Inverse obstacle scattering in two dimensions with multiple frequency data and multiple angles of incidence

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Abstract

We consider the problem of reconstructing the shape of an impenetrable sound-soft obstacle from scattering measurements. The input data is assumed to be the far-field pattern generated when a plane wave impinges on an unknown obstacle from one or more directions and at one or more frequencies. It is well known that this inverse scattering problem is both ill posed and nonlinear.

Direct scattering problem

We want to solve the direct acoustic scattering problem for sound-soft obstacles. For the incident plane wave $u^{inc}(x) = e^{ikx\cdot d}$ with direction d, we want to find the solution $u = u^{inc} + u^s$ to:

$$\begin{aligned} \Delta u + k^2 u &= 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{D} \\ u &= 0 \quad \text{on} \quad \partial D, \end{aligned}$$

where u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \qquad r = \|x\|.$$

We use two approaches as [1]. Using potential theory, we have:

$$\left[\left(\frac{1}{2}I + D_{\Gamma,k} - i\eta S_{\Gamma,k}\right)\varphi\right] = -u^{\text{inc}}, \tag{1}$$

and using Green's second identity, we have:

$$\left[\left(\frac{1}{2}I - S'_{\Gamma,k} + i\eta S_{\Gamma,k}\right)\frac{\partial u}{\partial \nu}\right] = \frac{\partial u^{\text{inc}}}{\partial \nu} + i\eta u^{\text{inc}}. \tag{2}$$

where $S_{\gamma,k}$, $S'_{\Gamma,k}$ and $D_{\Gamma,k}$ are respectively the single , its derivative, and double layer operators at wavenumber k and γ is a parameterization of the obstacle D, and φ is a layer potential.

We define the far-field pattern $u_\infty(\hat{x})$ as:

$$u(x) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right)\right\}.$$

We use the following equation to obtain the far-field pattern:

$$u^\infty(\hat{x}) = \left[\left(D^\infty_{\Gamma,k} - i\eta S^\infty_{\Gamma,k}\right)\varphi\right](\hat{x}). \tag{3}$$

where $S^\infty_{\Gamma,k}$ and $D^\infty_{\Gamma,k}$ are respectively the single and double layer far field operators(or its respective counterpart in the case of using Equation (1)).

Numerical Implementation

We discretize the boundary using a Nyström method [1], with N equispaced points.Because $S_{\Gamma,k}$ and the principal value for $D_{\Gamma,k}$ and $S'_{\Gamma,k}$ are logarithmically singular, however, we employ the hybrid Gauss-trapezoidal rule of order 16 due to Alpert [2].

For small N, we use direct LU factorization to solve the discretized versions of (1) and (2). For large N, we will employ the fast direct solver of Ambikasaran and Darve [3], whose cost scales as $\mathcal{O}(N \log^2 N)$ for a fixed frequency.

Inverse scattering problem

Given the measured far-field pattern, we wish to determine the shape Γ of the scatterer such that

$$F(\Gamma) = u^\infty .$$

We use the Newton-like method to solve for the unknown Γ, based on the approximation

$$F(\Gamma_j + P_j) \approx F(\Gamma_j) + F'(\Gamma_j)P_j = u^\infty ,$$

where Γ_j is the jth guess for Γ, $F'(\Gamma_j)$ denotes the Fréchet derivative of F, and P_j is the update. The $(j+1)$st iterate is then given by

$$\Gamma_{j+1} = \Gamma_j + P_j . \tag{4}$$

The approximation above leads to the linearized problem:

$$F'(\Gamma_j)P_j = u^\infty - F(\Gamma_j), \tag{5}$$

Theorem [4] Let v denote the solution to the *forward* scattering problem with Dirichlet data

$$v(t) = -\nu_j(t)\cdot P_j(t)\frac{\partial u_j}{\partial \nu}(t) \quad \text{on} \quad \Gamma_j,$$

where $P_j(t)$ is some two-dimensional perturbation of the boundary and $\nu_j(t)$ denotes the normal vector to Γ_j, and let v^∞ denote its far-field pattern. Then the Fréchet derivative of F is

$$F'(\Gamma_j)P_j = v^\infty. \tag{6}$$

Remark 1: We assume that $P_j(t)$ lies in the normal direction:

$$P_j(t) = \nu_j(t)p_j(t),$$

where $p_j(t)$ is a bandlimited scalar function.

Inverse scattering using multiple frequency

Since the number of nontrivial measurements that can be made in the far field is proportional to the size of the object in wavelengths, it is reasonable to expect that greater resolution should be obtained as the frequency increases.

We have the following behavior associated to Newton's method when using a single frequency to reconstruct the object

Frequency	Reconstruction	Initial Guess
Low	Fuzzy	Simple
High	Sharp	Closer to object

This interplay between easy recovery at low frequencies of a blurry reconstruction and the need for a good initial guess at high frequency to achieve higher fidelity reconstruction led Chen [5] to introduce the *recursive linearization algorithm* (RLA).

Recursive Linearization Algorithm

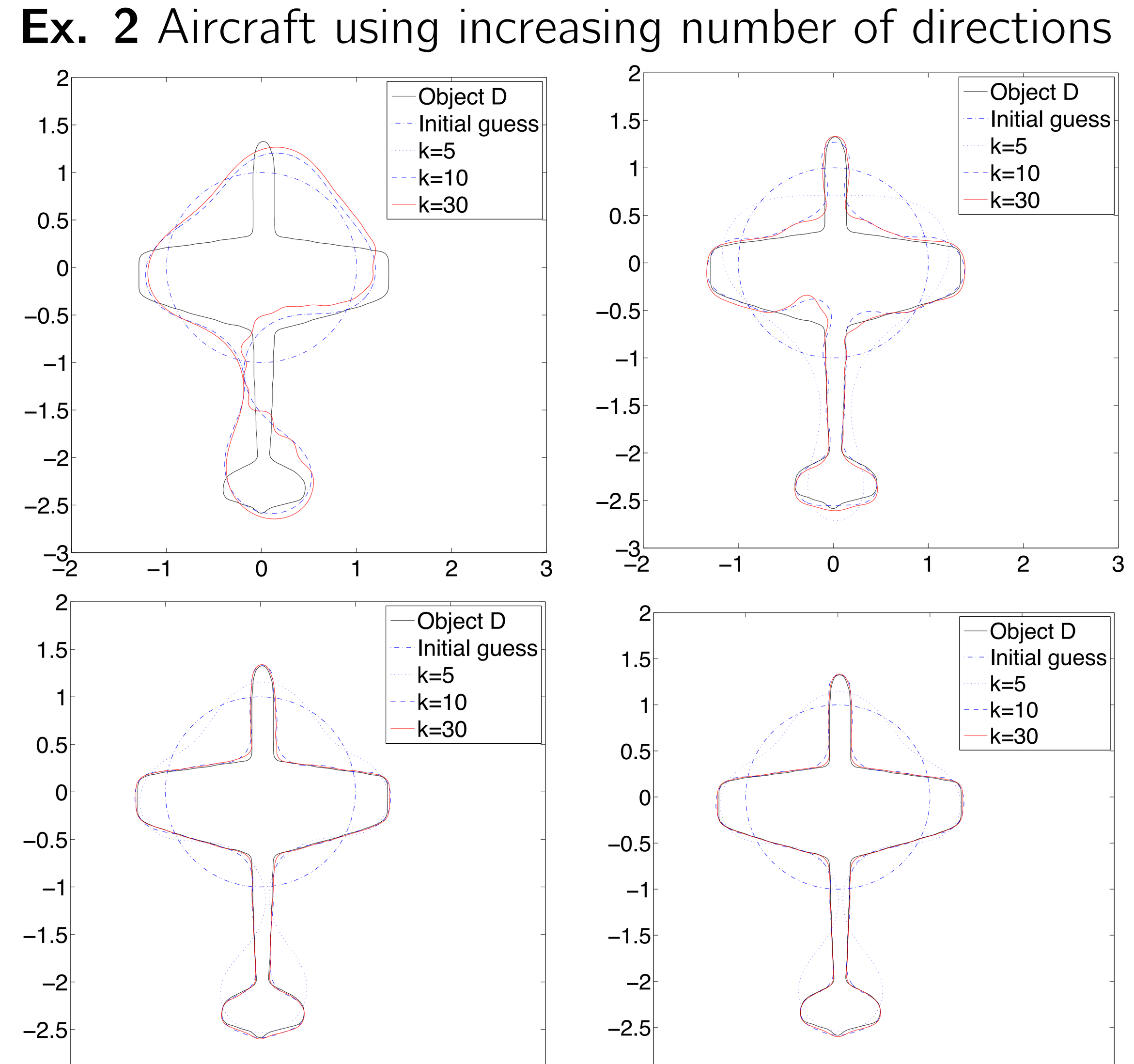
– Choose a initial guess Γ_0 for the domain.

– For $j = 1, 2, \ldots, N_k$

- Take $\Gamma_j = \Gamma_{j-1}$ as initial guess and use Newton's method for a single frequency to obtain the approximation for the domain for the wavenumber k_j.

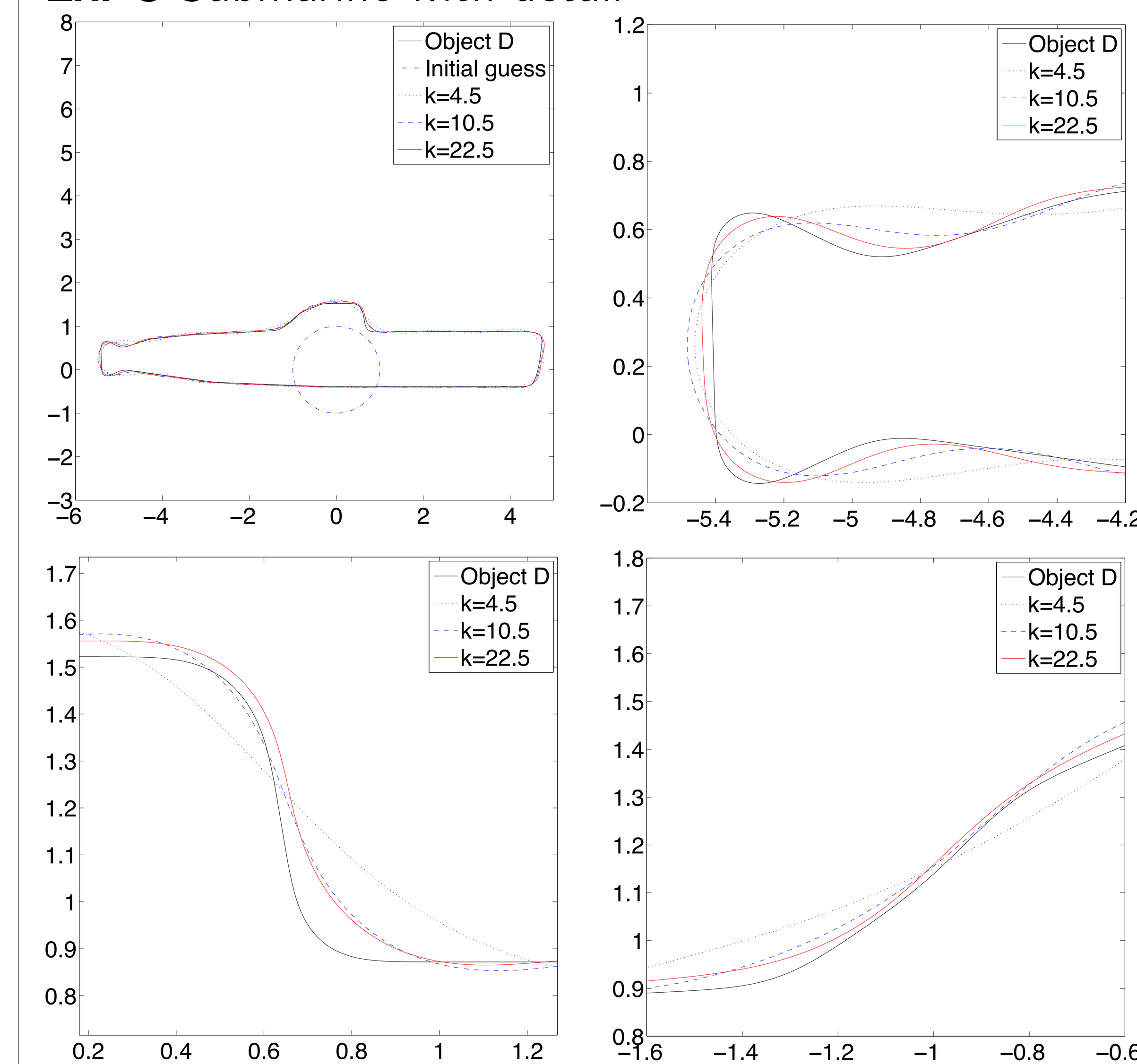
Numerical Examples

Ex. 2 Aircraft using increasing number of directions



Numerical Examples(Cont.)

Ex. 3 Submarine with detail



Conclusions and future work

– The method produces sharp approximations of the object from simple approximations. [6]

– We developed a fast solver for the solution of the penetrable media problem [7]. This solver will be used in the solution of the inverse medium problem.

– This algorithm can be extended for the 3D case.

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