# Transformed and Generalized Localization for Ensemble Methods in Data Assimilation

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Deutscher Wetterdienst

Wetter und Klima aus einer Hand

## Goal

The goal of this work is to suggest a general algorithm which allows localization in the case of non-local observation operators which is not restricted by this limit, and to test its applicability by studying the effect of the transformed localization when applied to a infrared atmospheric sounding interferometer(IASI)retrieval problem which is known to have a strong non-locality and is of high interest to both the research community and operational centers of weather prediction.

## Ensemble Kalman Filter(EnKF)

- Propagates the ensembles of states $x_k^{(\ell)} \in X$. from $t_{k-1}$ to time $t_k$ by applying the model dynamics $M : X \to X$, i.e.,
$$x_k^{(\ell,b)} = M(x_{k-1}^{(\ell,a)}),\ \ell = 1, ..., L.$$
- Ensemble matrix $Q$ is defined by
$$Q_k^b := \frac{1}{\sqrt{L-1}}\left(x_k^{(1,b)} - x_k^{(b)}, ..., x_k^{(L,b)} - x_k^{(b)}\right),\ k \in \mathbb{N}.$$
- Set covariance matrix $B = Q^{(b)}(Q^{(b)})^*$, then Kalman gain
$$K_k = (Q_k^{(b)})(Q_k^{(b)})^* H^*(R + H(Q_k^{(b)})(Q_k^{(b)})^* H^*)^{-1},\ k \in \mathbb{N}. \quad (1)$$
- The Kalman Update equation
$$x_k^{(a)} = x_k^{(b)} + K_k(y_k - Hx_k^{(b)}).$$
- The ensemble analysis
$$Q_k^{(a)} = Q_k^{(b)}\mathcal{L}$$
where $\mathcal{L} = \sqrt{I - (Q_k^{(b)})^* H_k^*\left(R + H_k Q_k^{(b)}(Q_k^{(b)})^* H_k^*\right)^{-1} H_k Q_k^{(b)}}$.
- An analysis covariance matrix $B^{(a)}$ is calculated by
$$B_k^{(a)} = (I - K_k H_k)B_k^{(b)},$$

## Transformed Localization

- Input is our measurements in a space $Y$, the observation operator $H$ from the state space $X$ into $Y$ with
$$Hx = y, \quad (2)$$
and the ensemble $Q^{(b)}$ defined on $X$.
- We first calculate a transformation $T : X \to X$ and $S : Y \to Y$ such that $\tilde{B} = TBT^*$ and $\tilde{H} = SHT^{-1}$ are either diagonal or have small elements in off-diagonal matrix elements far away from the diagonal.
$$\tilde{H}\tilde{x} = \tilde{y}. \quad (3)$$
The influence of $T\&S$ on EnKF with square root is
$$\tilde{Q} = TQ,\quad \tilde{Q}^* = Q^*T^*, \tilde{R} = SRS^*.$$
The transformed Kalman gain matrix as given by (1) ,
$$\tilde{K}_k = Q^{\tilde{(b)}}(Q^{\tilde{(b)}})^*\tilde{H}^*(\tilde{R} + \tilde{H}Q^{\tilde{(b)}}(Q^{\tilde{(b)}})^*\tilde{H}^*)^{-1},\ k \in \mathbb{N}. \quad (4)$$
It has been shown by [1, Lemma 6.1, Theorem 6.2] that,
$$\tilde{K}_k(\tilde{y}_k - \tilde{H}x_k^{\tilde{(b)}}) = TK(y_k - Hx_k^b),$$
where the *analysis ensemble* is given by,
$$Q_k^{\tilde{(a)}} = Q_k^{\tilde{(b)}}\tilde{\mathcal{L}},\ \text{with a matrix}\ \tilde{\mathcal{L}} = \mathcal{L}.$$
Analysis increment is
$$x^{\tilde{(a)}} - x^{\tilde{(b)}} = TK(y - Hx^{(b)}),\ \text{with}\ \tilde{K} = TKS^{-1}.$$
- Then, we solve the localized transformed equations in each area or block given by the transformations $T, S$.
- The solution in each area or block are composed into a global analysis $\tilde{x}^{(a)}$ in transformed space.
- The transformed analysis $\tilde{x}^{(a)}$ is mapped back into the original state space $X$, i.e. we calculate $x^{(a)}$. If we denote the localized transformed matrix by $\tilde{B}_{k,gl}$ (where $gl$ stands for *generalized localization*), this means we calculate
$$\tilde{x}_{gl}^{(a)} := \tilde{x}^{(b)} + \tilde{K}_{k,gl}(\tilde{y}_k - \tilde{H}\tilde{x}_k^{(b)}),$$
where
$$\tilde{K}_{k,gl} := \tilde{B}_{k,gl}\tilde{H}^*(\tilde{R} + \tilde{H}\tilde{B}_{k,gl}\tilde{H}^*)^{-1}$$
with $\tilde{R}$ given in (4). According to [1, Theorem 6.2, (A.7)], it is equivalent to
$$x_{gl}^{(a)} := x^{(b)} + K_{k,gl}(y_k - Hx_k^{(b)}), \quad (5)$$
with
$$K_{k,gl} = B_{k,gl}H^*(R + HB_{k,gl}H^*)^{-1},$$
for $B_{k,gl} = T^{-1}\tilde{B}_{k,gl}(T^*)^{-1}$.in the original space, transformed into each other by $T$ and $S$.
- By [1, Cor 3.4] ***Singular value decomposition*** of the operator $H$ provides a transformation of both the spaces $X$ and $Y$ such that the matrix equation (3) is diagonal and, thus, can be fully localized in the spaces $(\tilde{X}, \tilde{Y})$.

## Generalized Localization

***Localization*** is to solve equations on some local domain $\mathbf{B}_\rho(\mathbf{p})$ around some point $p$ and with localization radius $\rho$, This means, we solve local linear systems
$$H_{p,\rho}x = y_{p,\rho}, \quad (6)$$
with the operator $H_{p,\rho}$ which is obtained from the operator $H$ by restricting it to data defined on $\mathbf{B}_\rho(\mathbf{p})$, with $y_{p,\rho}$ defined as the subset of data defined on $\mathbf{B}_\rho(\mathbf{p})$, and with background $x^{(b)}$. It leads to a local analysis $x_j^{(a)}$ and to a local analysis ensemble $x_j^{(a,\ell)}$ which is valid in a neighborhood $B_{\rho_j}(p_j)$. We can write the *local equations* (6) in the form
$$P_{p,\rho}Hx = P_{p,\rho}y,$$
where $P_{p,\rho}$ is a projection operator restricting to the ball defined by $p$ and $\rho$.

In a ***Generalized Localization Concept*** we replace the simple operator $P$ by a general family $P_j,\ j = 1, ..., J$ of *projection operators* and solve equations
$$H_jx = y_j,\ \text{where}\ H_j := P_jH,\ y_j := P_jy,$$
such that
$$X_j \cap X_i = \{0\}, i \neq j,\ i, j = 1, ..., J.$$
Now, we solve the equation
$$H_jx = y_j \ \Leftrightarrow\ P_jHx = P_jy$$
- In order to enhance the numbers of degrees of freedom of the ensemble & cease the false correlation to use localization on ensemble
- If we localize non-local operators, the approximation quality can be very poor and assimilation will not give the desired result.
- In general, if the observation operator $H$ is non-local, then even if the $B$ matrix is local, the term $HBH^*$ is non-local and in (1) the inversion will need to solve a full system. In this case, localization will lead to large errors.
- However, if we transform the state space $X$ and the observation space $Y$ in a way such that $\tilde{H}$ is local and $\tilde{B}$ is local as well, then we can achieve locality of the terms $\tilde{H}\tilde{B}\tilde{H}^*$, $\tilde{B}\tilde{H}^*$ and $\tilde{R}$, such that $\tilde{K}$ remains local. Then, localization applied to the transformed version of the EnKF will yield small approximation errors.
- The full regularized solution $x_\alpha$ equals to the sum of the regularized solutions $x_{proj,\alpha}$ under generalized localization.

## Atmospheric Radiance Inversion

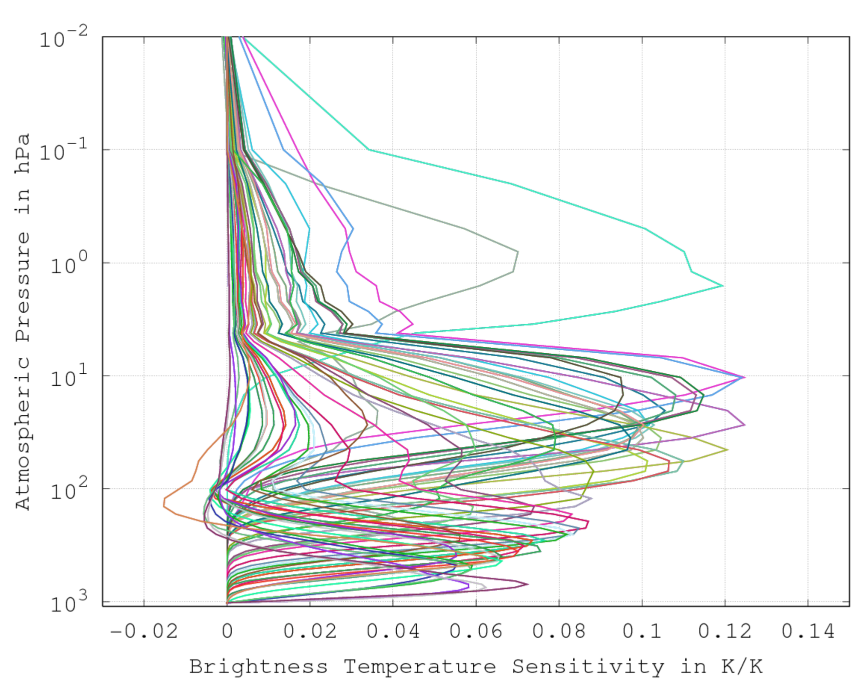


Figure 1: The sensitivity functions of the temperature sensitive IASI channels of the RTTOV operator $H$, i.e. the rows $(H_{j1}, ..., H_{j,n})$ for $j = 1, ..., m$.

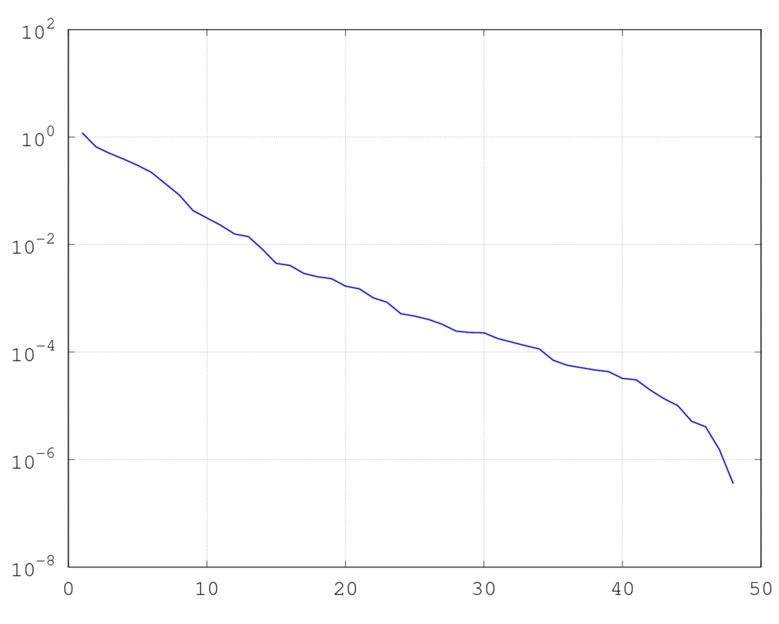

Figure 2: Exponential decay of the singular values of $H$.

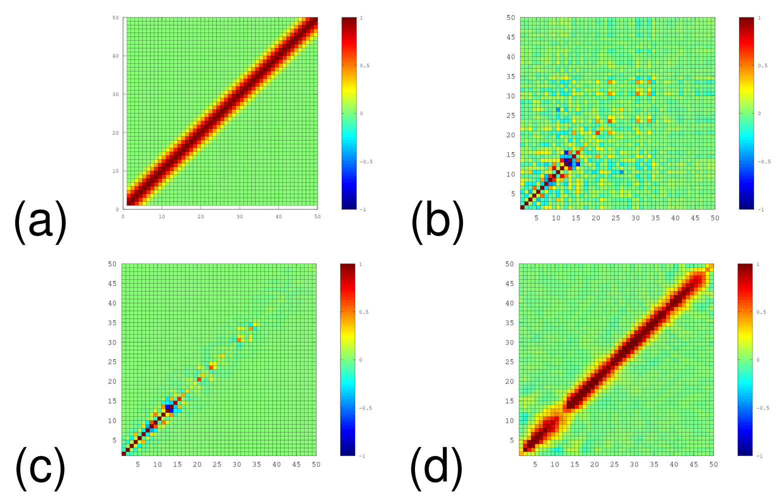


Figure 3: We display a local B-matrix in (a), which has been constructed according to as a Gaussian matrix in the column. The transformed matrix $\tilde{B}$ is displayed in Figure (b). We localize the matrix $\tilde{B}$ by multiplication with the matrix shown in (a), the result is displayed in (c). Figure (d) shows the matrix $B_{gl}$ defined as the backtransformed matrix from $\tilde{B}_{gl}$.

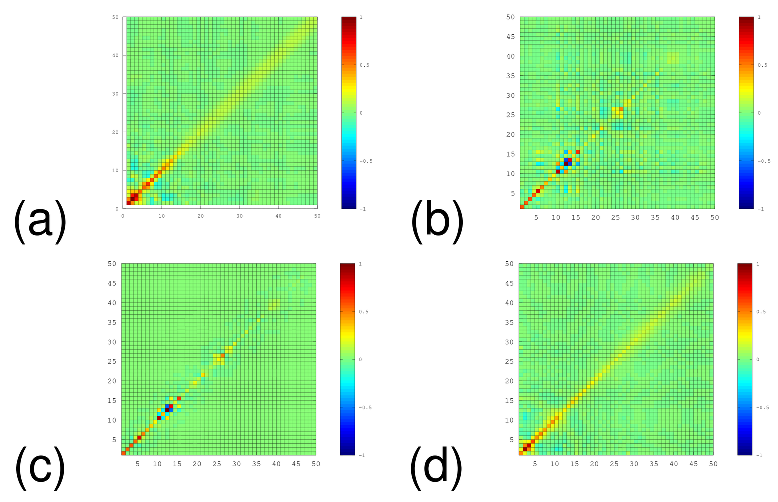


Figure 4: We display a B-matrix constructed from the experimental global LETKF of DWD in (a), which has been constructed by taking the standard stochastic estimator as in (2) and adding a small constant times the Gaussian covariance matrix displayed in Figure (a). The transformed matrix $\tilde{B}$ is displayed in Figure (b). We localize the matrix $\tilde{B}$ by multiplication with the matrix shown in Figure (a), the result is displayed in (c). Figure (d) shows the matrix $B_{gl}$ defined as the backtransformed matrix from $\tilde{B}_{gl}$.

The difference between a background atmospheric temperature profile and some given temperature profile are displayed in the below figures.

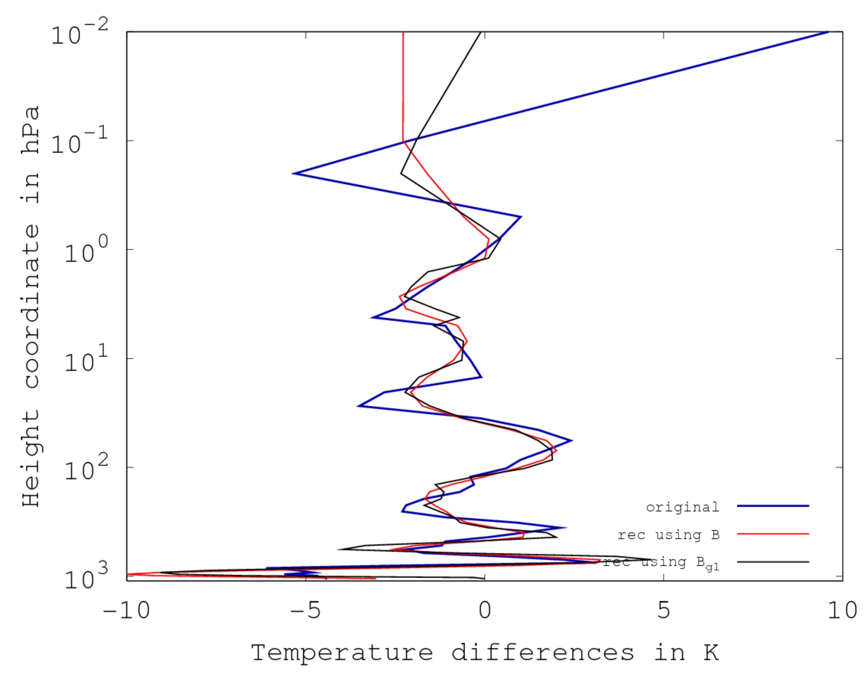


Figure 5: The true difference is shown in thick blue, the reconstruction using the Gaussian $B$-matrix from Figure (a) with regularization parameter $\alpha = 0.0001$ is displayed in red, the reconstruction with $B_{gl}$ in black.

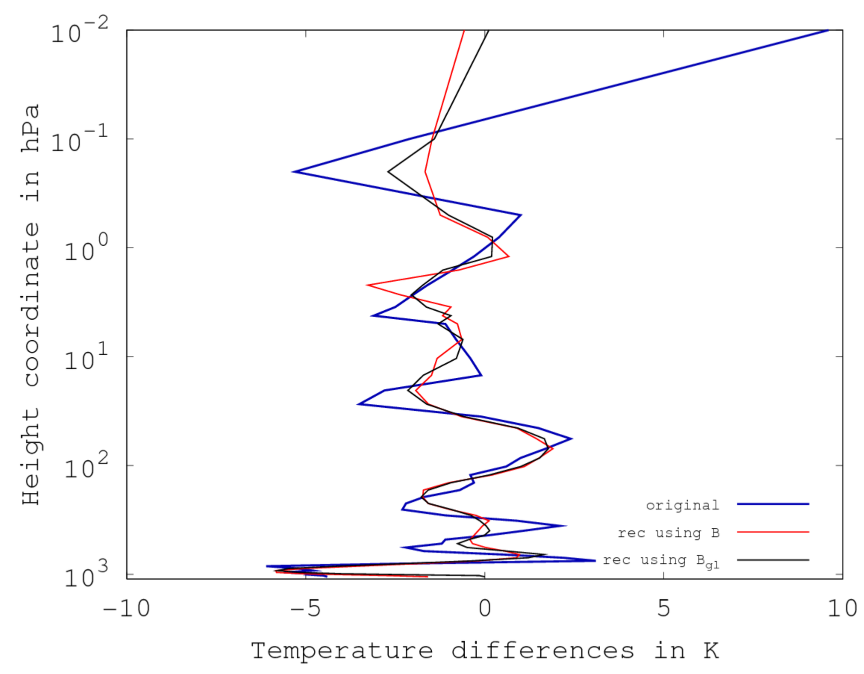


Figure 6: The true difference is shown in thick blue, the reconstruction using the LETKF $B$-matrix displayed in Figure with regularization parameter $\alpha = 0.0001$ is displayed in red, the reconstruction with the corresponding $B_{gl}$ in black.

## References

[1] A. Nadeem and R. Potthast. Mathematical Methods in the Applied Science, 16, 2015(to appear).

## Acknowledgement

This work was supported by Federal Urdu University of Art, Science and Technology,Islamabad Pakistan.