Longleftrightarrow \quad \widehat{u}^\varepsilon(\cdot,\xi) \in H^2_\xi(\Omega) \end{array}}$$

$$D\big(A(\xi)\big) = \big\{ v \in H^2_\xi(\mathcal{C}) \;/\; \partial_\nu v = 0 \text{ on } \partial\mathcal{C} \cap \partial\Omega \big\} \qquad A(\xi)\,v = -\,n^{-2}\,\Delta v$$

We finally obtain
$$\widehat{u}^\varepsilon(\cdot,\xi) = \sum_{n=0}^{+\infty} \frac{\widehat{f}_n(\xi)}{\mu_n(\xi) - (\omega^2 + i\varepsilon\omega)}\,\psi_n(\cdot,\xi)$$

where the series converges in $H^2(\mathcal{C})$. By inverse FB-transform, we get

$$u^\varepsilon(\cdot + p\,e_1) = (2\pi)^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\,\psi_n(\cdot,\xi)}{\mu_n(\xi) - (\omega^2 + i\varepsilon\omega)}\,e^{ip\xi}\,d\xi$$

The limiting absorption principle

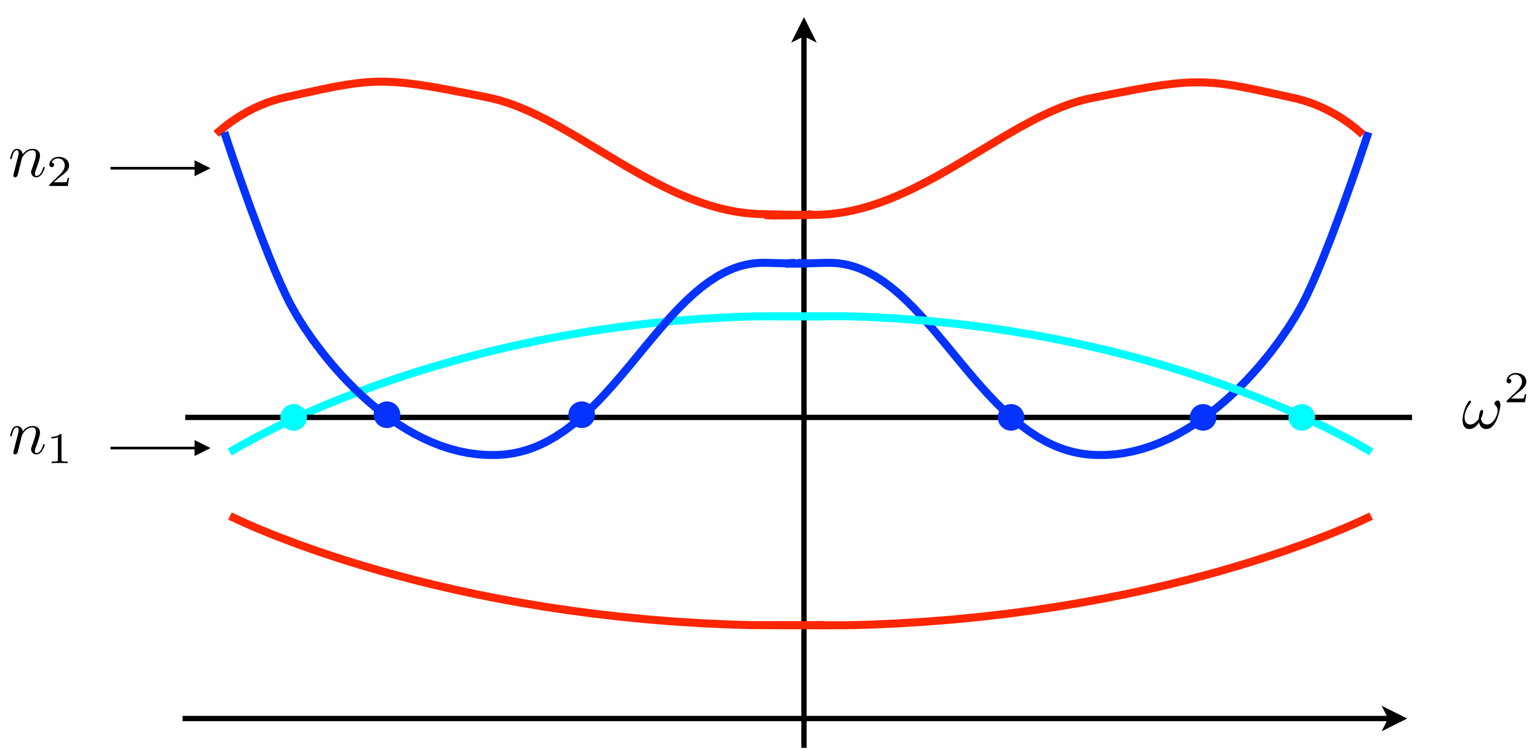
$$(\mathcal{P}_\varepsilon) \qquad -\Delta u^\varepsilon - n^2\,(\omega^2 + i\varepsilon\omega)\,u^\varepsilon = f \quad \text{in } \Omega \qquad \partial_\nu u^\varepsilon = 0 \quad \text{on } \partial\Omega \qquad u^\varepsilon \in H^2(\Omega)$$

$$u^\varepsilon(\cdot + p\,e_1) = (2\pi)^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\,\psi_n(\cdot,\xi)}{\mu_n(\xi) - (\omega^2 + i\varepsilon\omega)}\, e^{ip\xi}\, d\xi$$

It is natural to introduce $\quad I(\omega) = \big\{ n \in \mathbb{N} \;/\; \omega^2 \in \mathcal{I}m\,\mu_n \big\}$ (finite set)

$$I(\omega) = \emptyset \quad \Longleftrightarrow \quad \omega^2 \notin \sigma(A)$$

For $n \in I(\omega)$, we introduce the finite set $\;\Xi_n(\omega) = \big\{ \xi \in [-\pi, \pi[\;/\; \mu_n(\xi) = \omega^2 \big\}$



$I(\omega) = \{n_1, n_2\}$

$\Xi_{n_1}(\omega) = \{\bullet\}$

$\Xi_{n_2}(\omega) = \{\bullet\}$

The limiting absorption principle

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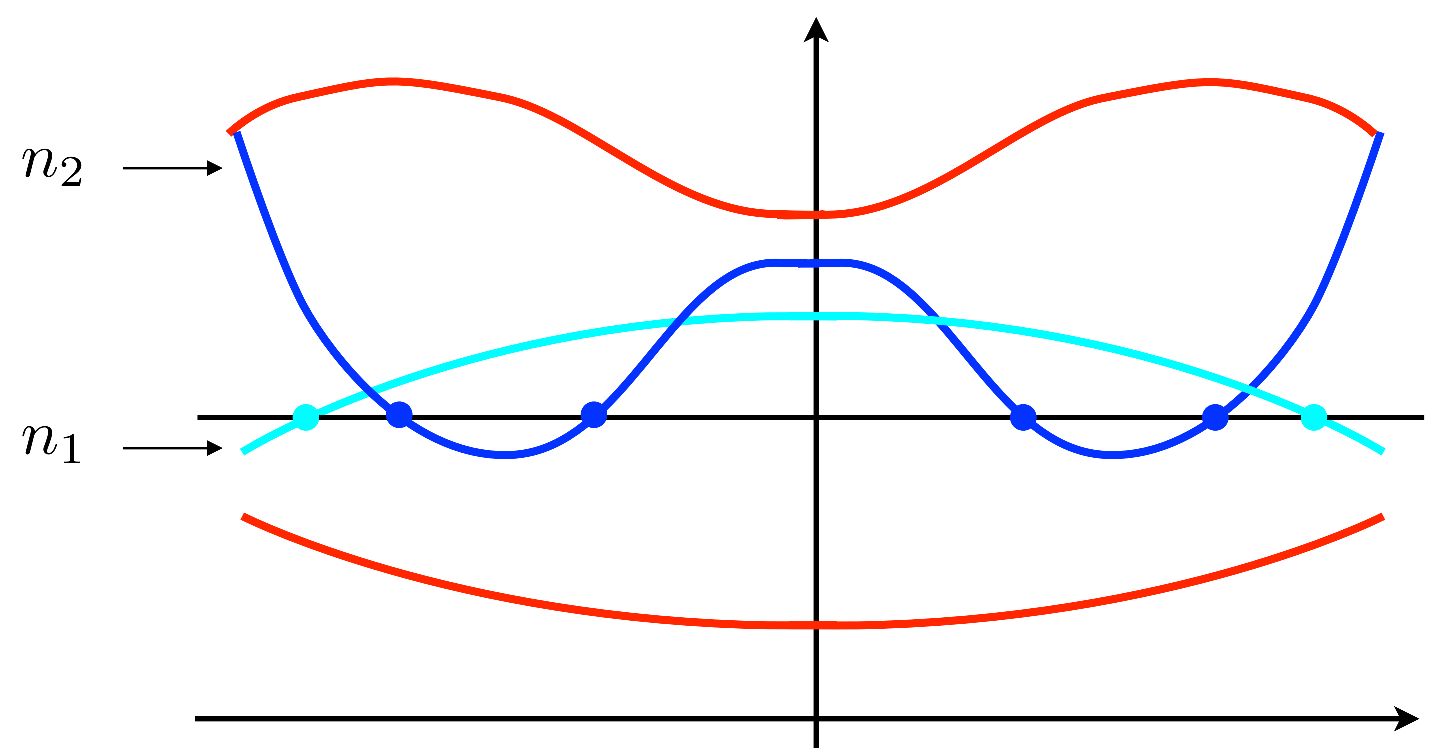
$$u^\varepsilon(\cdot + p\,e_1) = (2\pi)^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\,\psi_n(\cdot,\xi)}{\mu_n(\xi) - (\omega^2 + i\varepsilon\omega)}\, e^{ip\xi}\, d\xi$$

For $n \in I(\omega)$, we introduce the finite set $\Xi_n(\omega) = \{\xi \in [-\pi, \pi[\ /\ \mu_n(\xi) = \omega^2\}$

$\Xi(\omega) = \bigcup_{n \in I(\omega)} \Xi_n(\omega)$: the set of propagative wave numbers at frequency

(the union being understood in the sense $\{\xi_1, \xi_2\} \cup \{\xi_1, \xi_3\} = \{\xi_1, \xi_1, \xi_2, \xi_3\}$)



$I(\omega) = \{n_1, n_2\}$

$\Xi_{n_1}(\omega) = \{\bullet\}$

$\Xi_{n_2}(\omega) = \{\bullet\}$

$\Xi(\omega) = \{\bullet\ \bullet\}$

The limiting absorption principle

$$(\mathcal{P}_\varepsilon) \quad -\Delta u^\varepsilon - n^2\,(\omega^2 + i\varepsilon\omega)\,u^\varepsilon = f \quad \text{in } \Omega \quad \partial_\nu u^\varepsilon = 0 \text{ on } \partial\Omega \quad u^\varepsilon \in H^2(\Omega)$$

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For $n \in I(\omega)$, we introduce the finite set $\Xi_n(\omega) = \{\xi \in [-\pi,\pi[\ /\ \mu_n(\xi) = \omega^2\}$

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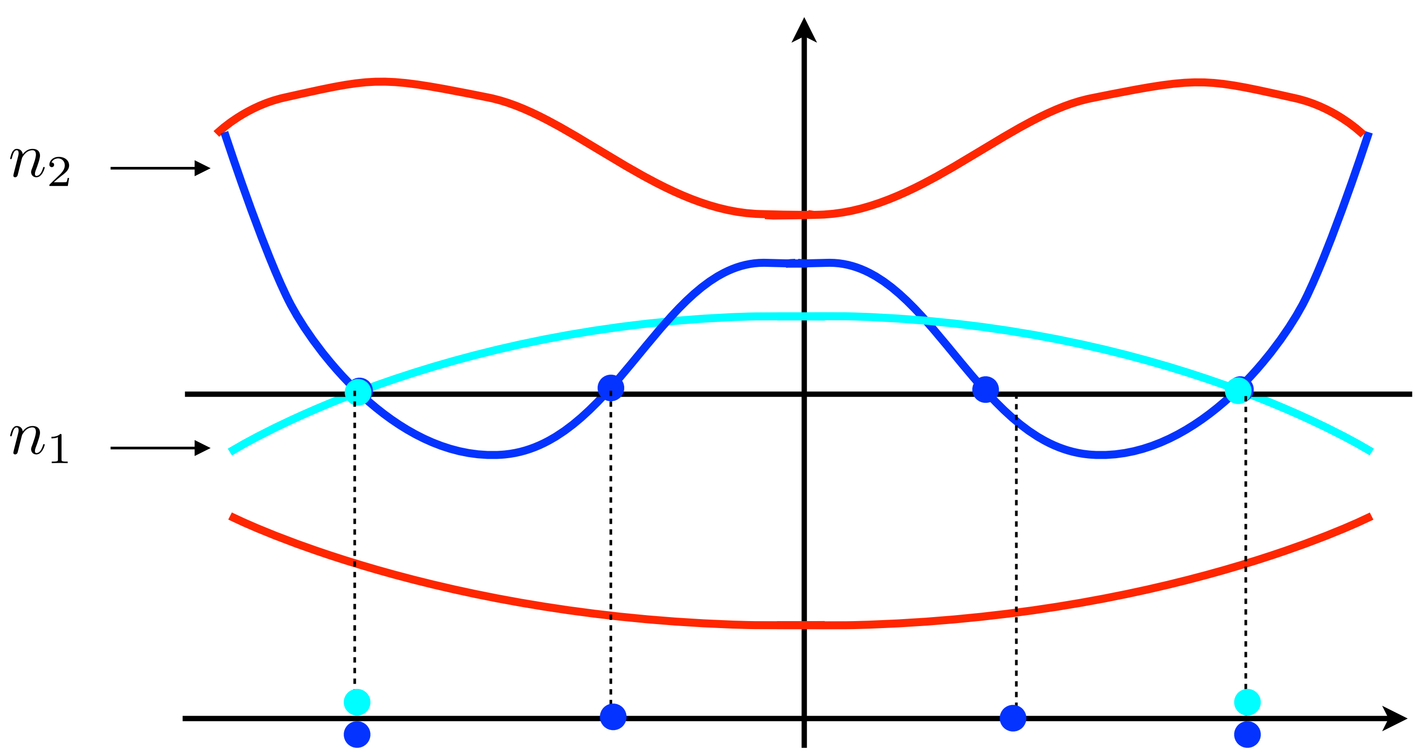
: the set of propagative wave numbers at frequency

(the union being understood in the sense $\{\xi_1,\xi_2\}\cup\{\xi_1,\xi_3\} = \{\xi_1,\xi_1,\xi_2,\xi_3\}$)



$I(\omega) = \{n_1, n_2\}$

$\Xi_{n_1}(\omega) = \{\bullet\}$

$\Xi_{n_2}(\omega) = \{\bullet\}$

$\Xi(\omega) = \{\bullet\ \bullet\}$

Convergence : the evanescent part

$(\mathcal{P}_\varepsilon)$ $\quad -\Delta u^\varepsilon - n^2\,(\omega^2 + i\varepsilon\omega)\,u^\varepsilon = f$ in Ω $\quad \partial_\nu u^\varepsilon = 0$ on $\partial\Omega$ $\quad u^\varepsilon \in H^2(\Omega)$

$$u^\varepsilon(\cdot + p\,e_1) = (2\pi)^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\,\psi_n(\cdot,\xi)}{\mu_n(\xi) - (\omega^2 + i\varepsilon\omega)}\, e^{ip\xi}\, d\xi$$

$$\boxed{I(\omega) = \left\{ n \in \mathbb{N} \;/\; \omega^2 \in \mathcal{I}m\, \mu_n \right\}}$$

$$\boxed{u^\varepsilon = u^\varepsilon_{evan} + u^\varepsilon_{prop} \qquad u^\varepsilon_{evan} = \sum_{n \notin I(\omega)} \cdots \qquad u^\varepsilon_{prop} = \sum_{n \in I(\omega)} \cdots}$$

For $\; n \notin I(\omega)$, one can define $u_{evan} \in H^2(\Omega)$, cell by cell, as

$$u_{evan}(\cdot + p\,e_1) := (2\pi)^{-\frac{1}{2}} \sum_{n \notin I(\omega)} \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\,\psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi$$

and one can prove that $\quad \boxed{\|u^\varepsilon_{evan} - u_{evan}\|_{H^2(\Omega)} \leq C\,\varepsilon}$

Key argument : $\quad \inf_{n \notin I(\omega)} \; \inf_{\xi \in [-\pi,\pi]} |\mu_n(\xi) - \omega^2| > 0$

$\mathcal{I}m\,\mu_n$ $\quad n \notin I(\omega)$

Convergence : the evanescent part

Proof of the L^2 estimate : we use FB-Plancherel's theorem

$$\left\| u^{\varepsilon}_{evan} - u_{evan} \right\|^2_{n^2} = \int_{-\pi}^{\pi} \left\| \left(\widehat{u}^{\varepsilon}_{evan} - \widehat{u}_{evan} \right)(\cdot,\xi) \right\|^2_{n^2}$$

where by definition

$$\boxed{\widehat{u}^{\varepsilon}_{evan}(\cdot,\xi) = \sum_{n \notin I(\omega)} \frac{\widehat{f}_n(\xi)}{\mu_n(\xi) - (\omega^2 + i\varepsilon\omega)} \, \psi_n(\cdot,\xi)} \qquad \boxed{\widehat{u}_{evan}(\cdot,\xi) = \sum_{n \notin I(\omega)} \frac{\widehat{f}_n(\xi)}{\mu_n(\xi) - \omega^2} \, \psi_n(\cdot,\xi)}$$

so that $\left\| \left[\widehat{u}^{\varepsilon}_{evan} - \widehat{u}_{evan} \right](\cdot,\xi) \right\|^2_{n^2} = \sum_{n \notin I(\omega)} |d^{\varepsilon}_n(\xi)|^2 \, |\widehat{f}_n(\xi)|^2$ where we have set

$$\widehat{d}^{\varepsilon}_n(\xi) := \left(\mu_n(\xi) - (\omega^2 + i\varepsilon\omega) \right)^{-1} - \left(\mu_n(\xi) - \omega^2 \right)^{-1} = \frac{i\varepsilon\,\omega}{\left(\mu_n(\xi) - (\omega^2 + i\varepsilon\omega) \right) \left(\mu_n(\xi) - \omega^2 \right)}$$

In particular $|d^{\varepsilon}_n(\xi)| \leq \dfrac{\varepsilon\,\omega}{|\mu_n(\xi) - \omega^2|^2} \leq C\,\varepsilon$ since $\inf\limits_{n \notin I(\omega)} \inf\limits_{\xi \in [-\pi,\pi]} |\mu_n(\xi) - \omega^2| > 0$

$$\left\| \left[\widehat{u}^{\varepsilon}_{evan} - \widehat{u}_{evan} \right](\cdot,\xi) \right\|^2_{n^2} \leq C^2\,\varepsilon^2 \sum_{n \notin I(\omega)} |\widehat{f}_n(\xi)|^2 \leq C^2\,\varepsilon^2\, \|\widehat{f}(\cdot,\xi)\|^2_{n^2}$$

After integration in ξ, we get (Plancherel)

$$\boxed{\left\| u^{\varepsilon}_{evan} - u_{evan} \right\|^2_{n^2} \leq C^2\,\varepsilon^2\, \|f\|^2}$$

Convergence : the propagative part

$\boxed{\omega^2 \notin \sigma_0}$

$$(\mathcal{P}_\varepsilon) \quad -\Delta u^\varepsilon - n^2\,(\omega^2 + i\varepsilon\omega)\,u^\varepsilon = f \quad \text{in } \Omega \quad \partial_\nu u^\varepsilon = 0 \quad \text{on } \partial\Omega \quad u^\varepsilon \in H^2(\Omega)$$

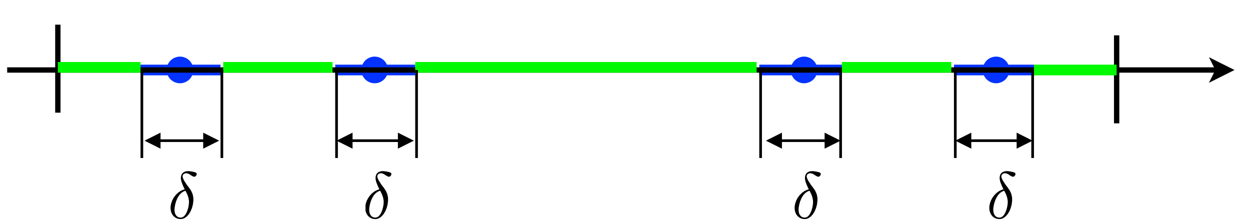
$$u^\varepsilon_{prop}(\cdot + p\,e_1) = (2\pi)^{-\frac{1}{2}} \sum_{n\in I(\omega)} \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\,\psi_n(\cdot,\xi)}{\mu_n(\xi) - (\omega^2 + i\varepsilon\omega)}\, e^{ip\xi}\, d\xi$$

$$I(\omega) = \{ n \in \mathbb{N} \;/\; \omega^2 \in \mathcal{I}m\,\mu_n \} \qquad \Xi_n(\omega) = \{ \xi \in [-\pi,\pi[\;/\;\mu_n(\xi) = \omega^2 \}$$

For $n \in I(\omega)$, $\displaystyle\int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\,\psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi$ does not exist but we can define

$$p.v. \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\,\psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi \;=\; \lim_{\delta\downarrow 0} \int_{I_n^\delta(\omega)} \frac{\widehat{f}_n(\xi)\,\psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi \qquad \in H^2(\mathcal{C})$$

where by definition $\displaystyle I_n^\delta(\omega) = \,]-\pi,\pi[\;\setminus \bigcup_{\xi^*\in\Xi_n(\omega)}]\,\xi^* - \delta\,,\xi^* + \delta\,[$



$\Xi_n(\omega) = \{\bullet\}$ $\quad I_n^\delta(\omega)$

Key point : $\mu_n'(\xi^*) \neq 0, \quad \forall\, \xi^* \in \Xi_n(\omega) \quad (\omega^2 \notin \sigma_0)$

Plemelj-Privalov theorem

Let I be an open bounded interval of $\mathbb{R}$ containing 0 and X a Banach space.

Let $V \in C^r(I;X)$ for some $r \in]0,1]$, which means that (Hölder continuity)

$$\forall\ (t,t') \in I, \qquad \|V(t) - V(t')\| \leq C\ |t-t'|^r$$

Then $p.v. \int_I \frac{V(t)}{t}\, dt := \lim_{\delta \to 0} \int_{I \setminus [-\delta,\delta]} \frac{V(t)}{t}\, dt$ exists in X and for $\varepsilon > 0$

$$\boxed{\left\| \int_I \frac{V(t)}{t - i\,\varepsilon}\, dt - \left(p.v. \int_I \frac{V(t)}{t}\, dt + i\pi\, V(0) \right) \right\| \ \leq\ C\ \varepsilon^r\ \|V\|_{C^r(I,X)}}$$

where by definition $\|V\|_{C^r(I,X)} := \sup_{t \in I} \|V(t)\| + \sup_{(t,t') \in I} \frac{\|V(t) - V(')\|}{|t-t'|^r}$

Remark : Hölder regularity and $\varepsilon > 0$ are important. For $\varepsilon < 0$

$$+\, i\pi\, V(0) \quad \longrightarrow \quad -\, i\pi\, V(0)$$

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Corollary : if $V \in H^s(I;X),\ s > 1/2$ $\qquad$ ($H^s(I,X) \hookrightarrow C^{s-\frac{1}{2}}(I,X)$)

$$\boxed{\left\| \int_I \frac{V(t)}{t - i\,\varepsilon}\,dt - \left(p.v. \int_I \frac{V(t)}{t}\,dt + i\pi\,V(0)\right) \right\| \ \leq\ C\ \varepsilon^{s-\frac{1}{2}}\ \|V\|_{H^s(I,X)}}$$

Proof of the theorem : without ant loss of generality $I = \,]-a, a\,[$

$$\int_{-a}^{a} \frac{V(t)}{t - i\,\varepsilon}\, dt = \int_{-a}^{a} \frac{t\, V(t)}{t^2 + \varepsilon^2}\, dt + i\varepsilon \int_{-a}^{a} \frac{V(t)}{t^2 + \varepsilon^2}\, dt$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$p.v. \int_I \frac{V(t)}{t}\, dt \qquad\qquad \pi\, V(0)$$

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$$\int_{-a}^{a} \frac{t\,V(t)}{t^2+\varepsilon^2}\, dt \longrightarrow p.v. \int_I \frac{V(t)}{t}\, dt \qquad \int_{-a}^{a} \frac{V(t)}{t^2+\varepsilon^2}\, dt \longrightarrow \pi\, V(0)$$

$$\int_{-a}^{a} \frac{V(t)}{t^2+\varepsilon^2}\, dt = \varepsilon^{-1} \int_{-\frac{a}{\varepsilon}}^{\frac{a}{\varepsilon}} \frac{V(\varepsilon\,\tau)}{\tau^2+1}\, d\tau \qquad \pi\, V(0) = \int_{-\infty}^{\infty} \frac{V(0)}{\tau^2+1}\, d\tau$$

$$\boxed{\varepsilon \int_{-a}^{a} \frac{V(t)}{t^2+\varepsilon^2}\, dt - \pi\, V(0) = \int_{-\frac{a}{\varepsilon}}^{\frac{a}{\varepsilon}} \frac{V(\varepsilon\,\tau) - V(0)}{\tau^2+1}\, d\tau + V(0) \int_{|\tau|>\varepsilon} \frac{d\tau}{1+\tau^2}}$$

$$\left\| \int_{-\frac{a}{\varepsilon}}^{\frac{a}{\varepsilon}} \frac{V(\varepsilon\,\tau) - V(0)}{\tau^2+1}\, d\tau \right\| \leq \varepsilon^r \left(\int_{-\infty}^{+\infty} \frac{\tau^r}{\tau^2+1}\, d\tau \right) \|V\|_{C^r(I,X)}$$

$$\left\| V(0) \int_{|\tau|>\frac{a}{\varepsilon}} \frac{d\tau}{1+\tau^2} \right\| \leq 2\, \frac{\varepsilon}{a}\, \|V\|_{C^r(I,X)}$$

Proof of the theorem : without ant loss of generality $I =]-a,a[$

$$\int_{-a}^{a} \frac{V(t)}{t+i\,\varepsilon}\,dt = \int_{-a}^{a} \frac{t\,V(t)}{t^2+\varepsilon^2}\,dt - i\varepsilon \int_{-a}^{a} \frac{V(t)}{t^2+\varepsilon^2}\,dt$$

$$\int_{-a}^{a} \frac{t\,V(t)}{t^2+\varepsilon^2}\,dt \longrightarrow p.v. \int_I \frac{V(t)}{t}\,dt \qquad\qquad \int_{-a}^{a} \frac{V(t)}{t^2+\varepsilon^2}\,dt \longrightarrow \pi\, V(0)$$

By a symmetry argument we observe that

$\in L^1$ since $V \in C^r(I;X)$

$$\int_{I\setminus[-\delta,\delta]} \frac{V(t)}{t}\,dt = \int_{I\setminus[-\delta,\delta]} \frac{V(t)-V(0)}{t}\,dt \quad\Longrightarrow\quad p.v. \int_I \frac{V(t)}{t}\,dt = \int_I \frac{V(t)-V(0)}{t}\,dt$$

Using again the symmetry argument, we can write

$$\boxed{\int_{-a}^{a} \frac{t\,V(t)}{t^2+\varepsilon^2}\,dt - p.v. \int_{-a}^{a} \frac{V(t)}{t}\,dt = \int_{-a}^{a} \frac{t\,\big(V(t)-V(0)\big)}{t^2+\varepsilon^2}\,dt - \int_{-a}^{a} \frac{V(t)-V(0)}{t}\,dt}$$

We compute that $|t/(t^2+\varepsilon^2) - 1/t| = |t|^{-1}\,[\varepsilon^2/(t^2+\varepsilon^2)]$ which implies

$$\left| \int_{-a}^{a} \frac{t\,V(t)}{t^2+\varepsilon^2}\,dt - p.v. \int_{-a}^{a} \frac{V(t)}{t}\,dt \right| \leq \varepsilon^2 \left(\int_{-a}^{a} \frac{|t|^{r-1}}{t^2+\varepsilon^2}\,dt \right) \|V\|_{C^r(I,X)}$$

Proof of the theorem : without ant loss of generality $I = \,]-a, a[$

$$\int_{-a}^{a} \frac{V(t)}{t + i\,\varepsilon}\, dt = \int_{-a}^{a} \frac{t\, V(t)}{t^2+\varepsilon^2}\, dt - i\varepsilon \int_{-a}^{a} \frac{V(t)}{t^2+\varepsilon^2}\, dt$$

$$p.v. \int_I \frac{V(t)}{t}\, dt \qquad\qquad \pi\, V(0)$$

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$$\int_{I \setminus [-\delta,\delta]} \frac{V(t)}{t}\, dt = \int_{I\setminus[-\delta,\delta]} \frac{V(t)-V(0)}{t}\, dt \quad \Longrightarrow \quad p.v. \int_I \frac{V(t)}{t}\, dt = \int_I \frac{V(t)-V(0)}{t}\, dt$$

$$\left| \int_{-a}^{a} \frac{t\,V(t)}{t^2+\varepsilon^2}\, dt - p.v. \int_{-a}^{a} \frac{V(t)}{t}\, dt \right| \le \varepsilon^2 \left(\int_{-a}^{a} \frac{|t|^{r-1}}{t^2+\varepsilon^2}\, dt \right) \|V\|_{C^r(I,X)}$$

To conclude, it suffices to notice that $(\, \tau = \varepsilon\, t\,)$

$$\varepsilon^2 \int_{-a}^{a} \frac{|t|^{r-1}}{t^2+\varepsilon^2}\, dt = \varepsilon^r \int_{-\frac{a}{\varepsilon}}^{\frac{a}{\varepsilon}} \frac{|\tau|^{r-1}}{\tau^2+1}\, d\tau \le C_r\, \varepsilon^r, \qquad C_r := \int_{-\infty}^{+\infty} \frac{|\tau|^{r-1}}{\tau^2+1}\, d\tau$$

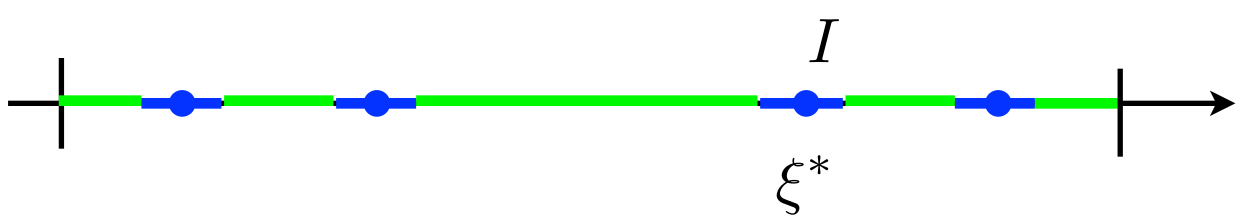
Plemelj-Privalov theorem

Application to $u_n^\varepsilon(\cdot + p, e_1) := (2\pi)^{-\frac{1}{2}} \displaystyle\int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\,\psi_n(\cdot,\xi)}{\mu_n(\xi) - (\omega^2 + i\varepsilon\omega)}\, e^{ip\xi}\, d\xi\,, \quad n \in I(\omega)$

The problems only come from the points ξ^* in the set

$$\Xi_n(\omega) = \big\{\xi \in [-\pi, \pi[\ /\ \mu_n(\xi) = \omega^2\big\}$$

One decomposes the integral into integrals over small neighborhoods of such ξ^* plus the rest that does not pose any difficulty



In the neighborhood I of ξ^*, $\quad \mu_n(\xi) - \omega^2 \ \sim\ \mu_n'(\xi^*)\,(\xi - \xi^*)$

$$\longrightarrow \qquad \frac{1}{\mu_n'(\xi^*)} \int_I \frac{\widehat{f}_n(\xi)\,\psi_n(\cdot,\xi)}{(\xi - \xi^*) - i\varepsilon\,\omega\,\mu_n'(\xi^*)^{-1}}\, e^{ip\xi}\, d\xi$$

To apply the Plemelj-Privalov's theorem with $X = H^2(\mathcal{C})$ it suffices to check that

$$\xi \longrightarrow \widehat{f}_n(\xi)\,\psi_n(\cdot,\xi) \qquad\qquad \widehat{f}_n(\cdot,\xi) := \int_{\mathcal{C}} f(\cdot,\xi)\,\overline{\psi_n(\cdot,\xi)}\, dx$$

belongs to $H^s\big(-\pi, \pi; H^2(\mathcal{C})\big)$ for some $s > 1/2$

Plemelj-Privalov theorem

Application to $u_n^\varepsilon(\cdot + p, e_1) := (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot,\xi)}{\mu_n(\xi) - (\omega^2 + i\varepsilon\omega)}\, e^{ip\xi}\, d\xi\,, \quad n \in I(\omega)$

$$\longrightarrow \qquad \frac{1}{\mu_n'(\xi^*)} \int_I \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot,\xi)}{(\xi - \xi^*) - i\varepsilon\, \omega\, \mu_n'(\xi^*)^{-1}}\, e^{ip\xi}\, d\xi$$

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belongs to $H^s\big(-\pi,\pi; H^2(\mathcal{C})\big)$ for some $s > 1/2$

Since $\psi_n(\cdot,\xi) : D_n \longrightarrow H^2(\mathcal{C})$ is analytic, the only limitation in regularity comes from $\widehat{f}_n(\xi)$. More precisely, the desired regularity will be obtained as soon as

$$\widehat{f}(\cdot,\xi) \in H^s\big(-\pi,\pi; L^2(\mathcal{C})\big)$$

According to the properties of the Floquet-Bloch transform, this is guaranteed if

$$f \in L^2_s(\Omega), \quad s > \frac{1}{2}$$

Plemelj-Privalov theorem

Application to $u_n^\varepsilon(\cdot + p, e_1) := (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot,\xi)}{\mu_n(\xi) - (\omega^2 + i\varepsilon\omega)}\, e^{ip\xi}\, d\xi\,, \quad n \in I(\omega)$

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Assuming that $f \in L^2_s(\Omega), \quad s > \dfrac{1}{2}$, it follows that setting

$$\boxed{u_n(\cdot + p, e_1) := (2\pi)^{-\frac{1}{2}}\, p.v. \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi \; + i\, \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{\xi^* \in \Xi_n(\omega)} \frac{\widehat{f}_n(\xi^*)\, \psi_n(\cdot,\xi^*)}{\left|\mu_n'(\xi^*)\right|}\, e^{ip\xi^*}}$$

we have the following convergence estimate

$$\|u_n^\varepsilon(\cdot + p, e_1) - u_n(\cdot + p, e_1)\|_{H^2(\mathcal{C})} \leq \; C(n,p)\, \varepsilon^{s - \frac{1}{2}}\, \|f\|_{L^2_s(\Omega)}$$

Plemelj-Privalov theorem

Application to $u_n^\varepsilon(\cdot+p,e_1) := (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\,\psi_n(\cdot,\xi)}{\mu_n(\xi)-(\omega^2+i\varepsilon\omega)}\, e^{ip\xi}\, d\xi\,, \quad n\in I(\omega)$

$$\longrightarrow \quad \frac{1}{\mu_n'(\xi^*)} \int_I \frac{\widehat{f}_n(\xi)\,\psi_n(\cdot,\xi)}{(\xi-\xi^*) - i\varepsilon\,\omega\,\mu_n'(\xi^*)^{-1}}\, e^{ip\xi}\, d\xi$$

The rigorous proof uses the change of variable

$$\tau = \mu_n(\xi) - \omega^2$$

valid if the interval I is small enough

Assuming that $f\in L^2_s(\Omega), \quad s>\frac{1}{2}$, it follows that setting

$$u_n(\cdot+p,e_1) := (2\pi)^{-\frac{1}{2}}\ p.v. \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\,\psi_n(\cdot,\xi)}{\mu_n(\xi)-\omega^2}\, e^{ip\xi}\, d\xi \;+ i\,\Big(\frac{\pi}{2}\Big)^{\frac{1}{2}} \sum_{\xi^*\in\Xi_n(\omega)} \frac{\widehat{f}_n(\xi^*)\,\psi_n(\cdot,\xi^*)}{|\mu_n'(\xi^*)|}\, e^{ip\xi^*}$$

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$$\|u_n^\varepsilon(\cdot+p,e_1) - u_n(\cdot+p,e_1)\|_{H^2(\mathcal{C})} \le\; C(n,p)\,\varepsilon^{s-\frac{1}{2}}\ \|f\|_{L^2_s(\Omega)}$$

The convergence of u_n^ε towards u_n only holds in $H^2_{loc}(\Omega)$.

We shall see later that u_n does not belong to $L^2(\Omega)$.

Convergence : the propagative part

Since by definition $u^{\varepsilon}_{prop} = \sum_{n \in I(\omega)} u^{\varepsilon}_{n}$, defining $u_{prop} = \sum_{n \in I(\omega)} u_{n}$

we have thus shown the convergence of u^{ε}_{prop} towards u_{prop} in $H^2_{loc}(\Omega)$

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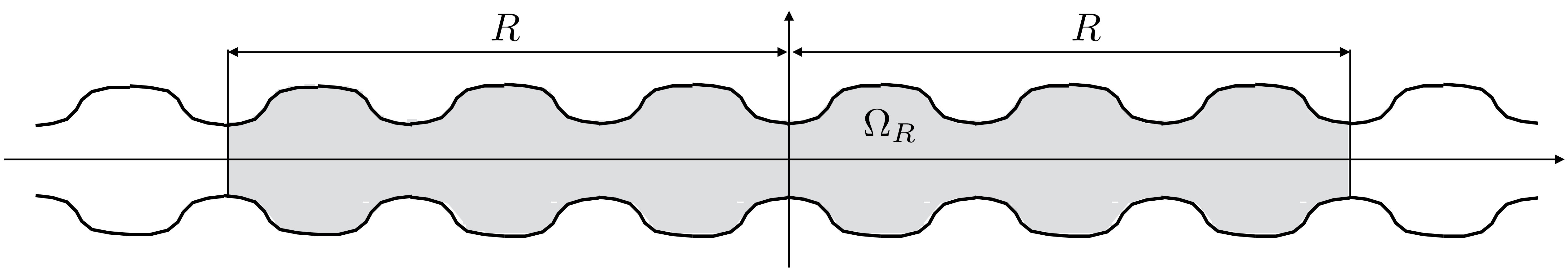
Since by definition $u^\varepsilon_{prop} = \sum_{n\in I(\omega)} u^\varepsilon_n$, defining $u_{prop} = \sum_{n\in I(\omega)} u_n$

we have thus shown the convergence of u^ε_{prop} towards u_{prop} in $H^2_{loc}(\Omega)$

More precisely, for any $R > 0$, setting $\Omega_R = \{x = (x_1, x_T) \in \Omega \,/\, |x_1| < R\}$

$$\boxed{\|u^\varepsilon_{prop} - u_{prop}\|_{H^2(\Omega_R)} \leq C_R(\omega)\, \varepsilon^{s-\frac{1}{2}}\, \|f\|_{L^2_s(\Omega)}}$$



The limiting absorption principle : final theorem

Assume $\omega^2 \notin \sigma_0$ and $f \in L^2_s(\Omega),\ s > 1/2$. Define cell by cell, for each $n \geq 0$, the function

$$u_n(\cdot + p\, e_1) := (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot, \xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi$$

$n \in I(\omega)$

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$n \notin I(\omega)$

$$+\, i\, \Big(\frac{\pi}{2}\Big)^{\frac{1}{2}} \sum_{\xi^* \in \Xi_n(\omega)} \frac{\widehat{f}_n(\xi^*)\, \psi_n(\cdot, \xi^*)}{\big|\mu_n'(\xi^*)\big|}\, e^{ip\xi^*}$$

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Then the function u given by

$$u := u_{prop} + u_{evan}\ ,\quad u_{prop} = \sum_{n \in I(\omega)} u_n \in H^2_{loc}(\Omega)\ ,\quad u_{evan} = \sum_{n \notin I(\omega)} u_n \in H^2(\Omega)\ ,$$

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is well-defined in and is a solution, in the sense of distributions, of

$$(\mathcal{P}) \qquad -\Delta u - n^2\,\omega^2 u = f \quad \text{in } \Omega\ , \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega\,.$$

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$n \notin I(\omega)$

Then the function u given by

$$u := u_{prop} + u_{evan}\ , \quad u_{prop} = \sum_{n \in I(\omega)} u_n \in H^2_{loc}(\Omega)\ , \quad u_{evan} = \sum_{n \notin I(\omega)} u_n \in H^2(\Omega)\ ,$$

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$$(\mathcal{P}) \qquad -\Delta u - n^2\, \omega^2 u = f \quad \text{in } \Omega\ , \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega\, .$$

It is the limit when $\varepsilon > 0 \longrightarrow 0$ of the solution u^ε of the damped Helmholtz equation

$$(\mathcal{P}_\varepsilon) \quad -\Delta u^\varepsilon - n^2\, (\omega^2 + i\varepsilon\omega)\, u^\varepsilon = f \quad \text{in } \Omega \quad \partial_\nu u^\varepsilon = 0 \ \text{ on } \partial\Omega \quad u^\varepsilon \in H^2(\Omega)$$

with the error estimates $\|u^\varepsilon - u\|_{H^2(\Omega_R)} \leq C_R(\omega)\, \varepsilon^{s-\frac{1}{2}}\, \|f\|_{L^2_s(\Omega)}, \quad \forall\, R > 0,$

The limiting absorption principle : final theorem

Assume $\omega^2 \notin \sigma_0$ and $f \in L^2_s(\Omega),\ s > 1/2$. Define cell by cell, for each $n \geq 0$, the function

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$$+\, i \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{\xi^* \in \Xi_n(\omega)} \frac{\widehat{f}_n(\xi^*)\, \psi_n(\cdot,\xi^*)}{\left|\mu_n'(\xi^*)\right|}\, e^{ip\xi^*}$$

Then the function u given by

$$u := u_{prop} + u_{evan}\ , \quad u_{prop} = \sum_{n \in I(\omega)} u_n \in H^2_{loc}(\Omega)\ , \quad u_{evan} = \sum_{n \notin I(\omega)} u_n \in H^2(\Omega)\ ,$$

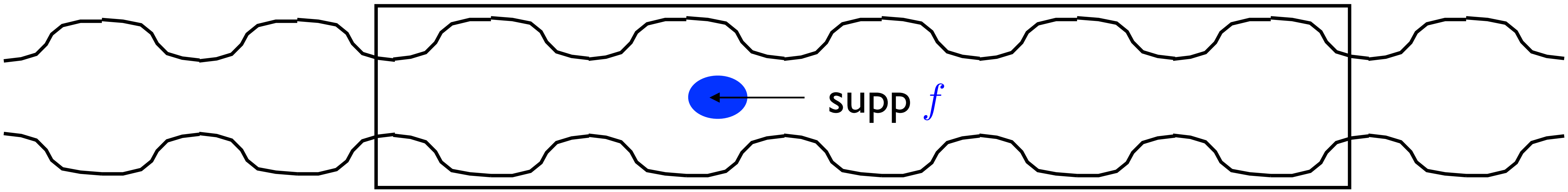
is well-defined in and is a solution, in the sense of distributions, of

$$(\mathcal{P}) \qquad -\Delta u - n^2\, \omega^2 u = f \quad \text{in } \Omega\ , \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega\ .$$

By definition u is the outgoing solution of $(\mathcal{P})$

Objective of the course

$$(\mathcal{P}) \qquad -\Delta u - n^2\omega^2 u = f \quad \text{in } \Omega\,, \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega\,.$$



$$0 < n_- < n(x_1, x_T) < n_+ < +\infty \qquad n(x_1+1, x_T) = n(x_1, x_T)$$

1. Define and construct properly the good solution of $(\mathcal{P})$

   Floquet-Bloch transform, Plemelj-Privalov theorem

2. Describe the properties of this solution, in particular its behaviour at infinity

   Complex variable methods, contour integrals

3. Find radiations condition at infinity that characterize this solution

   Energy like arguments

The propagative wave numbers

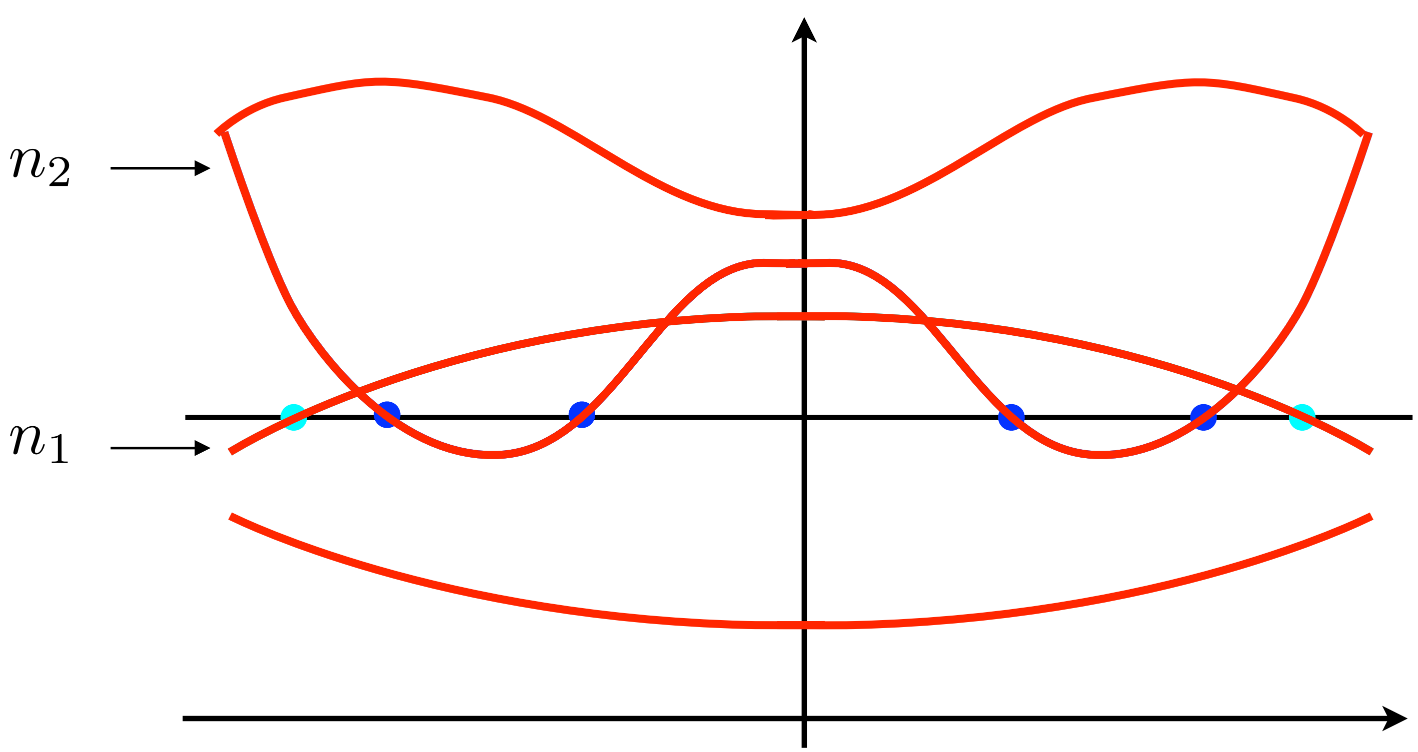
For $n \in I(\omega)$, we introduce the finite set $\Xi_n(\omega) = \{\xi \in [-\pi, \pi[\; / \; \mu_n(\xi) = \omega^2\}$

$$\Xi(\omega) = \bigcup_{n \in I(\omega)} \Xi_n(\omega)$$

: the set of propagative wave numbers at frequency

$$\Xi_n(\omega) = \Xi_n^+(\omega) \cup \Xi_n^-(\omega)$$

$$\Xi_n^+(\omega) = \{\xi \in \Xi_n(\omega) \; / \; \mu_n'(\xi) > 0\} \qquad \Xi_n^-(\omega) = \{\xi \in \Xi_n(\omega) \; / \; \mu_n'(\xi) < 0\}$$



$$I(\omega) = \{n_1, n_2\}$$

$$\Xi_{n_1}(\omega) = \{\bullet\}$$

$$\Xi_{n_2}(\omega) = \{\bullet\}$$

$$\Xi(\omega) = \{\bullet \; \bullet\}$$

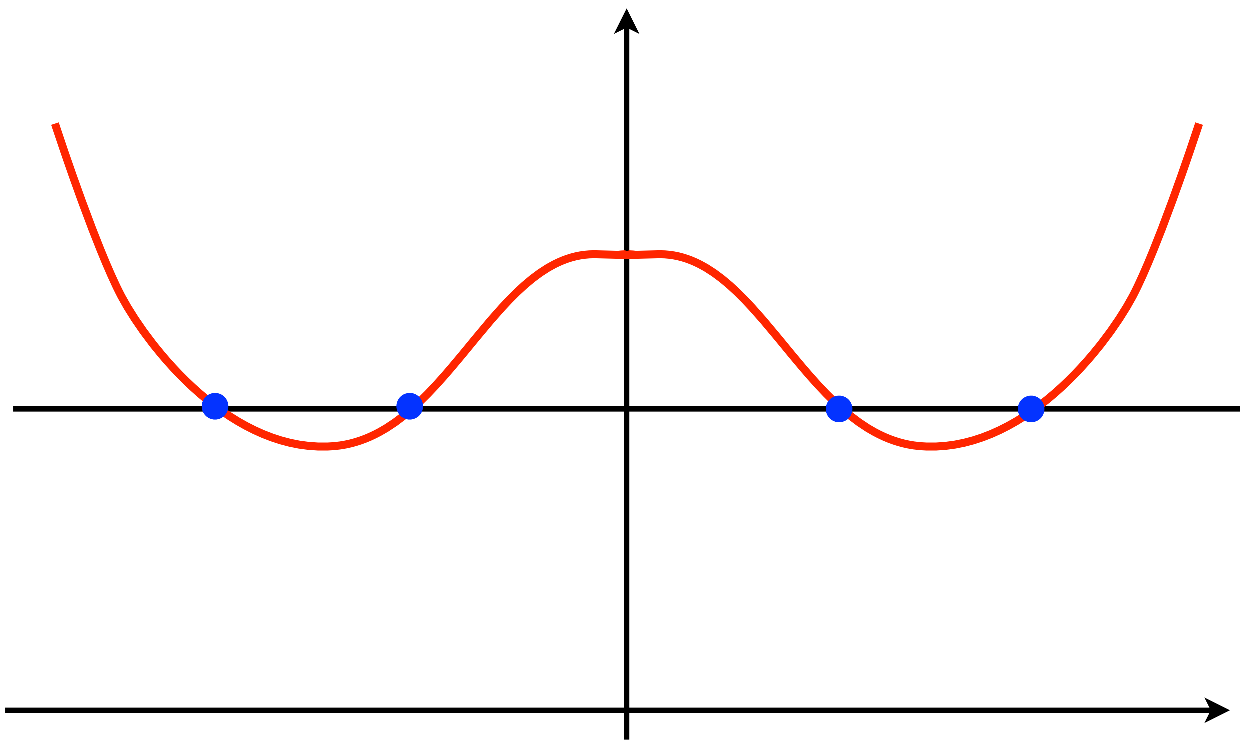
The propagative wave numbers

For $n \in I(\omega)$, we introduce the finite set $\Xi_n(\omega) = \left\{\xi \in [-\pi, \pi[\ / \ \mu_n(\xi) = \omega^2\right\}$

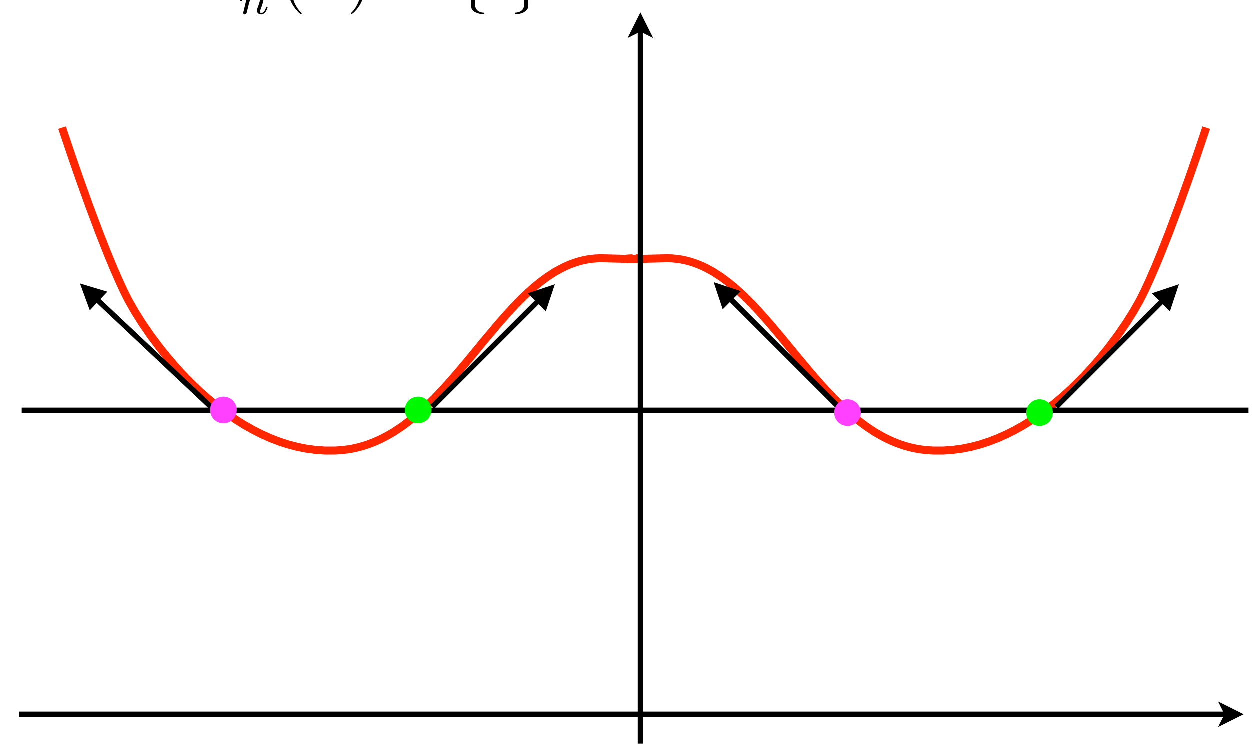
$$\Xi_n(\omega) = \Xi_n^+(\omega) \cup \Xi_n^-(\omega)$$

$$\Xi_n^+(\omega) = \left\{\xi \in \Xi_n(\omega) \ / \ \mu_n'(\xi) > 0\right\} \qquad \Xi_n^-(\omega) = \left\{\xi \in \Xi_n(\omega) \ / \ \mu_n'(\xi) < 0\right\}$$

$\Xi_n(\omega) = \{\bullet\}$



$\Xi_n^+(\omega) = \{\bullet\}$ $\Xi_n^-(\omega) = \{\bullet\}$

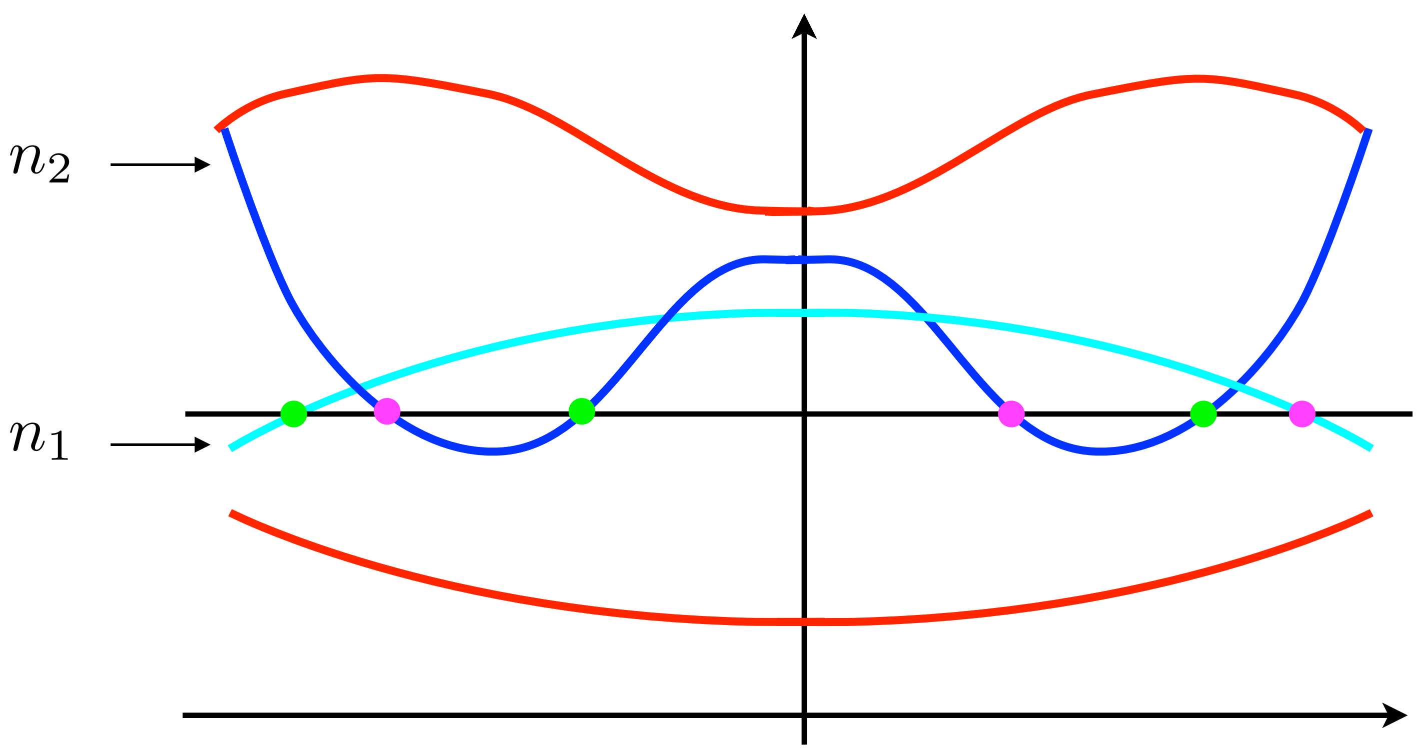


The propagative wave numbers

For $n \in I(\omega)$, we introduce the finite set $\Xi_n(\omega) = \{\xi \in [-\pi, \pi[\ / \ \mu_n(\xi) = \omega^2\}$

$$\Xi(\omega) = \bigcup_{n \in I(\omega)} \Xi_n(\omega)$$ (propagative wave numbers) $$\Xi_n(\omega) = \Xi_n^+(\omega) \cup \Xi_n^-(\omega)$$

$$\boxed{\begin{aligned} &\Xi(\omega) = \Xi^+(\omega) \cup \Xi^-(\omega) \\ &\Xi^+(\omega) = \bigcup_{n \in I(\omega)} \Xi_n^+(\omega) \\ &\Xi^-(\omega) = \bigcup_{n \in I(\omega)} \Xi_n^-(\omega) \end{aligned}}$$



$$I(\omega) = \{n_1, n_2\}$$

$$\Xi^+(\omega) = \{\bullet\}$$

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The propagative wave numbers

For $n \in I(\omega)$, we introduce the finite set $\Xi_n(\omega) = \{\xi \in [-\pi, \pi[\ / \ \mu_n(\xi) = \omega^2\}$

$$\Xi(\omega) = \bigcup_{n \in I(\omega)} \Xi_n(\omega)$$ (propagative wave numbers) $$\Xi_n(\omega) = \Xi_n^+(\omega) \cup \Xi_n^-(\omega)$$

$$\boxed{\Xi(\omega) = \Xi^+(\omega) \cup \Xi^-(\omega) \qquad \Xi^+(\omega) = \bigcup_{n \in I(\omega)} \Xi_n^+(\omega) \qquad \Xi^-(\omega) = \bigcup_{n \in I(\omega)} \Xi_n^-(\omega)}$$

Property : the set $\Xi(\omega)$ is symmetric in the sense that

$$\xi \in \Xi^+(\omega) \quad \Longleftrightarrow \quad \xi \in \Xi^-(\omega)$$

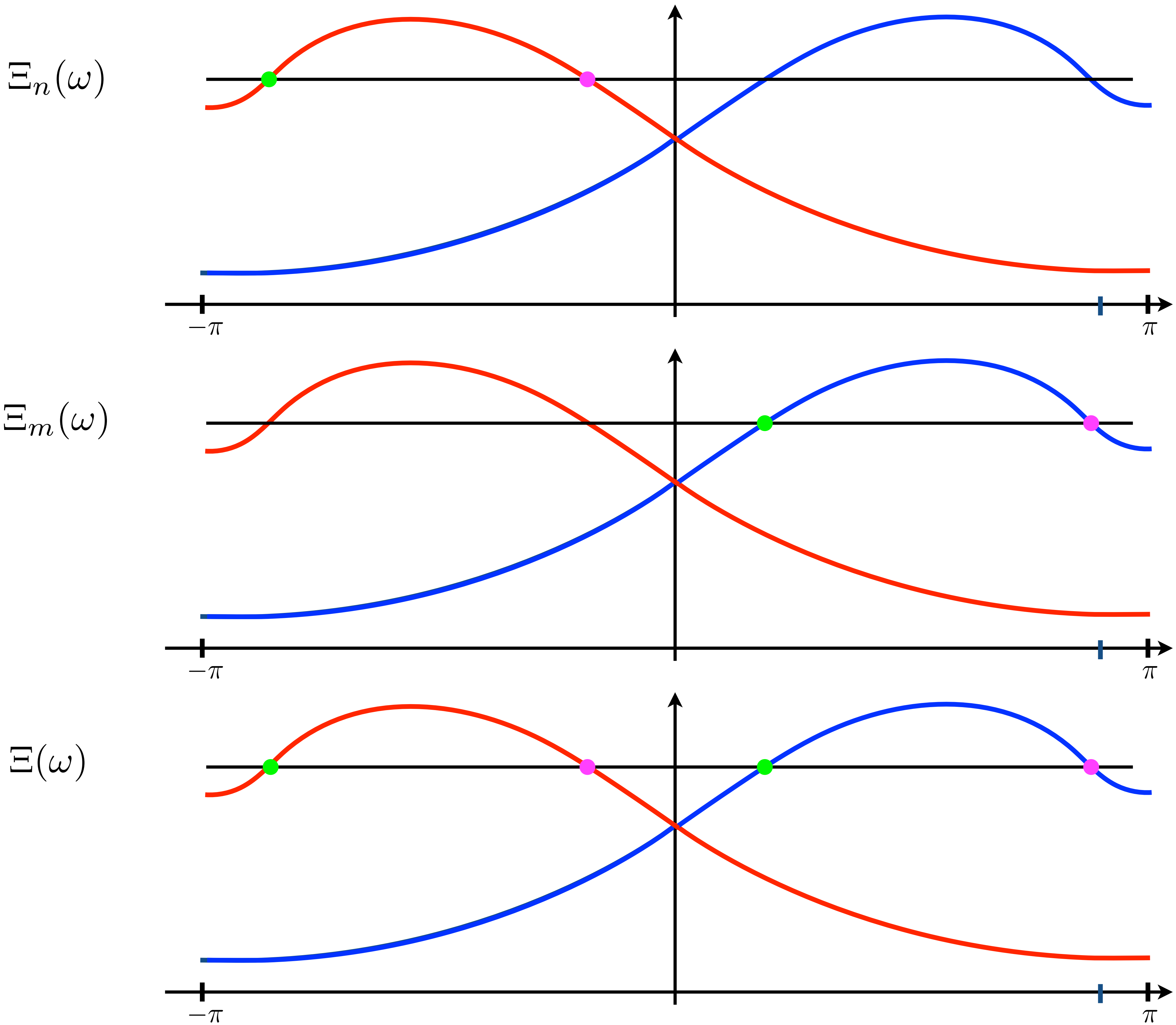
Proof : use the λ_n's instead of the μ_n's and the evenness of $\lambda_n(\xi)$

$$\lambda_n(\xi) = \omega^2 \quad \text{and} \quad \lambda_n'(\xi) > 0 \qquad\qquad \lambda_n(-\xi) = \omega^2 \quad \text{and} \quad \lambda_n'(-\xi) < 0$$

$$\boxed{\Xi^+(\omega) = \{\xi_1^+, \xi_2^+, \cdots, \xi_N^+\} \qquad \Xi^+(\omega) = \{\xi_1^-, \xi_2^-, \cdots, \xi_N^-\} \qquad \xi_\ell^- = -\xi_\ell^+}$$

The propagative wave numbers

$$I(\omega) = \{\underline{n}, \underline{m}\}$$



Asymptotic behaviour at infinity

Theorem : Assume that $\omega^2 \notin \sigma_0$ and $e^{\alpha|x_1|}\, f \in L^2(\Omega),\ \alpha > 0.$

Asymptotic behaviour at $+\infty$:

$$u(\cdot + p\, e_1) = \ i\, (2\pi)^{\frac{1}{2}} \sum_{n \in I(\omega)} \sum_{\xi^* \in \Xi_n^+(\omega)} \frac{\widehat{f}_n(\xi^*)\, \psi_n(\cdot, \xi^*)}{\left|\mu_n'(\xi^*)\right|}\, e^{ip\xi^*} \ + w^+(\cdot + p\, e_1)$$

where $w^+ \in H^2_{loc}(\mathcal{C})$ is exponentially decaying at $+\infty$ in the sense that

$$\exists\, 0 < \beta < \alpha \quad \text{such that} \quad \|w^+(\cdot + p\, e_1)\|_{H^2(\mathcal{C})} \ \leq \ C\, e^{-\beta\, |p|}, \quad \forall\, p > 0.$$

Asymptotic behaviour at $-\infty$:

$$u(\cdot + p\, e_1) = \ i\, (2\pi)^{\frac{1}{2}} \sum_{n \in I(\omega)} \sum_{\xi^* \in \Xi_n^-(\omega)} \frac{\widehat{f}_n(\xi^*)\, \psi_n(\cdot, \xi^*)}{\left|\mu_n'(\xi^*)\right|}\, e^{ip\xi^*} \ + w^-(\cdot + p\, e_1)$$

where $w^- \in H^2_{loc}(\mathcal{C})$ is exponentially decaying at $-\infty$ in the sense that

$$\exists\, 0 < \beta < \alpha \quad \text{such that} \quad \|w^-(\cdot + p\, e_1)\|_{H^2(\mathcal{C})} \ \leq \ C\, e^{-\beta\, |p|}, \quad \forall\, p < 0.$$

Asymptotic behaviour at infinity

Proof when $\omega^2 \notin \sigma(A)$: we have to prove that u is exponentially decreasing at $\pm\infty$

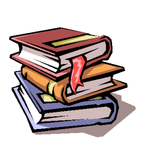
Since $I(\omega) = \emptyset$, we have

$$u(\cdot + p\, e_1) = (2\pi)^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot, \xi)}{\mu_n(\xi) - \omega^2}\; e^{ip\xi}\; d\xi$$

which can be rewritten in an abstract way

$$u(\cdot + p\, e_1) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \left(A(\xi) - \omega^2\right)^{-1} \widehat{f}(\cdot, \xi)\;\; e^{ip\xi}\; d\xi$$

We shall use again results from the theory of analytic families of operators



T. Kato. Perturbation theory for linear operators.
Springer Verlag, (1994 , reprint od the edition of 1980)

For the proof, we shall make the (inessential) technical assumption that

$$-\pi \notin \Xi(\omega) \quad \text{and} \quad \pi \notin \Xi(\omega)$$

Fredholm analytic theory

Let $\mathcal{A}(\xi)$ denote an analytic family of operators (of class (B)) in H

Theorem 1 : Assume that

- $\mathcal{A}(\xi_0)$ is invertible $\iff$ $0 \notin \sigma\big(\mathcal{A}(\xi_0)\big)$
- $\mathcal{A}(\xi)$ has a compact resolvent for all ξ

Then, there exists a complex neighborhood $\mathcal{V}(\xi_0)$ of ξ_0 such that

- $\mathcal{A}(\xi)$ is invertible for all $\xi \in \mathcal{V}(\xi_0)$
- $\xi \mapsto \mathcal{A}(\xi)^{-1}$ is bounded analytic in $\mathcal{V}(\xi_0)$

Corollary 1 : Assume that $\mathcal{A}(\xi_0)$ is invertible for all ξ_0 in K, compact $\subset \mathbb{C}$, then there exists a complex neighborhood $\mathcal{V}(K)$ of K such that

$$\xi \mapsto \mathcal{A}(\xi)^{-1} \quad \text{is bounded analytic in } \mathcal{V}(K)$$

Asymptotic behaviour at infinity

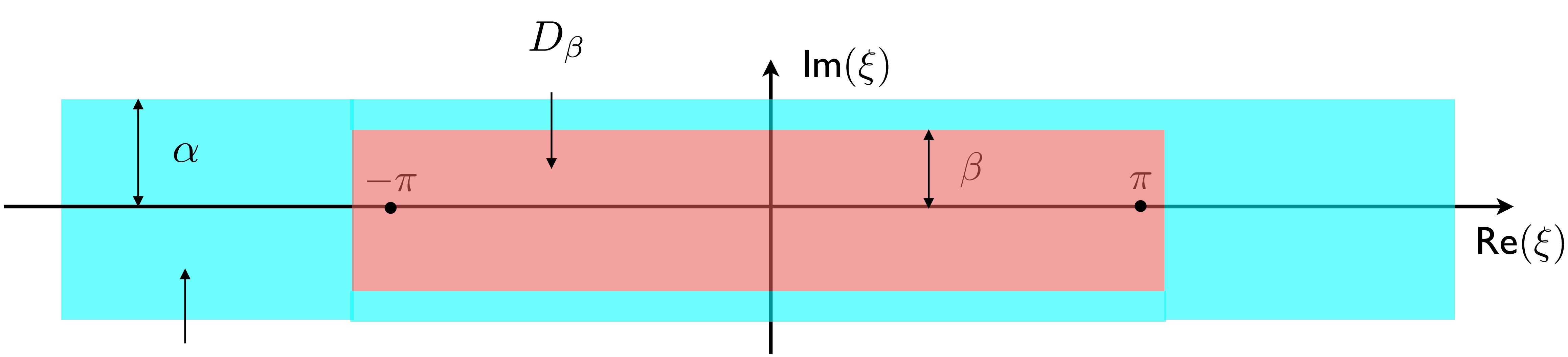
Proof when $\omega^2 \notin \sigma(A)$: we have to prove that u is exponentially decreasing at $\pm\infty$

$$u(\cdot + p\, e_1) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \big(A(\xi) - \omega^2\big)^{-1}\, \widehat{f}(\cdot,\xi)\;\; e^{ip\xi}\, d\xi$$

Applying the corollary with $H := L^2(\mathcal{C}; n^2 dx)$, $\mathcal{A}(\xi) := A(\xi) - \omega^2$, and $K = [-\pi,\pi]$

$$\xi \mapsto \big(A(\xi) - \omega^2\big)^{-1}\, \widehat{f}(\cdot,\xi)\;\; e^{ip\xi} \text{ is analytic in } D_\beta$$

and is, in addition, 2π - periodic



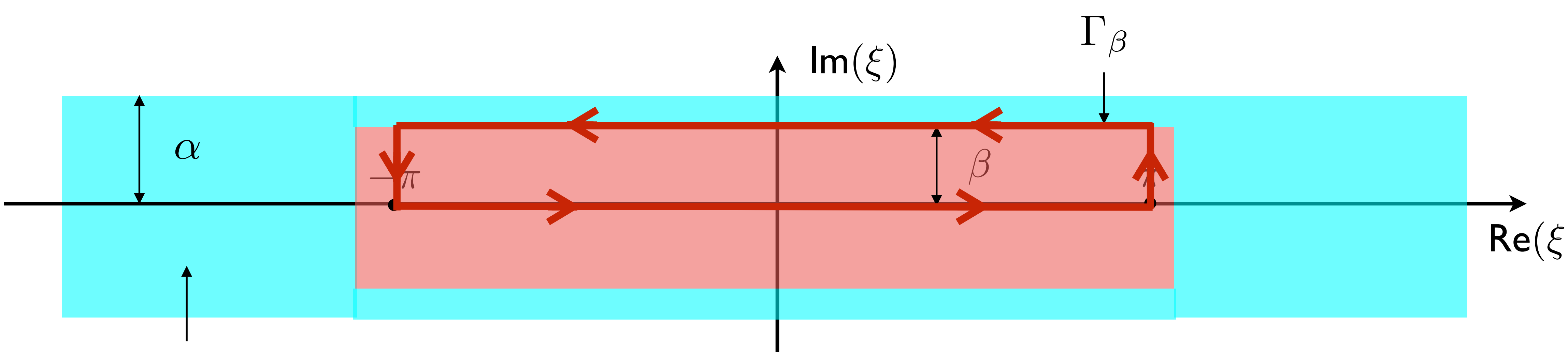
$\boxed{\widehat{f}(\cdot,\xi) \text{ is analytic}}$ $\quad e^{\alpha|x_1|}\, f \in L^2(\Omega),\;\; \alpha > 0.$

Asymptotic behaviour at infinity

Proof when $\omega^2 \notin \sigma(A)$: let us prove that u is exponentially decreasing at $+\infty$

$$u(\cdot + p\, e_1) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \left(A(\xi) - \omega^2\right)^{-1} \widehat{f}(\cdot, \xi)\; e^{ip\xi}\, d\xi \qquad p > 0$$

We can use complex variable techniques : $\displaystyle\int_{\Gamma_\beta} \left(A(\xi) - \omega^2\right)^{-1} \widehat{f}(\cdot, z)\; e^{ipz}\, dz = 0$



$\boxed{\widehat{f}(\cdot, \xi) \text{ is analytic}}$ $\quad e^{\alpha|x_1|}\, f \in L^2(\Omega),\ \alpha > 0.$

Asymptotic behaviour at infinity

Proof when $\omega^2 \notin \sigma(A)$: let us prove that u is exponentially decreasing at $+\infty$

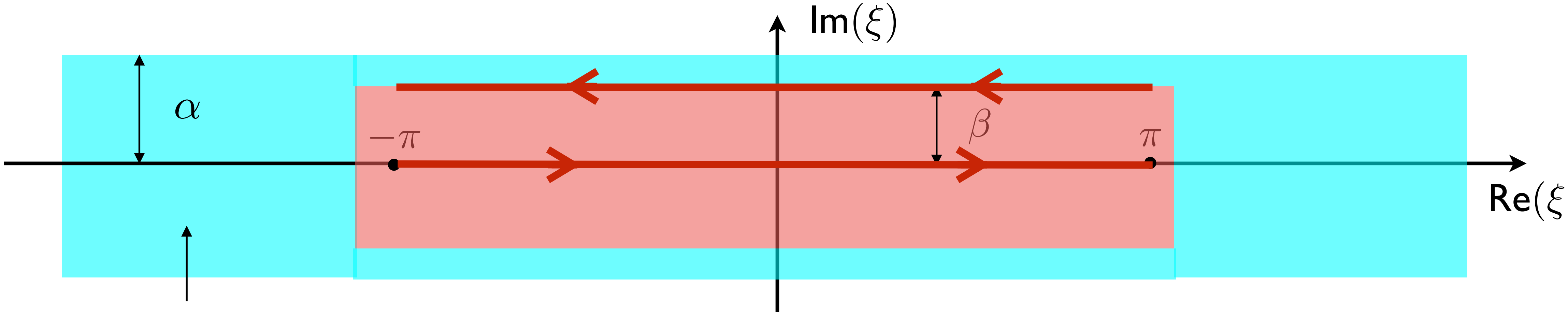
$$u(\cdot + p\, e_1) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \left(A(\xi) - \omega^2\right)^{-1} \widehat{f}(\cdot, \xi)\ e^{ip\xi}\, d\xi \qquad p > 0$$

We can use complex variable techniques : $\displaystyle\int_{\Gamma_\beta} \left(A(\xi) - \omega^2\right)^{-1} \widehat{f}(\cdot, z)\ e^{ipz}\, dz = 0$

Then using the periodicity argument

$$u(\cdot + p\, e_1) = (2\pi)^{-\frac{1}{2}} \int_{[-\pi,\pi]+i\beta} \left(A(z) - \omega^2\right)^{-1} \widehat{f}(\cdot, z)\ e^{ipz}\, dz$$

One concludes noticing that along $\mathcal{I}m\, z = \beta\ :\ \left|e^{ipz}\right| \leq e^{-\beta\,|p|}$



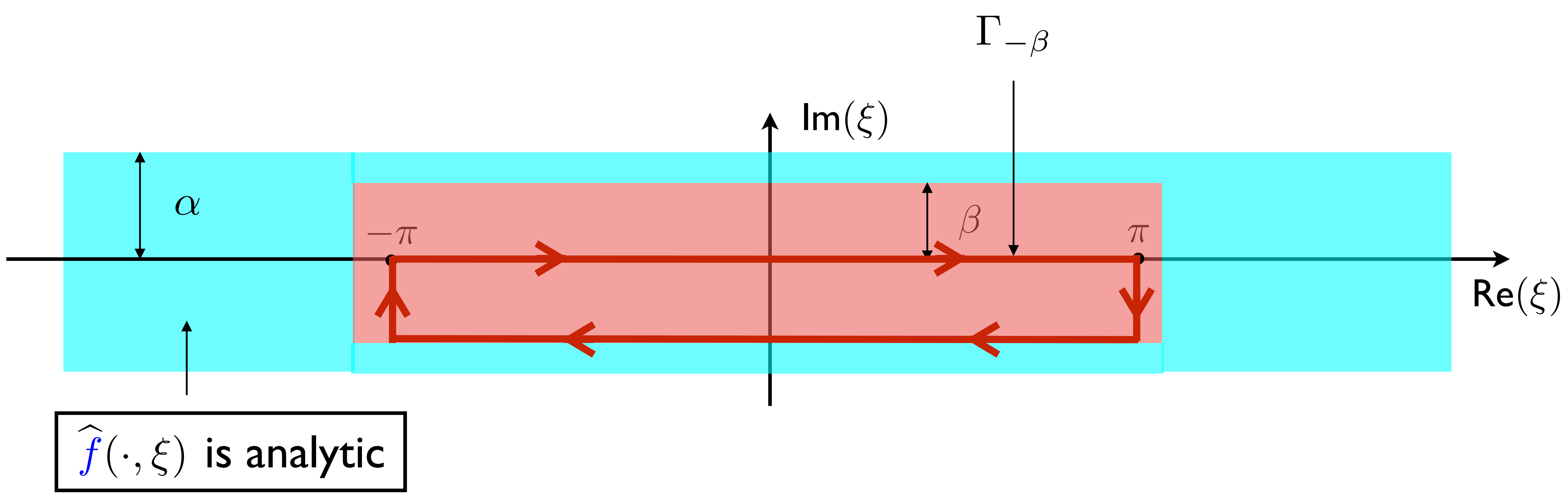
$\widehat{f}(\cdot, \xi)$ is analytic

Asymptotic behaviour at infinity

Proof when $\omega^2 \notin \sigma(A)$: to prove that u is exponentially decreasing at $-\infty$

$$u(\cdot + p\, e_1) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \left(A(\xi) - \omega^2\right)^{-1} \widehat{f}(\cdot, \xi)\; e^{ip\xi}\, d\xi \qquad p < 0$$

We simply have to change the contour $\Gamma_\beta \longrightarrow \Gamma_{-\beta}$



Asymptotic behaviour at infinity

Proof when $\omega^2 \in \sigma(A)$: we first look at the evanescent part

$$u_{evan}(\cdot + p\, e_1) = (2\pi)^{-\frac{1}{2}} \sum_{n \notin I(\omega)} \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\; e^{ip\xi}\; d\xi$$

Let us introduce the orthogonal projectors in $L^2(\mathcal{C}; n^2 dx)$

$$\mathbb{P}(\xi)\widehat{v} := \sum_{n \in I(\omega)} \Big(\int_{\mathcal{C}} \widehat{v}\; \overline{\psi_n(\cdot,\xi)}\, n^2\; dx \Big)\; \psi_n(\cdot,\xi) \qquad \mathbb{Q}(\xi) := I - \mathbb{P}(\xi)$$

By construction of $\mathbb{Q}(\xi)$

$$0 \notin \sigma\big(A(\xi)\mathbb{Q}(\xi) - \omega^2\big) = \big\{\mu_n(\xi) - \omega^2, n \notin I(\omega)\big\} \,\cup\, \{-\omega^2\}$$

and we can write

$$u_{evan}(\cdot + p\, e_1) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \big(A(\xi)\mathbb{Q}(\xi) - \omega^2\big)^{-1}\, \mathbb{Q}(\xi)\widehat{f}(\cdot,\xi)\;\; e^{ip\xi}\; d\xi$$

We wish to apply the corollary with $\mathcal{A}(\xi) = A(\xi)\mathbb{Q}(\xi) - \omega^2$

Asymptotic behaviour at infinity

Proof when $\omega^2 \in \sigma(A)$: we first look at the evanescent part

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where we have introduced the orthogonal projectors in $L^2(\mathcal{C}; n^2 dx)$

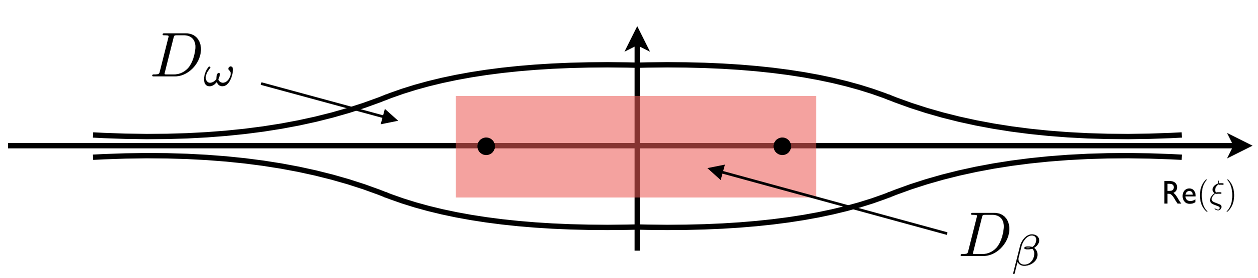
$$\mathbb{P}(\xi)\widehat{v} := \sum_{n \in I(\omega)} \left(\int_{\mathcal{C}} \widehat{v}\, \overline{\psi_n(\cdot,\xi)}\, n^2\, dx\right) \psi_n(\cdot,\xi) \qquad \mathbb{Q}(\xi) := I - \mathbb{P}(\xi)$$

We wish to apply the corollary with $\mathcal{A}(\xi) = A(\xi)\mathbb{Q}(\xi) - \omega^2$

Introducing $D_\omega = \bigcap_{n \in I(\omega)} D_n$, a symmetric neighborhood of the real axis, we define

$$\mathbb{P}(\xi)\widehat{v} := \sum_{n \in I(\omega)} \left(\int_{\mathcal{C}} \widehat{v}\, \overline{\psi_n(\cdot,\boxed{\overline{\xi}})}\, n^2\, dx\right) \psi_n(\cdot,\xi) \qquad \mathbb{Q}(\xi) := I - \mathbb{P}(\xi)$$

as bounded analytic families of operators in D_ω and can assume that $D_\omega \supset D_\beta$.



Asymptotic behaviour at infinity

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as bounded analytic families of operators in D_ω and can assume that $D_\omega \supset D_\beta$.

Key point : $a(z)$ analytic $\Longrightarrow$ $\overline{a(\overline{z})}$ analytic ($\sum_n a_n\, z^n \longrightarrow \sum_n \overline{a_n}\, z^n$)

Asymptotic behaviour at infinity

Proof when $\omega^2 \in \sigma(A)$: we first look at the evanescent part

$$u_{evan}(\cdot + p\, e_1) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \left(A(\xi)\mathbb{Q}(\xi) - \omega^2\right)^{-1} \mathbb{Q}(\xi)\widehat{f}(\cdot, \xi) \;\; e^{ip\xi} \, d\xi$$

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as bounded analytic families of operators in D_ω and can assume that $D_\omega \supset D_\beta$.

Note that these are no longer orthogonal projectors as soon as ξ is

Asymptotic behaviour at infinity

Proof when $\omega^2 \in \sigma(A)$: we first look at the evanescent part

$$u_{evan}(\cdot + p\, e_1) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \left(A(\xi)\mathbb{Q}(\xi) - \omega^2\right)^{-1} \mathbb{Q}(\xi)\widehat{f}(\cdot,\xi) \;\; e^{ip\xi}\, d\xi$$

where we have introduced the orthogonal projectors in $L^2(\mathcal{C}; n^2 dx)$

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as bounded analytic families of operators in D_ω and can assume that $D_\omega \supset D_\beta$.

Moreover $\mathbb{P}(\xi)$ and $\mathbb{Q}(\xi)$ are not 2π - periodic.

Asymptotic behaviour at infinity

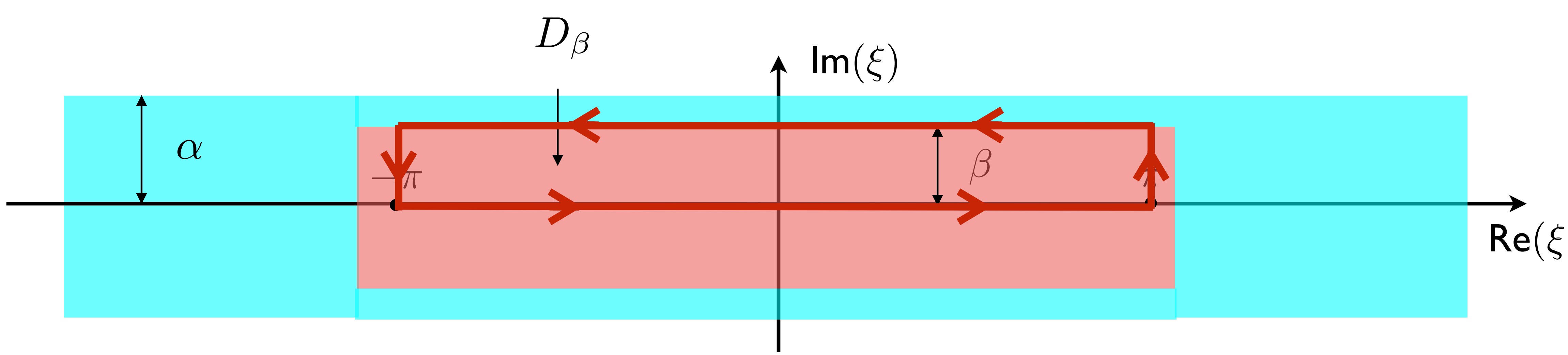
Proof when $\omega^2 \in \sigma(A)$: we first look at the evanescent part

$$u_{evan}(\cdot + p\, e_1) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \left(A(\xi)\mathbb{Q}(\xi) - \omega^2\right)^{-1} \mathbb{Q}(\xi)\widehat{f}(\cdot, \xi) \;\; e^{ip\xi} \, d\xi$$

where $\xi \mapsto \left(A(\xi)\mathbb{Q}(\xi) - \omega^2\right)^{-1}$ and $\xi \mapsto \mathbb{Q}(\xi)$ are analytic in D_β

$$u_{evan}(\cdot + p\, e_1) = (2\pi)^{-\frac{1}{2}} \int_{[-\pi,\pi]+i\beta} \left(A(\xi)\mathbb{Q}(\xi) - \omega^2\right)^{-1} \mathbb{Q}(\xi)\widehat{f}(\cdot, \xi) \;\; e^{ip\xi} \, d\xi$$

$$+ \, (2\pi)^{-\frac{1}{2}} \int_{\downarrow\uparrow} \left(A(\xi)\mathbb{Q}(\xi) - \omega^2\right)^{-1} \mathbb{Q}(\xi)\widehat{f}(\cdot, \xi) \;\; e^{ip\xi} \, d\xi$$

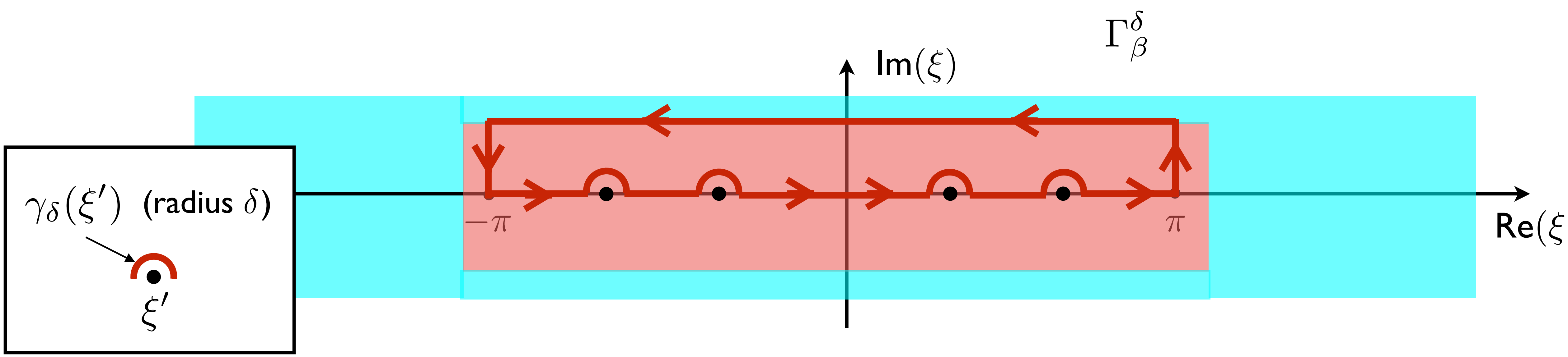


Asymptotic behaviour at infinity

Proof when $\omega^2 \in \sigma(A)$: we next look at the propagative part $u_{prop} = \sum_{n \in I(\omega)} u_n$ where

$$u_n(\cdot + p, e_1) := (2\pi)^{-\frac{1}{2}} \, p.v. \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot, \xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi + i\, \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{\xi^* \in \Xi_n(\omega)} \frac{\widehat{f}_n(\xi^*)\, \psi_n(\cdot, \xi^*)}{\left|\mu_n'(\xi^*)\right|}\, e^{ip\xi^*}$$

$\{\xi' \in \Xi_n(\omega)\} \equiv$ zeroes of $\mu_n(\xi) - \omega^2 \equiv$ poles of $\left(\mu_n(\xi) - \omega^2\right)^{-1}$. They are simple.



Asymptotic behaviour at infinity

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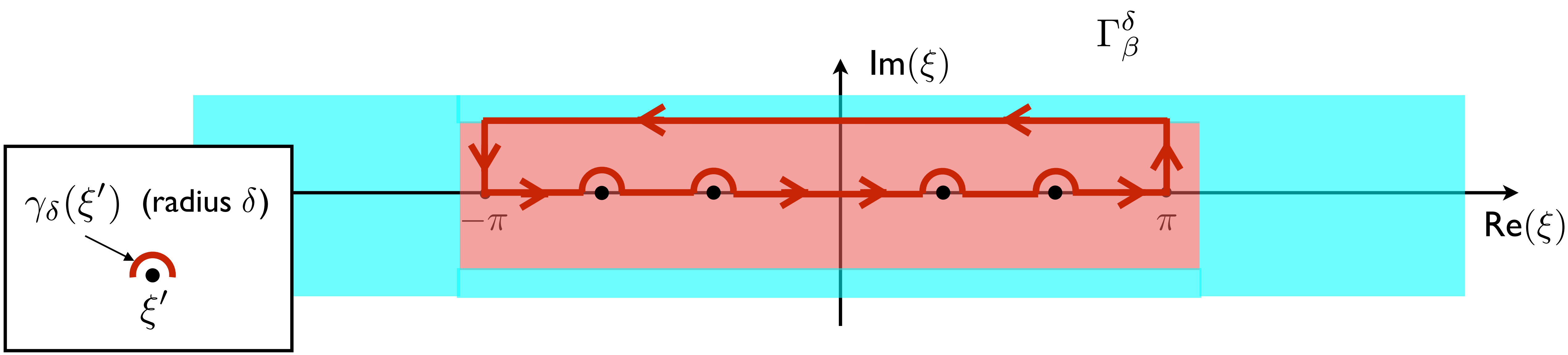
$$u_n(\cdot + p, e_1) := (2\pi)^{-\frac{1}{2}} \, p.v. \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi + i \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{\xi^* \in \Xi_n(\omega)} \frac{\widehat{f}_n(\xi^*)\, \psi_n(\cdot,\xi^*)}{\left|\mu_n'(\xi^*)\right|}\, e^{ip\xi^*}$$

$\{\xi' \in \Xi_n(\omega)\} \equiv$ zeroes of $\mu_n(\xi) - \omega^2 \equiv$ poles of $\left(\mu_n(\xi) - \omega^2\right)^{-1}$. They are simple.

$$p.v. \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi = \int_{[-\pi,\pi]+i\beta} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi - \int_{\downarrow\uparrow} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi$$

$$+ \lim_{\delta \to 0} \sum_{\xi' \in \Xi_n(\omega)} \int_{\gamma_\delta(\xi')} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi$$



Asymptotic behaviour at infinity

Proof when $\omega^2 \in \sigma(A)$: we next look at the propagative part $u_{prop} = \sum_{n \in I(\omega)} u_n$ where

$$u_n(\cdot + p, e_1) := (2\pi)^{-\frac{1}{2}}\, p.v. \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot, \xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi + i \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{\xi^* \in \Xi_n(\omega)} \frac{\widehat{f}_n(\xi^*)\, \psi_n(\cdot, \xi^*)}{\left|\mu_n'(\xi^*)\right|}\, e^{ip\xi^*}$$

$\{\xi' \in \Xi_n(\omega)\} \equiv$ zeroes of $\mu_n(\xi) - \omega^2 \equiv$ poles of $\left(\mu_n(\xi) - \omega^2\right)^{-1}$. They are simple.

$$p.v. \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot, \xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi = \int_{[-\pi,\pi]+i\beta} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot, \xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi \; - \int \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot, \xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi$$

$$+ \lim_{\delta \to 0} \sum_{\xi' \in \Xi_n(\omega)} \int_{\gamma_\delta(\xi')} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot, \xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi$$

Lemma : if $g(\xi)$ has a simple pole at ξ', $\lim_{\delta \to 0} \int_{\gamma_\delta(\xi')} g(\xi)\, d\xi = i\pi \lim_{\xi \to \xi'} (\xi - \xi')\, g(\xi)$

Corollary :
$$\lim_{\delta \to 0} \int_{\gamma_\delta(\xi')} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot, \xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi = i\pi \frac{\widehat{f}_n(\xi')\, \psi_n(\cdot, \xi')}{\mu_n'(\xi')}\, e^{ip\xi'}$$

Asymptotic behaviour at infinity

Proof when $\omega^2 \in \sigma(A)$: $$u_{prop} = \sum_{n \in I(\omega)} u_n$$

$$u_n(\cdot + p, e_1) := (2\pi)^{-\frac{1}{2}} \, p.v. \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot, \xi)}{\mu_n(\xi) - \omega^2} \, e^{ip\xi} \, d\xi \;\; + i \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{\xi^* \in \Xi_n(\omega)} \frac{\widehat{f}_n(\xi^*)\, \psi_n(\cdot, \xi^*)}{\left|\mu_n'(\xi^*)\right|} \, e^{ip\xi^*}$$

$$p.v. \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot, \xi)}{\mu_n(\xi) - \omega^2} \, e^{ip\xi} \, d\xi = \int_{[-\pi,\pi]+i\beta} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot, \xi)}{\mu_n(\xi) - \omega^2} \, e^{ip\xi} \, d\xi \; - \int_{\downarrow\uparrow} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot, \xi)}{\mu_n(\xi) - \omega^2} \, e^{ip\xi} \, d\xi$$

$$- \, i\pi \sum_{\xi' \in \Xi_n(\omega)} \frac{\widehat{f}_n(\xi')\, \psi_n(\cdot, \xi')}{\mu_n'(\xi')} \, e^{ip\xi'}$$

After summation over $n \in I(\omega)$, we get

$$u_{prop}(\cdot + p, e_1) = (2\pi)^{-\frac{1}{2}} \int_{[-\pi,\pi]+i\beta} \left(A(\xi)\mathbb{P}(\xi) - \omega^2\right)^{-1} \mathbb{P}(\xi) \widehat{f}(\cdot, \xi) \, e^{ip\xi} \, d\xi$$

$$+ \, (2\pi)^{-\frac{1}{2}} \int_{\downarrow\uparrow} \left(A(\xi)\mathbb{P}(\xi) - \omega^2\right)^{-1} \mathbb{P}(\xi) \widehat{f}(\cdot, \xi) \, e^{ip\xi} \, d\xi$$

$$+ \, i \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{n \in I(\omega)} \sum_{\xi^* \in \Xi_n(\omega)} \left(\frac{\widehat{f}_n(\xi^*)\, \psi_n(\cdot, \xi^*)}{\left|\mu_n'(\xi^*)\right|} + \frac{\widehat{f}_n(\xi^*)\, \psi_n(\cdot, \xi^*)}{\mu_n'(\xi^*)} \right) e^{ip\xi^*}$$

Asymptotic behaviour at infinity

Proof when $\omega^2 \in \sigma(A):$ $\qquad u_{prop} = \displaystyle\sum_{n \in I(\omega)} u_n$

$$u_n(\cdot + p, e_1) := (2\pi)^{-\frac{1}{2}}\; p.v. \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi \;+ i\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{\xi^* \in \Xi_n(\omega)} \frac{\widehat{f}_n(\xi^*)\, \psi_n(\cdot,\xi^*)}{\left|\mu_n'(\xi^*)\right|}\, e^{ip\xi^*}$$

$$\boxed{\begin{aligned} p.v. \int_{-\pi}^{\pi} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi = \int_{[-\pi,\pi]+i\beta} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi \;-\; \int_{\downarrow\uparrow} \frac{\widehat{f}_n(\xi)\, \psi_n(\cdot,\xi)}{\mu_n(\xi) - \omega^2}\, e^{ip\xi}\, d\xi \\ -\, i\pi \sum_{\xi' \in \Xi_n(\omega)} \frac{\widehat{f}_n(\xi')\, \psi_n(\cdot,\xi')}{\mu_n'(\xi')}\, e^{ip\xi'} \end{aligned}}$$

After summation over $n \in I(\omega)$, we get

$$\boxed{\begin{aligned} u_{prop}(\cdot + p, e_1) = (2\pi)^{-\frac{1}{2}} \int_{[-\pi,\pi]+i\beta} \left(A(\xi)\mathbb{P}(\xi) - \omega^2\right)^{-1} \mathbb{P}(\xi)\widehat{f}(\cdot,\xi)\, e^{ip\xi}\, d\xi \\ + (2\pi)^{-\frac{1}{2}} \int_{\downarrow\uparrow} \left(A(\xi)\mathbb{P}(\xi) - \omega^2\right)^{-1} \mathbb{P}(\xi)\widehat{f}(\cdot,\xi)\, e^{ip\xi}\, d\xi \\ -\, i\,(2\pi)^{\frac{1}{2}} \sum_{n \in I(\omega)} \sum_{\xi^* \in \Xi_n^+(\omega)} \frac{\widehat{f}_n(\xi^*)\, \psi_n(\cdot,\xi^*)}{\left|\mu_n'(\xi^*)\right|}\, e^{ip\xi^*} \end{aligned}}$$

Asymptotic behaviour at infinity

Proof when $\omega^2 \in \sigma(A)$:

$$u_{evan}(\cdot + p\, e_1) = (2\pi)^{-\frac{1}{2}} \int_{[-\pi,\pi]+i\beta} \left(A(\xi)\mathbb{Q}(\xi) - \omega^2\right)^{-1} \mathbb{Q}(\xi)\widehat{f}(\cdot,\xi) \;\; e^{ip\xi}\, d\xi$$

$$+ (2\pi)^{-\frac{1}{2}} \int_{\downarrow\uparrow} \left(A(\xi)\mathbb{Q}(\xi) - \omega^2\right)^{-1} \mathbb{Q}(\xi)\widehat{f}(\cdot,\xi) \;\; e^{ip\xi}\, d\xi$$

$$u_{prop}(\cdot + p, e_1) = (2\pi)^{-\frac{1}{2}} \int_{[-\pi,\pi]+i\beta} \left(A(\xi)\mathbb{P}(\xi) - \omega^2\right)^{-1} \mathbb{P}(\xi)\widehat{f}(\cdot,\xi)\, e^{ip\xi}\, d\xi$$

$$+ (2\pi)^{-\frac{1}{2}} \int_{\downarrow\uparrow} \left(A(\xi)\mathbb{P}(\xi) - \omega^2\right)^{-1} \mathbb{P}(\xi)\widehat{f}(\cdot,\xi)\, e^{ip\xi}\, d\xi$$

$$- i\, (2\pi)^{\frac{1}{2}} \sum_{n\in I(\omega)} \sum_{\xi^*\in\Xi_n^+(\omega)} \frac{\widehat{f}_n(\xi^*)\, \psi_n(\cdot,\xi^*)}{\left|\mu_n'(\xi^*)\right|}\, e^{ip\xi^*}$$

exponentially decaying at $+\infty$

Asymptotic behaviour at infinity

Proof when $\omega^2 \in \sigma(A)$:

$$u(\cdot + p\, e_1) = \; i\,(2\pi)^{\frac{1}{2}} \sum_{n \in I(\omega)} \sum_{\xi^* \in \Xi_n^+(\omega)} \frac{\widehat{f}_n(\xi^*)\,\psi_n(\cdot, \xi^*)}{\left|\mu_n'(\xi^*)\right|}\, e^{ip\xi^*} \; + w^+(\cdot + p\, e_1)$$

$$+\,(2\pi)^{-\frac{1}{2}} \int \left(A(\xi) - \omega^2\right)^{-1} \widehat{f}(\cdot, \xi)\;\; e^{ip\xi}\; d\xi$$

Lemma : There exists two balls centered at $-\pi$ and $+\pi$ inside which

(i) $\left(A(\xi) - \omega^2\right)^{-1}$ is well defined and bounded analytic

(ii) $\left(A(\xi)\mathbb{Q}(\xi) - \omega^2\right)^{-1} \mathbb{Q}(\xi) + \left(A(\xi)\mathbb{Q}(\xi) - \omega^2\right)^{-1} \mathbb{Q}(\xi) = \left(A(\xi) - \omega^2\right)^{-1}$

By periodicity of $\xi \mapsto A(\xi)$, $\left(A(i\lambda + \pi) - \omega^2\right)^{-1} = \left(A(i\lambda - \pi) - \omega^2\right)^{-1}$ for $|\lambda| \le \beta$

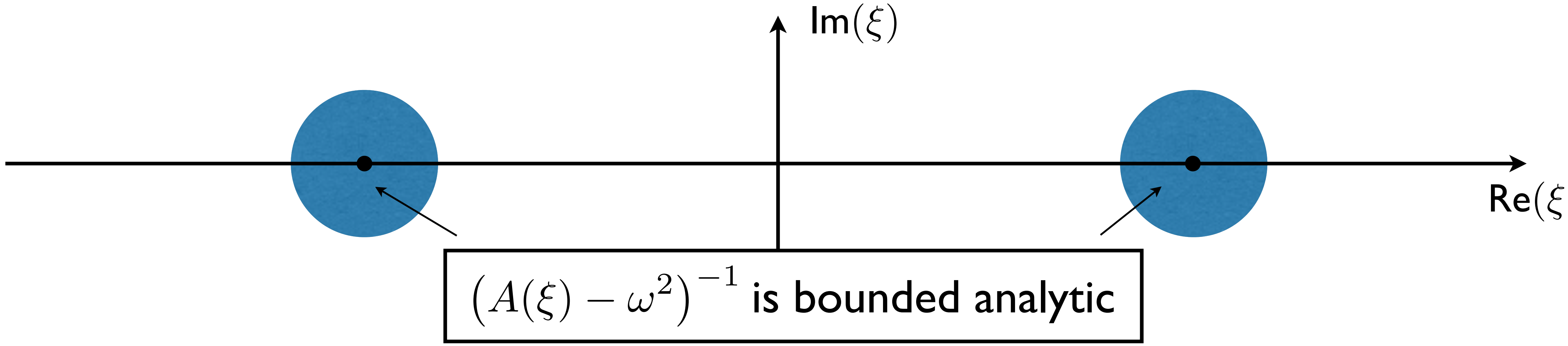
Asymptotic behaviour at infinity

Lemma : There exists two balls centered at $-\pi$ and $+\pi$ inside which

(i) $\left(A(\xi)-\omega^2\right)^{-1}$ is well defined and bounded analytic

(ii) $\left(A(\xi)\mathbb{Q}(\xi)-\omega^2\right)^{-1}\mathbb{Q}(\xi)+\left(A(\xi)\mathbb{Q}(\xi)-\omega^2\right)^{-1}\mathbb{Q}(\xi)=\left(A(\xi)-\omega^2\right)^{-1}$

$\pi \notin \Xi(\omega) \implies A(\pi)-\omega^2$ and $A(-\pi)-\omega^2$ are invertible

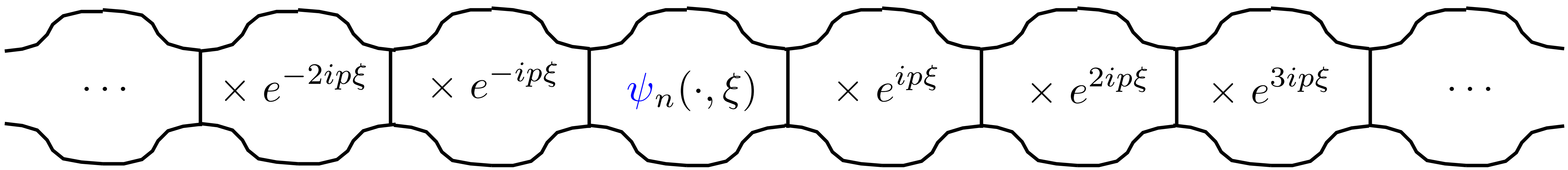


The identity (ii) holds along the red real segments : applied to $\psi_n(\cdot,\xi)$
both left and right hand sides give $(\mu_n(\xi)-\omega^2)^{-1}\,\psi_n(\cdot,\xi)$

By analyticity, (ii) also holds inside the two balls

Propagative Floquet modes

Given $n \neq 0,\ \xi \in [-\pi, \pi\,]$, we still denote $\psi_n(\cdot, \xi) \equiv E_\xi \psi_n(\cdot, \xi)$



Propagative Floquet modes

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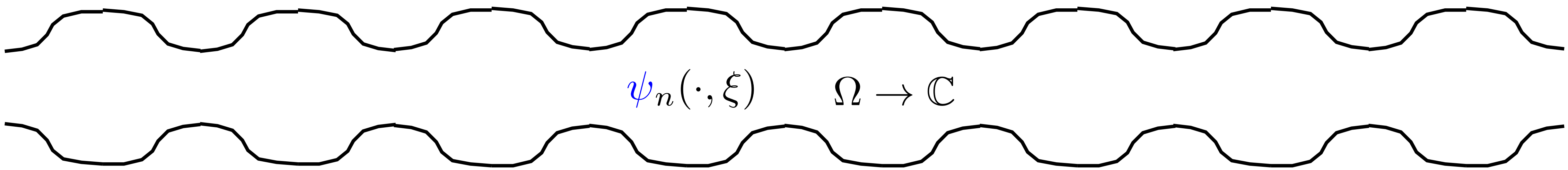


$$\boxed{\psi_n(x_1 + 1, x_T, \xi) = e^{i\xi}\ \psi_n(x_1, x_T, \xi)}$$

$$\boxed{\psi_n(x_1, x_T, \xi) = e^{i\xi x_1}\ \widetilde{\psi}_n(x_1, x_T, \xi) \qquad \widetilde{\psi}_n(\cdot, \xi) \ \text{periodic}}$$

Thanks to the equations and boundary conditions satisfied by $\psi_n(\cdot, \xi)$ in $\mathcal{C}$, it is easy to see that

$$\boxed{\psi_n(\cdot, \xi) \in H^2_{loc}(\Omega) \qquad \Delta \psi_n(\cdot, \xi) + \mu_n(\xi)\, n^2\, \psi_n(\cdot, \xi) = 0}$$

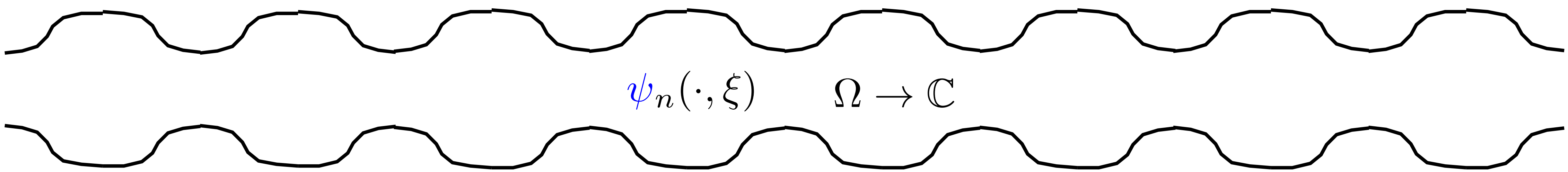
As a consequence, if $\mathcal{V}(\omega) := \{ v \in H^2_{loc} \ / \ \Delta v + n^2\, v = 0 \text{ in } \Omega, \partial_\nu v \text{ on } \partial\Omega \}$

$$\boxed{\text{span } \{ \psi_n(\cdot, \xi'), n \in I(\omega), \xi' \in \Xi_n(\omega) \} \subset \mathcal{V}(\omega)}$$

propagative Floquet modes

Propagative Floquet modes

Given $n \neq 0,\ \xi \in [-\pi, \pi\,]$, we still denote $\psi_n(\cdot, \xi) \equiv E_\xi \psi_n(\cdot, \xi)$



$$\boxed{\psi_n(x_1 + 1, x_T, \xi) = e^{i\xi}\, \psi_n(x_1, x_T, \xi)}$$

$$\psi_n(x_1, x_T, \xi) = e^{i\xi x_1}\, \widetilde{\psi}_n(x_1, x_T, \xi) \qquad \widetilde{\psi}_n(\cdot, \xi) \ \text{periodic}$$

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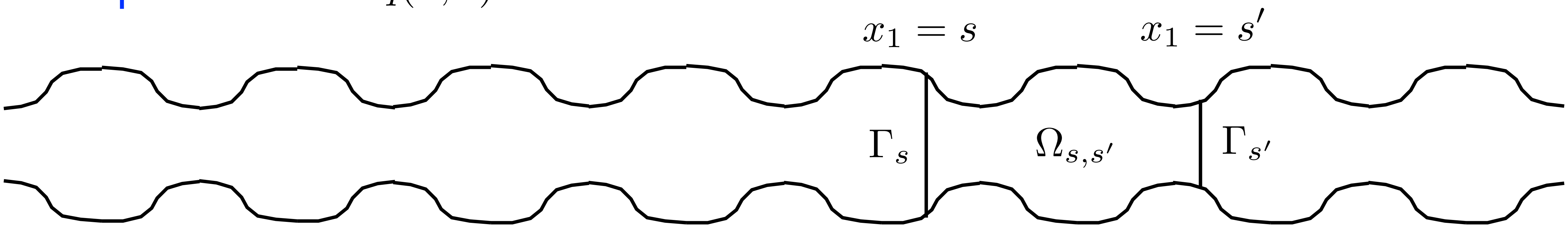
$$\boxed{\text{span}\ \{\psi_n(\cdot, \xi'), \xi' \in \Xi(\omega)\} = \mathcal{V}(\omega) \cap L^\infty(\Omega)}$$

propagative Floquet modes

Propagative Floquet modes

The sesquilinear form $q(u, v)$



$$q(s; u, v) := \int_{\Gamma_s} \left(\partial_{x_1} u\, \overline{v} - \partial_{x_1} \overline{v}\, u\right)\, dx_T$$

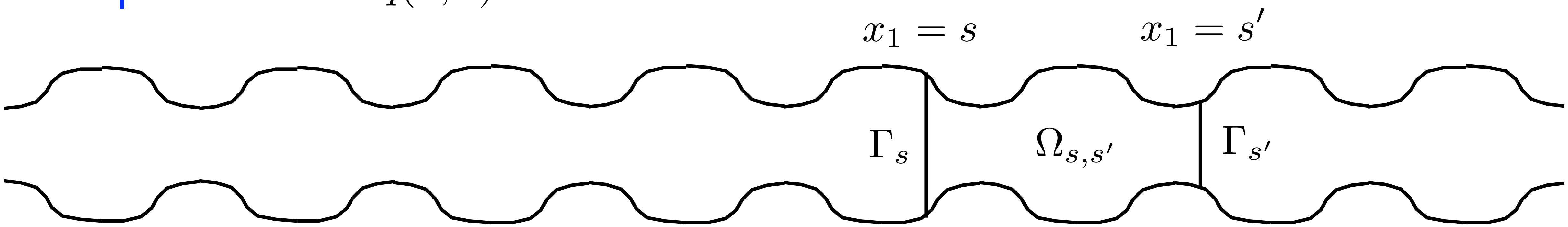
Lemma : For $(u, v) \in \mathcal{V}(\omega)$, $q(s; u, v) = q(u, v)$ is independent of s.

Proof : Using Green's formula

$$\begin{aligned} q(s'; u, v) - q(s; u, v) &= \int_{\Omega_{s,s'}} \left(\Delta u\, \overline{v} - \Delta \overline{v}\, u\right)\, dx \\ &= \int_{\Omega_{s,s'}} \left(n^2 \omega^2\, u\, \overline{v} - n^2 \omega^2\, \overline{v}\, u\right)\, dx = 0 \end{aligned}$$

Propagative Floquet modes

The sesquilinear form $q(u,v)$



$$q(s; u, v) := \int_{\Gamma_s} \left(\partial_{x_1} u \, \overline{v} - \partial_{x_1} \overline{v} \, u\right) \, dx_T$$

Lemma : For $(u,v) \in \mathcal{V}(\omega)$, $q(s; u, v) = q(u,v)$ is independent of s.

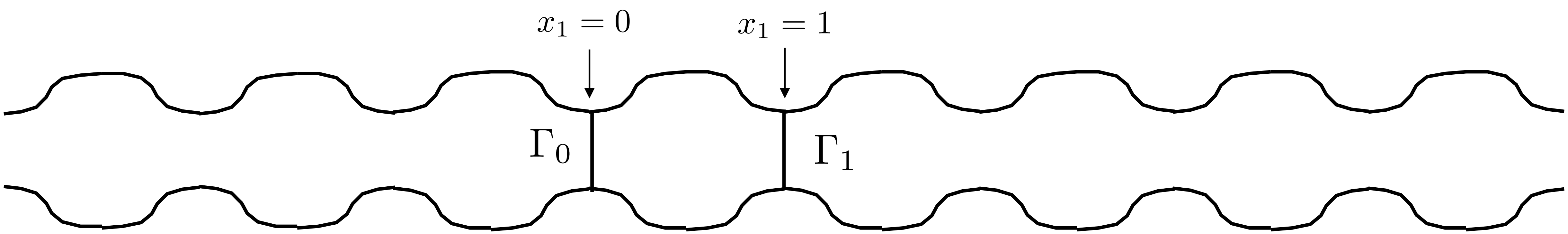
The sesquilinear form $q(u,v)$ "orthogonalizes" the propagative Floquet modes :

> Theorem : Let $(n,m) \in I(\omega)$, $\xi \in \Xi_n(\omega)$, $\xi' \in \Xi_m(\omega)$
>
> If $n \neq m$ or $\xi \neq \xi'$, $q\big(\psi_n(\cdot,\xi), \psi_m(\cdot,\xi')\big) = 0$
>
> Otherwise, $q\big(\psi_n(\cdot,\xi), \psi_n(\cdot,\xi)\big) = i\,\mu_n'(\cdot,\xi)$

Propagative Floquet modes

Proof of the theorem (1)

According to the lemma, $q\big(\psi_n(\cdot,\xi),\psi_m(\cdot,\xi')\big)$ is given indifferently by one of the following two expressions



$$\int \Big(\partial_{x_1}\psi_n(1,x_T,\xi)\,\overline{\psi}_m(1,x_T,\xi') - \partial_{x_1}\overline{\psi}_m(1,x_T,\xi)\,\overline{\psi}_n(1,x_T,\xi') \Big)\, dx_T$$

$$e^{i\xi} \qquad e^{-i\xi'} \qquad e^{i\xi} \qquad e^{-i\xi'}$$

$$= \int \Big(\partial_{x_1}\psi_n(0,x_T,\xi)\,\overline{\psi}_m(0,x_T,\xi') - \partial_{x_1}\overline{\psi}_m(0,x_T,\xi)\,\psi_n(0,x_T,\xi') \Big)\, dx_T$$

$$\Longrightarrow \quad (e^{\xi-\xi'}-1)\;\; q\big(\psi_n(\cdot,\xi),\psi_m(\cdot,\xi')\big) = 0$$

which proves $q\big(\psi_n(\cdot,\xi),\psi_m(\cdot,\xi')\big) = 0$ for $\xi \neq \xi'$

Propagative Floquet modes

Proof of the theorem (2)

The idea is to differentiate in ξ the equations in $\psi_n(\cdot,\xi)$ $\quad \Psi_n(\cdot,\xi) := \partial_\xi \psi_n(\cdot,\xi)$

$$\boxed{\begin{aligned} &-\Delta \psi_n(\cdot,\xi) = \mu_n(\xi)\, n^2\, \psi_n(\cdot,\xi) \qquad \partial_\nu \psi_n(\cdot,\xi) = 0 \\ &\partial_{x_1}\psi_n(\cdot,\xi)|_{\Gamma_1} = e^{i\xi}\, \partial_{x_1}\psi_n(\cdot,\xi)|_{\Gamma_0} \\ &\psi_n(\cdot,\xi)|_{\Gamma_1} = e^{i\xi}\, \psi_n(\cdot,\xi)|_{\Gamma_0} \end{aligned}}$$

$$\boxed{\begin{aligned} &-\Delta \Psi_n(\cdot,\xi) = \mu_n(\xi)\, n^2\, \Psi_n(\cdot,\xi) + \mu_n'(\xi)\, n^2\, \psi_n(\cdot,\xi) \qquad \partial_\nu \Psi_n(\cdot,\xi) = 0 \\ &\Psi_n(\cdot,\xi)|_{\Gamma_1} = e^{i\xi}\, \Psi_n(\cdot,\xi)|_{\Gamma_0} + i\, e^{i\xi}\, \psi_n(\cdot,\xi)|_{\Gamma_0} \\ &\partial_{x_1}\Psi_n(\cdot,\xi)|_{\Gamma_1} = e^{i\xi}\, \partial_{x_1}\Psi_n(\cdot,\xi)|_{\Gamma_0} + i\, e^{i\xi}\, \partial_{x_1}\psi_n(\cdot,\xi)|_{\Gamma_0} \end{aligned}}$$

$$q(1;\Psi_n(\cdot,\xi),\psi_m(\cdot,\xi)) - q(0;\Psi_n(\cdot,\xi),\psi_m(\cdot,\xi)) =$$

Green's formula

$$= \int_{\mathcal{C}} \big(\Delta\Psi_n(\cdot,\xi)\,\overline{\psi}_m(\cdot,\xi) - \Delta\overline{\psi}_m(\cdot,\xi)\,\Psi_n(\cdot,\xi)\big)\,dx$$

$$= \mu_n'(\xi)\int_{\mathcal{C}} \psi_n(\cdot,\xi)\,\overline{\psi}_m(\cdot,\xi)\,n^2 dx \;=\; \mu_n'(\xi)\,\delta_{nm}$$

Propagative Floquet modes

Proof of the theorem (2)

$$q(1; \Psi_n(\cdot,\xi), \psi_m(\cdot,\xi)) - q(0; \Psi_n(\cdot,\xi), \psi_m(\cdot,\xi)) = \mu_n'(\xi)\, \delta_{nm}$$

To conclude it suffices to observe that we also have

$$q(1; \Psi_n(\cdot,\xi), \psi_m(\cdot,\xi)) - q(0; \Psi_n(\cdot,\xi), \psi_m(\cdot,\xi)) = \; - i\, q\big(\psi_n(\cdot,\xi), \psi_m(\cdot,\xi)\big)$$

which results from a direct computation using

$$\boxed{\begin{aligned} \psi_n(\cdot,\xi)|_{\Gamma_1} &= e^{i\xi}\; \psi_n(\cdot,\xi)|_{\Gamma_0} \\ \partial_{x_1}\psi_n(\cdot,\xi)|_{\Gamma_1} &= e^{i\xi}\; \partial_{x_1}\psi_n(\cdot,\xi)|_{\Gamma_0} \end{aligned}}$$

$$\boxed{\begin{aligned} \Psi_n(\cdot,\xi)|_{\Gamma_1} &= e^{i\xi}\; \Psi_n(\cdot,\xi)|_{\Gamma_0} + i\, e^{i\xi}\; \psi_n(\cdot,\xi)|_{\Gamma_0} \\ \partial_{x_1}\Psi_n(\cdot,\xi)|_{\Gamma_1} &= e^{i\xi}\; \partial_{x_1}\Psi_n(\cdot,\xi)|_{\Gamma_0} + i\, e^{i\xi}\; \partial_{x_1}\psi_n(\cdot,\xi)|_{\Gamma_0} \end{aligned}}$$

Propagative Floquet modes

$$\{\psi_n(\cdot,\xi'), n \in I(\omega), \xi' \in \Xi_n(\omega)\}$$

$$\{\psi_n(\cdot,\xi'), n \in I(\omega),\ \xi' \in \Xi_n^+(\omega)\}$$

right propagating modes

$$\{\psi_n(\cdot,\xi'), n \in I(\omega),\ \xi' \in \Xi_n^-(\omega)\}$$

left propagating modes

New notation

Using the fact that

$$\Xi^+(\omega) = \bigcup_{n\in I(\omega)} \Xi_n^+(\omega) \ = \{\xi_1^+, \xi_2^+, \cdots, \xi_N^+\}$$

$$\Xi^-(\omega) = \bigcup_{n\in I(\omega)} \Xi_n^-(\omega) \ = \{\xi_1^-, \xi_2^-, \cdots, \xi_N^-\}$$

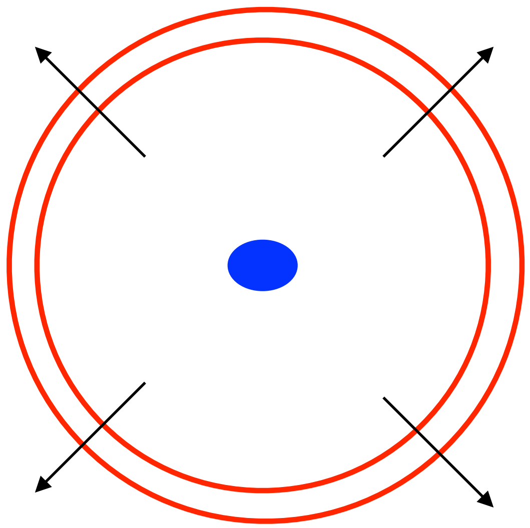
we can write accordingly

$$\{\psi_n(\cdot,\xi'), n \in I(\omega),\ \xi' \in \Xi_n^+(\omega)\} \ = \{\Phi_1^+, \Phi_2^+, \cdots, \Phi_N^+\}$$

$$\{\psi_n(\cdot,\xi'), n \in I(\omega),\ \xi' \in \Xi_n^-(\omega)\} \ = \{\Phi_1^-, \Phi_2^-, \cdots, \Phi_N^-\}$$

The radiation condition

Homogeneous free space



There exists $A(\Theta)$, $|\Theta| = 1$, such that

$$u(r\Theta) = A(\Theta)\frac{e^{i\omega r}}{r^{\frac{d-1}{2}}}\left(1 + O\left(\frac{1}{r}\right)\right)$$

$$\iff \quad \partial_r u + i\,\omega u = O(r^{-2})$$

shows that the outgoing solution u satisfies the outgoing radiation conditions

(CR+) Outgoing radiation condition at $+\infty$

There exists coefficients $\{a_\ell^+, 1 \le \ell \le N\}$ and w^+ exponentially decreasing at $+\infty$

such that $u = \sum a_\ell^+ \, \Phi_\ell^+ + w^+$

(CR-) Outgoing radiation condition at $-\infty$

There exists coefficients $\{a_\ell^-, 1 \le \ell \le N\}$ and w^- exponentially decreasing at $-\infty$

such that $u = \sum a_\ell^- \, \Phi_\ell^- + w^-$

The radiation condition

$$u(\cdot + p\, e_1) = \; i\,(2\pi)^{\frac{1}{2}} \sum_{n\in I(\omega)} \sum_{\xi^*\in\Xi_n^+(\omega)} \frac{\widehat{f}_n(\xi^*)\,\psi_n(\cdot,\xi^*)}{\left|\mu_n'(\xi^*)\right|}\, e^{ip\xi^*} \; + w^+(\cdot + p\, e_1)$$

$$u(\cdot + p\, e_1) = \; i\,(2\pi)^{\frac{1}{2}} \sum_{n\in I(\omega)} \sum_{\xi^*\in\Xi_n^-(\omega)} \frac{\widehat{f}_n(\xi^*)\,\psi_n(\cdot,\xi^*)}{\left|\mu_n'(\xi^*)\right|}\, e^{ip\xi^*} \; + w^-(\cdot + p\, e_1)$$

$$\boxed{\begin{aligned} \{\psi_n(\cdot,\xi'), n\in I(\omega),\ \xi'\in\Xi_n^+(\omega)\} &= \{\Phi_1^+,\Phi_2^+,\cdots,\Phi_N^+\} \\ \{\psi_n(\cdot,\xi'), n\in I(\omega),\ \xi'\in\Xi_n^-(\omega)\} &= \{\Phi_1^-,\Phi_2^-,\cdots,\Phi_N^-\} \end{aligned}}$$

shows that the outgoing solution u satisfies the outgoing radiation conditions

(CR+) Outgoing radiation condition at $+\infty$

There exists coefficients $\{a_\ell^+, 1\le \ell\le N\}$ and w^+ exponentially decreasing at $+\infty$

such that $$u = \sum a_\ell^+\, \Phi_\ell^+ + w^+$$

(CR-) Outgoing radiation condition at $-\infty$

There exists coefficients $\{a_\ell^-, 1\le \ell\le N\}$ and w^- exponentially decreasing at $-\infty$

such that $$u = \sum a_\ell^-\, \Phi_\ell^- + w^-$$

The uniqueness result

Theorem : Assume that $\omega^2 \notin \sigma_0$, $e^{\alpha|x_1|}$, $f \in L^2(\Omega),\ \alpha > 0$ and $\sigma_p(A) = \emptyset$.

There exists a unique function $u \in H^2_{loc}(\Omega)$ satisfying

$$(\mathcal{P}) \qquad -\Delta u - n^2\,\omega^2 u = f \quad \text{in } \Omega \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega$$

as well as the two outgoing radiation conditions (CR-) and (CR+)

Proof : The existence result has been proven by construction (by limiting absorption). Only the uniqueness result remains to be shown.

Assuming $f = 0$ means $u \in \mathcal{V}(\omega)$ so that for any integer $N \geq 0$

$$q(N;u,u) = q(-N;u,u) \qquad\qquad q(s;u,v) := \int_{\Gamma_s} \left(\partial_{x_1} u\,\overline{v} - \partial_{x_1}\overline{v}\,u\right)\,dx_T$$

Using $u = \sum a_\ell^+\,\Phi_\ell^+ + w^+$ and $u = \sum a_\ell^-\,\Phi_\ell^- + w^-$, we deduce that

$$q(N;u,u) = \sum\ |a_\ell^+|^2\,q(\Phi_\ell^+,\Phi_\ell^+) + q(N;w^+,w^+) + 2\,\mathcal{I}m\ q\Big(N;w^+,\sum a_\ell^+\,\Phi_\ell^+\Big)$$

$$q(-N;u,u) = \sum\ |a_\ell^-|^2\,q(\Phi_\ell^-,\Phi_\ell^-) + q(-N;w^-,w^-) + 2\,\mathcal{I}m\ q\Big(-N;w^-,\sum a_\ell^-\,\Phi_\ell^-\Big)$$

The uniqueness result

Theorem : Assume that $\omega^2 \notin \sigma_0$, $e^{\alpha|x_1|}$, $f \in L^2(\Omega),\ \alpha > 0$ and $\sigma_p(A) = \emptyset$.

There exists a unique function $u \in H^2_{loc}(\Omega)$ satisfying

$$(\mathcal{P}) \qquad -\Delta u - n^2\,\omega^2 u = f \quad \text{in } \Omega \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega$$

as well as the two outgoing radiation conditions (CR-) and (CR+)

Proof : $\quad q(N;u,u) = q(-N;u,u) \qquad u = \sum a_\ell^+\ \Phi_\ell^+ + w^+ \qquad u = \sum a_\ell^-\ \Phi_\ell^- + w^-$

$$q(N;u,u) = \sum\ |a_\ell^+|^2\, q(\Phi_\ell^+,\Phi_\ell^+) + \underbrace{q(N;w^+,w^+) + 2\,\mathcal{I}m\ q\Big(N;w^+,\sum a_\ell^+\ \Phi_\ell^+}_{\longrightarrow 0 \quad (N \longrightarrow +\infty)}\Big)$$

$$q(-N;u,u) = \sum\ |a_\ell^-|^2\, q(\Phi_\ell^-,\Phi_\ell^-) + q(-N;\underbrace{w^-,w^-) + 2\,\mathcal{I}m\ q\Big(-N;w^-,\sum a_\ell^-\ \Phi_\ell^-}_{\longrightarrow 0 \quad (N \longrightarrow +\infty)}\Big)$$

$$\left.\begin{array}{l} \sum\ |a_\ell^+|^2\, q(\Phi_\ell^+,\Phi_\ell^+) = \sum\ |a_\ell^-|^2\, q(\Phi_\ell^-,\Phi_\ell^-) \\[2ex] q(\Phi_\ell^+,\Phi_\ell^+) = i\ q_\ell^+,\ q_\ell^+ > 0 \qquad q(\Phi_\ell^-,\Phi_\ell^-) = i\ q_\ell^-,\ q_\ell^- < 0 \end{array}\right\} \quad a_\ell^+ = a_\ell^- = 0, \quad 1 \le \ell \le N$$

The uniqueness result

Theorem : Assume that $\omega^2 \notin \sigma_0$, $e^{\alpha|x_1|}$, $f \in L^2(\Omega),\ \alpha > 0$ and $\sigma_p(A) = \emptyset$.

There exists a unique function $u \in H^2_{loc}(\Omega)$ satisfying

$$(\mathcal{P}) \qquad -\Delta u - n^2\,\omega^2 u = f \quad \text{in } \Omega \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega$$

as well as the two outgoing radiation conditions (CR-) and (CR+)

Proof : $u = \sum a_\ell^+\ \Phi_\ell^+ + w^+ \qquad u = \sum a_\ell^-\ \Phi_\ell^- + w^- \qquad a_\ell^+ = a_\ell^- = 0, \quad 1 \leq \ell \leq N$

thus $u \equiv w^+ \equiv w^-$ is exponentially decreasing at both $\pm\infty$, which implies $u \in D(A)$.

If u were not identically 0, u would be an eigenvector of A (for the eigenvalue ω^2)

This is impossible since $\sigma_p(A) = \emptyset$. Thus $u = 0$, which concludes the proof.