Verification of a variational source condition for inverse medium scattering problems

Frederic Weidling

1. Introduction

Regularization theory deals with the solution of ill-posed operator equations

$$F(f) = y^\delta$$

where $F : \mathrm{dom}(F) \subset \mathcal{X} \to \mathcal{Y}$. A prominent example to solve these equation given perturbed data y^δ with $\|F(f^\dagger) - y^\delta\|_{\mathcal{Y}} \le \delta$ is Tikhonov regularization

$$f_\alpha^\delta \in \underset{f \in \mathrm{dom}(F)}{\mathrm{argmin}} \left[\frac{1}{\alpha} \left\| F(f) - y^\delta \right\|_{\mathcal{Y}}^2 + \frac{1}{2} \|f\|_{\mathcal{X}}^2 \right].$$

For many interesting problems the convergence rates are unknown, while stability estimates exists. We try to answer the question:

Can stability estimates be sharpened to variational source conditions?

We consider acoustic medium scattering given by

$$\Delta u + \kappa^2 u = fu \qquad \text{in } \mathbb{R}^3,$$
$$\frac{\partial u^{\mathrm{s}}}{\partial r} - i\kappa u^{\mathrm{s}} = O\left(\frac{1}{r^2}\right) \qquad \text{as } r = |x| \to \infty,$$

where $\kappa > 0$, f is the contrast of the medium and

$$f \in \mathcal{D} := \left\{ f \in L^\infty(\mathbb{R}^3) \colon \Im(f) \le 0, \Re(f) \le 1, \mathrm{supp}(f) \subset B(\pi) \right\}.$$

The problem is to reconstruct f from the knowledge of the incident wave and measurements of the scattered field.
Employing methods used to proof the stability estimates (e.g. [4, 6]), for example geometrical optical solutions, we show that for this problem the answer is **yes**.

2. Why variational source conditions?

In our setting condition of the form: There exists $\beta \in (0, 1]$ and an index function ψ s.t.:

$$\forall f \in \mathrm{dom}(F): \quad \frac{\beta}{2} \left\| f - f^\dagger \right\|_{\mathcal{X}}^2 \le \frac{1}{2} \|f\|_{\mathcal{X}}^2 - \frac{1}{2} \left\| f^\dagger \right\|_{\mathcal{X}}^2 + \psi \left(\left\| F(f) - F(f^\dagger) \right\|_{\mathcal{Y}}^2 \right) \qquad \text{(VSC)}$$

Advantages over spectral source conditions:

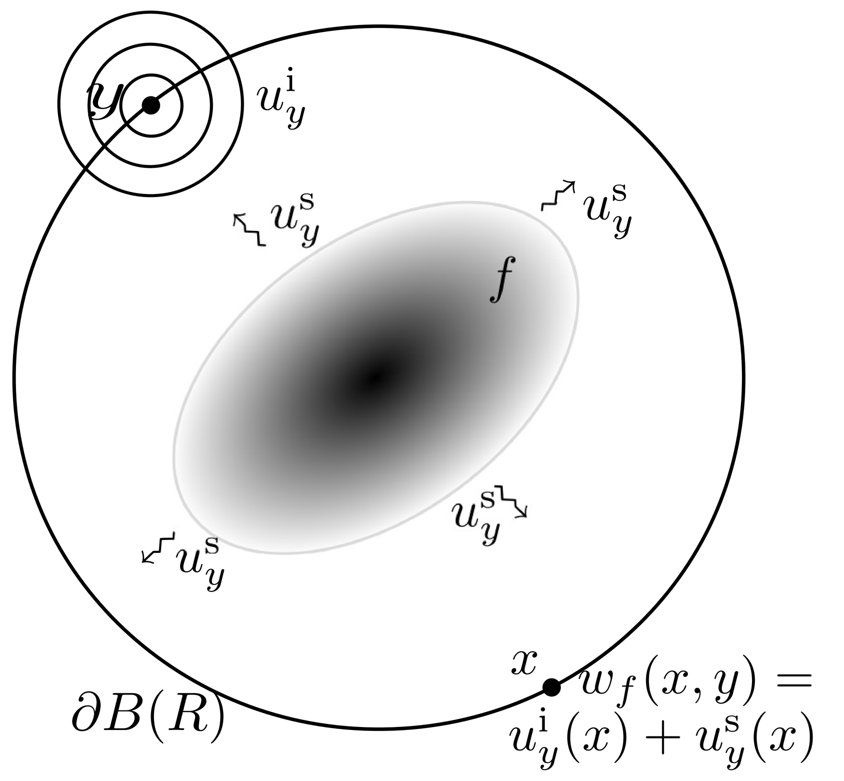
- Yields for Tikhonov regularization the convergence rate
$$\frac{\beta}{2} \left\| f_\alpha^\delta - f^\dagger \right\|_{\mathcal{X}}^2 \le 4\psi(\delta^2)$$
if the index function ψ is concave and the regularization parameter chosen as $-1/(2\alpha) \in \partial(-\psi(4\delta^2))$ (see [3, 7]).
- If (VSC) holds for all $f^\dagger \in \mathcal{K} \subset \mathcal{X}$ they imply the stability estimate
$$\frac{\beta}{2} \|f_1 - f_2\|_{\mathcal{X}}^2 \le \psi \left(\|F(f_1) - F(f_2)\|_{\mathcal{Y}}^2 \right), \qquad \forall f_1, f_2 \in \mathcal{K}.$$
- In Hilbert spaces with linear operators necessary and sufficient for certain rates of convergence for a big class of regularization methods (see [2]).
- No further differentiability assumptions on the forward operator $\rightsquigarrow$ no restrictive assumptions connecting the operators F and F' like the tangential cone condition (usually not verifiable).
- Usable in in a Banach space setting.
- Suitable for more general noise models and data fidelity terms/penalty terms.

Hence **popular in regularization theory** but **few results on the verification** of such conditions. [1, 5]

3. Near field inverse scattering

Assumption (Solution regularity) *Let $\frac{3}{2} < m < s$, $s \ne 2m + 3/2$. Suppose that the true contrast $f^\dagger$ satisfies $f^\dagger \in \mathcal{D} \cap H_0^s(B(\pi))$ with $\|f^\dagger\|_{H^s} \le C_s$ for some $C_s \ge 0$.*

- Choose $R > \pi$.
- For each $y \in \partial B(R)$:
  - Use a point source as incident wave
  $$u_y^{\mathrm{i}}(x) = \frac{1}{4\pi} \frac{e^{i\kappa|x-y|}}{|x - y|}.$$
  - For all $x \in \partial B(R)$ measure the total field $w_f(x, y)$.
- Define the operator
$$F_{\mathrm{n}} \colon \mathcal{D} \to L^2(\partial B(R)^2), \quad f \mapsto w_f.$$

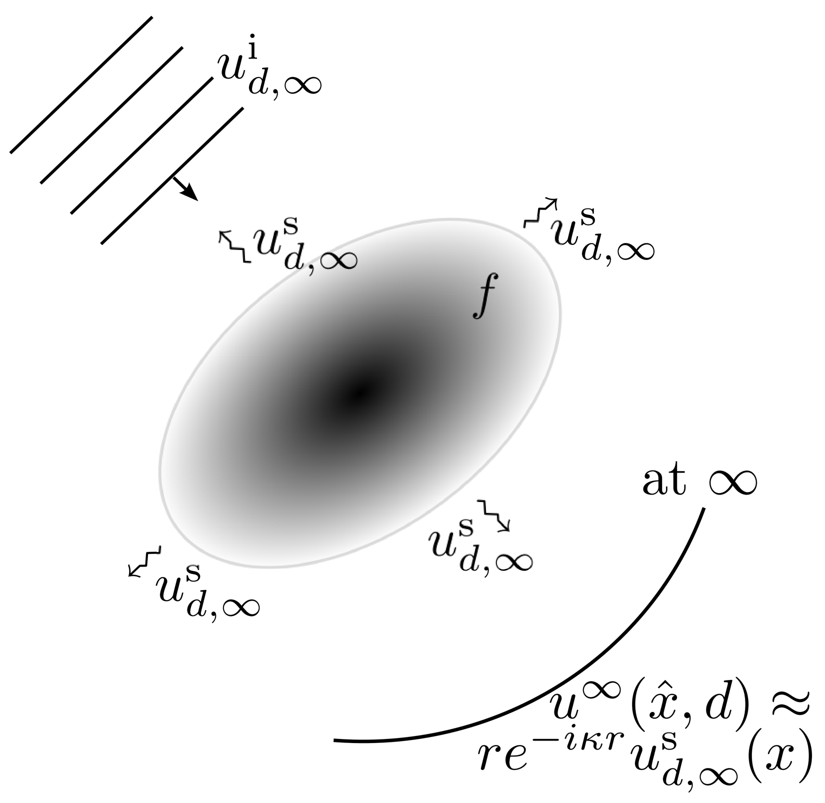


Theorem (VSC for near field) *Let $R > \pi$ and the assumption be fulfilled. Then* (VSC) *holds true for the operator F_{n} with $\mathrm{dom}(F_{\mathrm{n}}) := \mathcal{D} \cap H_0^m(B(\pi))$ with*

$$\psi_{\mathrm{n}}(t) := A \left(\ln(3 + t^{-1}) \right)^{-2\mu}, \qquad \mu := \min \left\{ 1, \frac{s - m}{m + 3/2} \right\}, \qquad \beta = \frac{1}{2},$$

where the constant $A > 0$ depends on m, s, C_s, κ and R.

4. Far field inverse scattering



- For each direction $d \in \partial B(1)$:
  - Use a incident plane wave propagating in direction d
  $$u_{d,\infty}^{\mathrm{i}}(x) = e^{i\kappa d \cdot x}.$$
  - For all directions $\hat{x} \in \partial B(1)$ measure the far field $u^\infty(\hat{x}, d)$ defined by
  $$u_{d,\infty}^{\mathrm{s}}(x) = \frac{e^{i\kappa|x|}}{|x|} u^\infty \left(\frac{x}{|x|}, d \right) + O(|x|^{-3})$$
- Define the operator
$$F_{\mathrm{f}} \colon \mathcal{D} \to L^2(\partial B(1)^2), \quad f \mapsto u^\infty.$$

Theorem (VSC for far field) *Let the assumption be satisfied and $0 < \theta < 1$. Then the F_{f} with $\mathrm{dom}(F_{\mathrm{f}}) := \mathcal{D} \cap H_0^m(B(\pi))$ fulfills* (VSC) *with*

$$\psi_{\mathrm{f}}(t) := B \left(\ln(3 + t^{-1}) \right)^{-2\mu\theta}, \qquad \beta = \frac{1}{2},$$

where μ is given as before and the constant $B > 0$ depends on $m, s, C_s, \kappa, \theta$ and R.

5. Outlook

- Extend the results to **other scattering problems**, for example electromagnetic inverse medium scattering.
- Specify the dependenc on the variational source condition of the parameter κ to obtain **Hölder-logarithmic** results (see [6]).
- Use **Banach space** penalty functional.

6. Reference

[1] M. Burger, J. Flemming, and B. Hofmann. Inverse Problems, 29 (2013), p. 025013.
[2] J. Flemming, B. Hofmann, and P. Mathé. Inverse Problems, 27 (2011), p. 025006.
[3] M.Grasmair. Inverse Problems, 26 (2010), p. 115014.
[4] P. Hähner and T. Hohage. SIAM J. Math. Anal., 33 (2001), pp. 670–685.
[5] B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. Inverse Problems, 23 (2007), p. 987–1010.
[6] M. I. Isaev and R. G. Novikov. Inverse Problems, 30 (2014), p. 095006.
[7] F. Werner and T. Hohage. Inverse Problems, 28 (2012), p. 104004.

Any questions?
Just ask me:

Frederic Weidling
Institute for Numerical and Applied Mathematics
University of Goettingen
Email: f.weidling@math.uni-goettingen.de

joint work with:
Thorsten Hohage

accepted for puplication in Inverse Problems
available on arxiv 1503.05763