Black-Scholes model and the implied volatility

- If Black-Scholes model were a good model for market data, we would have

\[ \sigma_{BS}(T, K) = \sigma \quad \forall T, K \]

- In practice, we observe \( \sigma_{BS}(\cdot, \cdot) \) that depends on \( T \) and \( K \): the *smile* phenomenon.
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- Drawback of the BS model: underestimates important fluctuations of \( S_T \) around \( S_0 \)

- Nevertheless:
  - explicit formulas for Calls/Puts with intuitive parameters
  - often used only as a reference to express prices in terms of their volatilities
  - but BS models sometimes used as a model to quote option prices on assets with small liquidity.
Model Calibration: general considerations

- **Objective**: price/hedge a complex product using simpler products (called the *hedging instruments*)
  
  Example:
  
  - hedge an exotic option with maturity $T$
  - using Vanillas (Calls and Puts) with maturities $\leq T$

Two steps:

1. Identify the hedging instruments: liquid products with listed and reliable prices (small bid-ask spread).
2. Choose a model.

   The role of the model is: reproduce the quoted prices of the simple instruments, then allow to manage the complex product.
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Quoting Emmanuel Derman:

*If you want to know the value of a security, use the price of another security that is as similar to it as possible. All the rest is modeling.*
Model Calibration: general considerations

- **Model calibration** = determine the parameters of a model from market data of liquid instruments.

- Some theory, lots of practice.

- One can distinguish two different approaches:
  - Statistical estimation
    - from historical data
    - under the historical measure
  - Calibration
    - based on quotes of option prices (Calls/Puts)
    - reflects the market view on the future evolution of the asset
    - directly under the risk-neutral measure
Model Calibration: general considerations

Nature of market data

- In statistical estimation:
  - observations \((y_i)_{i=1, \ldots, n}\)
  - Build an estimator as \(n = \# \text{ observations} \to \infty\).

- In calibration:
  - The number of observations is fixed by the market
    - Option prices \(\{P(T_i, K_j)\}_{i,j}\) for some maturities \(T_i\) and some strikes \(K_j\).
    - Typically a few hundreds prices at most.
  - Which price? Bid, Ask?
    - Have to make a choice
    - Typically one calibrates to the mid price \(\text{Mid} = \frac{\text{Ask} + \text{Bid}}{2}\)
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In practice

- If the goal is option pricing/hedging, the calibration approach is more appropriate.
- Statistical estimation procedures are rather used to identify trends in assets (↗ or ↘) and exploit possible arbitrage.
Model Calibration

Heuristical ideas/guidelines

- Number of model parameters: the higher
  - The better the model fits to the market
  - But:
    - the more difficult to interpret each parameter.
    - the more sensitive the parameters are to changes in market data.
Model Calibration

Heuristical ideas/guidelines

- Number of model parameters: the higher
  - The better the model fits to the market
  - But:
    - the more difficult to interpret each parameter.
    - the more sensitive the parameters are to changes in market data.

- In general, a “good” model should have
  - A parameterisation that is parsimonious and intuitive (for traders)
  - A calibration method that is stable and fast.
Model Calibration

**In practice**: given a model(\(\theta\)) that depends on some vector parameter \(\theta \in \mathbb{R}^d\)

- We want to find \(\theta\) such that

\[
P^{\text{model}(\theta)}(T_i, K_j) = P^{\text{market}}(T_i, K_j) \quad \forall i, j.
\]
Model Calibration

In practice: given a model(θ) that depends on some vector parameter θ ∈ ℝ^d

- We want to find θ such that

\[ P_{\text{model}}(θ) (\mathcal{T}_i, \mathcal{K}_j) = P_{\text{market}} (\mathcal{T}_i, \mathcal{K}_j) \quad \forall i, j. \]

- If such a θ does not exist, or if it cannot be computed explicitly (because there is no explicit formula for \( P_{\text{model}}(θ) \)), we replace the problem above with

\[ \min_\theta \sum_{i,j} \left( P_{\text{model}}(θ)(\mathcal{T}_i, \mathcal{K}_j) - P_{\text{market}} (\mathcal{T}_i, \mathcal{K}_j) \right)^2 \]

- A least square approach: minimize the \( L^2 \) distance between model prices and market prices.