Adaptive Plane Wave Discontinuous Galerkin Methods for the Helmholtz Equation

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Abstract

We present a study of an h-version a posteriori analysis for the Plane Wave Discontinuous Galerkin Method, with a fixed number of plane waves per element. We derive an error indicator using residuals, and present a numerical study of its efficiency. The condition number of the PWDG matrix deteriorates as the mesh is refined. We show that using scaled Bessel functions significantly improves conditioning.

Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz polyhedral domain with boundary $\partial\Omega := \Gamma_D \cup \Gamma_A$ consisting of two disjoint components. The problem is to approximate the solution u of

$$-\Delta u - k^2 u = 0 \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \nu} + iku = g \quad \text{on } \Gamma_A,$$
$$u = 0 \quad \text{on } \Gamma_D.$$

Here the wavenumber $k > 0$, $g \in L^2(\Gamma_A)$. This problem is often considered because the Robin boundary condition is a simple absorbing boundary condition, so the problem is a simplified model for scattering from a bounded domain. In the scattering case, g is determined by the incident field.

Purpose of Study

We are interested in deriving a posteriori error indicators based on residuals to drive the PWDG method adaptively to a solution.

The PWDG Method

The PWDG method is based on the use of plane waves propagating in different directions in each element. On each element K the local solution space are the plane waves

$$V_{p_K}^K := \operatorname{span}\left\{\exp(ik\mathbf{d}_j^K \cdot \mathbf{x}),\ 1 \le j \le p_K\right\}.$$

Then the solution space is

$$V_h = \left\{v_h \in L^2(\Omega) : v_h|_K \in V_{p_K}^K\right\}.$$

The PWDG Method is to find $u_h \in V_h$ such that $A_h(u_h, v_h) = \ell_h(v)$ for all $v \in V_h$, where

$$A_h(u,v) := \int_{\mathcal{E}_\mathcal{I}} \{\!\{u\}\!\}\,[\![\nabla_h \overline{v}]\!] - \int_{\mathcal{E}_\mathcal{I}} [\![\overline{v}]\!]\,\{\!\{\nabla_h u\}\!\}$$
$$-\frac{1}{ik}\int_{\mathcal{E}_\mathcal{I}} \beta\,[\![\nabla_h u]\!]\,[\![\nabla_h \overline{v}]\!] + ik\int_{\mathcal{E}_\mathcal{I}} [\![u]\!]\,[\![\overline{v}]\!]$$
$$-\int_{\mathcal{E}_\mathcal{D}} (\nabla_h u \cdot \nu)\,\overline{v} - \int_{\mathcal{E}_\mathcal{B}} \delta\,(\nabla_h u \cdot \nu)\,\overline{v}$$
$$+ik\int_{\mathcal{E}_\mathcal{B}} (1-\delta) u\overline{v} + \int_{\mathcal{E}_\mathcal{B}} (1-\delta) u\,(\nabla_h \overline{v} \cdot \nu)$$
$$-\frac{1}{ik}\int_{\mathcal{E}_\mathcal{B}} \delta\,(\nabla_h u \cdot \nu)\,(\nabla_h \overline{v} \cdot \nu) + ik\int_{\mathcal{E}_\mathcal{D}} \alpha u \overline{v},$$
$$\ell_h(v) := -\frac{1}{ik}\int_{\mathcal{E}_\mathcal{B}} \delta g\,(\nabla_h \overline{v} \cdot \nu) + \int_{\mathcal{E}_\mathcal{B}} (1-\delta) g \overline{v}$$

with penalty parameters $\alpha, \beta, \delta > 0$.

An a Posteriori Error Estimate

Theorem

For any sufficiently fine mesh, there is a constant C independent of h, u, u_h such that

$$\|u - u_h\|_{L^2(\Omega)} \le C\eta(u_h)$$

where

$$\eta(u_h)^2 := k^{2s-1}(d_\Omega k)^{1-2s}\left[\eta_J^2 + \eta_{J,\nu}^2 + \eta_B^2 + \eta_D^2\right]$$

with the residuals

$$\eta_J^2(u_h) := \|\alpha^{1/2} h_e^s\,[\![u_h]\!]\,\|_{L^2(\mathcal{E}_\mathcal{I})}^2,$$
$$\eta_{J,\nu}^2(u_h) := k^{-2}\|\beta^{1/2} h_e^s\,[\![\nabla_h u_h]\!]\,\|_{L^2(\mathcal{E}_\mathcal{I})}^2$$
$$\eta_B^2(u_h) := k^{-2}\|\delta^{1/2} h_e^s \left[g - \frac{\partial u_h}{\partial \nu} - iku_h\right]\|_{L^2(\mathcal{E}_\mathcal{B})}^2,$$
$$\eta_D^2(u_h) := \|h_e^s \alpha^{1/2} u_h\|_{L^2(\mathcal{E}_\mathcal{D})}^2$$

$0 \le s \le 1/2$ *is a parameter that depends on (re-entrant corners of) the domain. For convex domains choose* $s = 1/2$.

Solution: Case 1

Exact solution $u(\mathbf{x}) = J_\xi(kr)\sin(\xi\theta)$

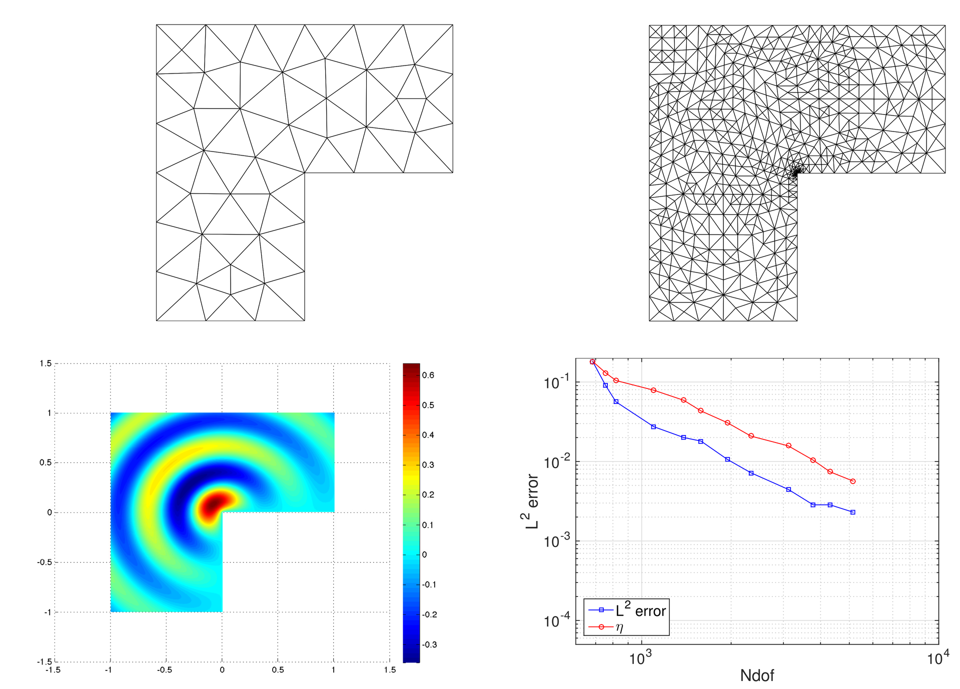

Figure 1: Singular Bessel, $\xi = 2/3$

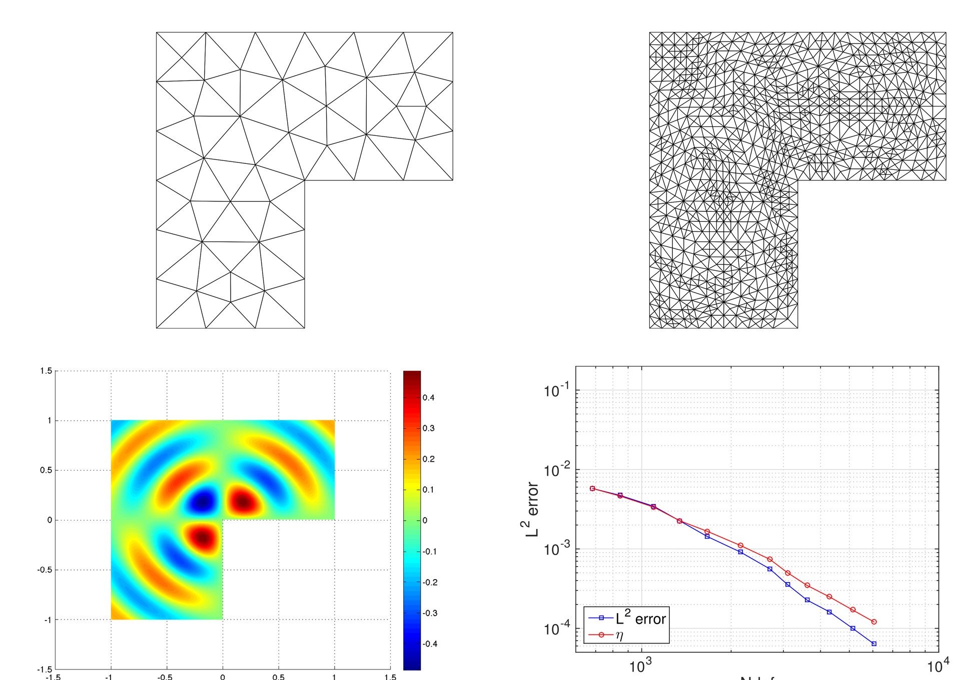

Figure 2: Smooth Bessel, $\xi = 2$. Top left: initial mesh. Top right: 12 iterations. Bottom left: Computed solution. Bottom right: L^2 norm of error and error indicator.

Solution: Case 2 (a)

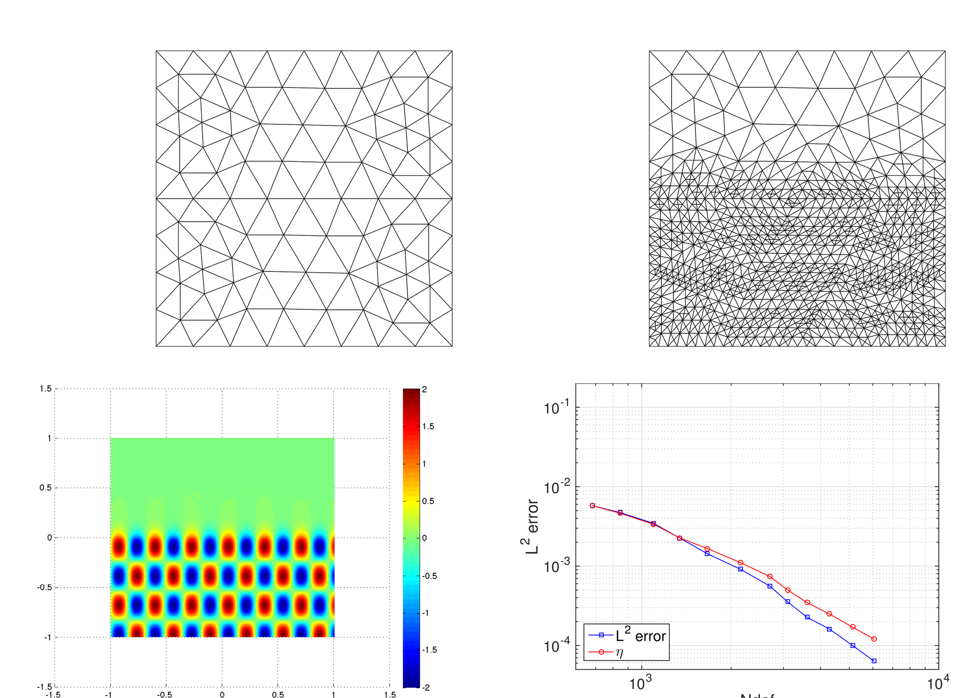

Figure 3: Total Internal Reflection. 9 plane wave directions

Solution: Case 2(b)

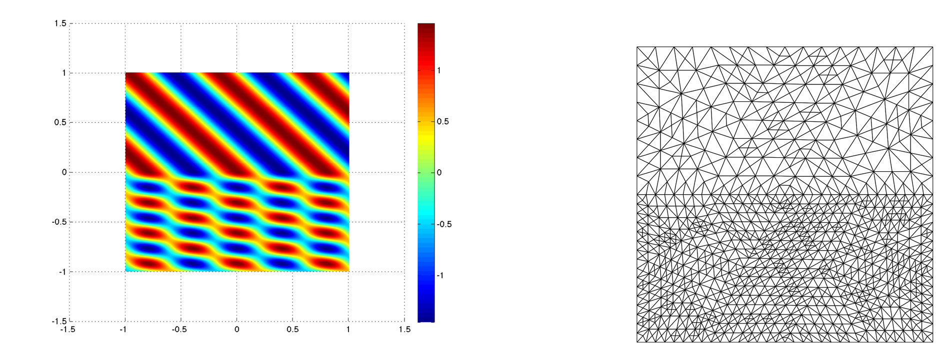

Figure 4: Partial Internal Reflection. 9 plane waves per element

Bessel Basis Functions

Numerical observations show that using basis functions of the form

$$u_h = \sum_{m=-p}^{p} \frac{J_m(k|\mathbf{x} - \mathbf{c_K}|)}{k\sqrt{(J_m'(kh_K))^2 + (J_m(kh_K))^2}}$$

significantly improves the conditioning of the problem.

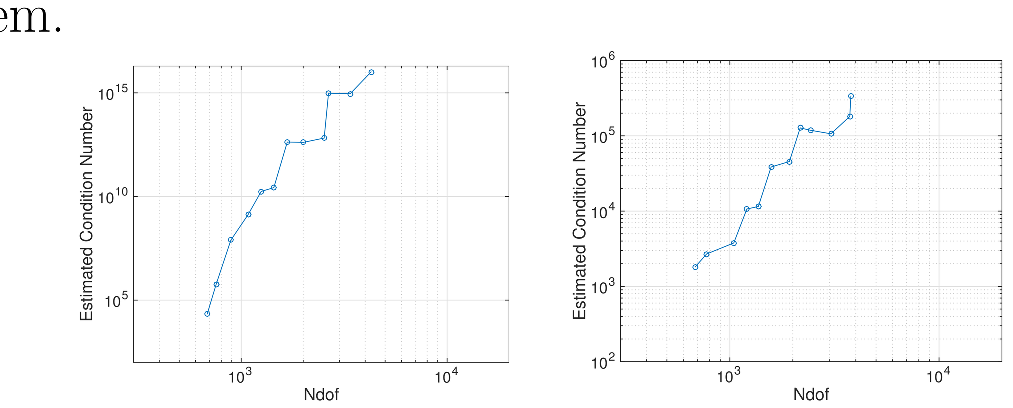

Figure 5: Condition numbers of the scheme. Left: 9 plane waves per element Right: 8 Bessel functions per element

References

[1] S. Kapita, P. Monk, T. Warburton, Residual Based Adaptivity and PWDG Methods for the Helmholtz, SIAM J. Sci. Comput., **37**(3), A1525-A1553.

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