**Dupire’s formula**

**Advantage**

- Calibrates to the market by construction.

For any price surface $P_{mkt}(T, K)$ with no arbitrage and smooth, we can construct $\sigma_{loc}(\cdot)$ that perfectly fits the prices.

**Limits**

1. In reality, we do not observe the whole surface but only a finite set of prices $(P_{mkt}(T_i, K_j))_{i,j}$
Dupire’s formula

**Advantage**

- Calibrates to the market by construction.

  For any price surface \( P^{\text{mkt}}(T, K) \) with no arbitrage and smooth, we can construct \( \sigma_{\text{loc}}(\cdot) \) that perfectly fits the prices.

**Limits**

i) In reality, we do not observe the whole surface but only a finite set of prices \( (P^{\text{mkt}}(T_i, K_j))_{i,j} \)

Possible approaches in practice:

1. Interpolate the points \( P^{\text{mkt}}(T_i, K_j) \) with a smooth function
   - then compute the derivatives on this smooth function.
   - one will choose a fct that satisfies the no-arbitrage conditions for a Put price surface (convexity, monotonicity in \( T,... \))

*Interest*: requires to choose an interpolation of market prices; but then, the only computation are the derivatives in Dupire’s formula.
2. Fix the values $\sigma_{i,j} = \sigma_{\text{loc}}(T_i, K_j)$ of the local vol on the maturity/strike grid.
2. Fix the values $\sigma_{i,j} = \sigma_{\text{loc}}(T_i, K_j)$ of the local vol on the maturity/strike grid.

$\rightarrow$ Interpolate the $\sigma_{i,j}$ with a function, for ex. local polynomials (splines).

$\Rightarrow$ No particular conditions related to no-arbitrage are required on this function!

We only ask condition $(H)$ (recall: $\sigma_{\text{loc}}(t,x) > 0$, $x \mapsto x\sigma_{\text{loc}}(t,x)$ Lipsch.)
Dupire’s formula

2. Fix the values $\sigma_{i,j} = \sigma_{\text{loc}}(T_i, K_j)$ of the local vol on the maturity/strike grid.

$\rightarrow$ Interpolate the $\sigma_{i,j}$ with a function, for ex. local polynomes (splines).

$\rightsquigarrow$ No particular conditions related to no-arbitrage are required on this function!

We only ask condition (H) (recall : $\sigma_{\text{loc}}(t,x) > 0$, $x \mapsto x\sigma_{\text{loc}}(t,x)$ Lipsch.)

$\rightarrow$ Compute $P_{\text{model}}(T, K)$ by numerical solution (ex. finite diff) of the forward PDE

$$
\begin{cases}
\partial_T P + (r - d)K\partial_K P - \frac{1}{2}\sigma_{\text{loc}}(T,K)^2 K^2 \partial_{KK} P + dP = 0, & \forall T, K \\
P(0, K) = (K - S_0)^+
\end{cases}
$$

→ Calibrate the values $\sigma_{i,j}$ with a least square approach:

$$
\min_{\sigma_{i,j}} \sum_{i,j} (P_{\text{model}}(T_i, K_j) - P_{\text{mkt}}(T_i, K_j))^2
$$

\text{Interest: method } #2 \text{ contains a numerical solver of a PDE, but there is only one PDE to solve, for all } (T_i, K_j).
Dupire’s formula

2. Fix the values $\sigma_{i,j} = \sigma_{\text{loc}}(T_i, K_j)$ of the local vol on the maturity/strike grid.

→ Interpolate the $\sigma_{i,j}$ with a function, for ex. local polynomials (splines).

~~ No particular conditions related to no-arbitrage are required on this function!

We only ask condition (H) (recall : $\sigma_{\text{loc}}(t, x) > 0$, $x \mapsto x\sigma_{\text{loc}}(t, x)$ Lipsch.)

→ Compute $P_{\text{model}}(T, K)$ by numerical solution (ex. finite diff) of the forward PDE

\[
\begin{aligned}
\partial_T P + (r - d)K\partial_K P - \frac{1}{2}\sigma_{\text{loc}}(T, K)^2K^2\partial_{KK} P + d P &= 0, \quad \forall T, K \\
P(0, K) &= (K - S_0)^+ 
\end{aligned}
\]

→ Calibrate the values $\sigma_{i,j}$ with a least square approach :

\[
\min_{\sigma_{i,j}} \sum_{i,j} (P_{\text{model}}(T_i, K_j) - P_{\text{mkt}}(T_i, K_j))^2
\]

Interest: method #2 contains a numerical solver of a PDE, but there is only one PDE to solve, for all $(T_i, K_j)$. 
Dupire’s formula and local volatility

Limits

ii) Dupire’s formula ensures that

\[ S_t^{\text{loc}} \overset{d}{=} S_t^{\text{mkt}}, \quad \forall t \geq 0. \]

\[ \forall t \geq 0, \quad \max_{s \leq t} |W_s| \text{ has non zero quadratic variation a.s.,} \]

\[ \max_{s \leq t} W_s \text{ is increasing (hence with finite variation) a.s.} \]

But there exist several examples of stochastic processes with same marginal laws for every \( t \geq 0 \), but very different trajectories.
Dupire’s formula and local volatility

Limits

ii) Dupire’s formula ensures that

\[ S_t^{\text{loc}} \overset{d}{=} S_t^{\text{mkt}}, \quad \forall t \geq 0. \]

\[ \iff \text{But there exist several examples of stochastic processes with same marginal laws for every } t \geq 0, \text{ but very different trajectories.} \]

**Exemple 1**: If \( (W_t)_{t \geq 0} \) is a Brownian motion, we know that, by the reflection principle

\[ \forall t \geq 0, \quad \max_{s \leq t} W_s \overset{d}{=} |W_t| \]

But: \(|W_t|\) has non zero quadratic variation a.s., while \(\max_{s \leq t} W_s\) is increasing (hence with finite variation) a.s.
Dupire’s formula and local volatility

Exemple 2:

\[ X_t = \int_0^t s \, dW_s \quad Y_t = \int_0^t W_s \, ds \]

\( X \) et \( Y \) are Gaussian processes, such that \( X_t \) \( \overset{d}{=} \) \( Y_t \) \( \overset{d}{=} \) \( \mathcal{N} \left( 0, \frac{t^3}{3} \right) \) for every \( t \), but

- \((X_t)_t\) is a martingale with non-zero quadratic variation
- \((Y_t)_t\) is a process of finite variation (hence, with zero quadr. variation)
Dupire’s formula and local volatility

Exemple 2 :

\[ X_t = \int_0^t s \, dW_s \quad Y_t = \int_0^t W_s \, ds \]

\( X \) et \( Y \) are Gaussian processes, such that \( X_t \overset{d}{=} Y_t \overset{d}{=} \mathcal{N}\left(0, \frac{t^3}{3}\right) \) for every \( t \), but

- \((X_t)_t\) is a martingale with non-zero quadratic variation
- \((Y_t)_t\) is a process of finite variation (hence, with zero quadr. variation)

Rmk : we can construct an infinite family of processes that have the same marginal laws for every \( t \geq 0 \) by observing that, if \((Z_t)_{t \geq 0}\) is a Brownian motion independent of \((W_t)_t\), then

\[ \forall \rho \in (0, 1) \quad \rho \int_0^t s \, dW_s + \sqrt{1 - \rho^2} \int_0^t Z_s \, ds \overset{d}{=} X_t, \quad \forall t \]
Dupire’s formula and local volatility

Exemple 2:

\[ X_t = \int_0^t s \, dW_s \quad Y_t = \int_0^t W_s \, ds \]

\( X \) et \( Y \) are Gaussian processes, such that \( X_t \overset{d}{=} Y_t \overset{d}{=} \mathcal{N}(0, \frac{t^3}{3}) \) for every \( t \), but

\begin{itemize}
  \item (\( X_t \))\textsubscript{t} is a martingale with non-zero quadratic variation
  \item (\( Y_t \))\textsubscript{t} is a process of finite variation (hence, with zero quadr. variation)
\end{itemize}

Rmk: we can construct an infinite family of processes that have the same marginal laws for every \( t \geq 0 \) by observing that, if \((Z_t)\textsubscript{t \geq 0}\) is a Brownian motion independent of \((W_t)\textsubscript{t}\), then

\[
\forall \rho \in (0, 1) \quad \rho \int_0^t s \, dW_s + \sqrt{1 - \rho^2} \int_0^t Z_s \, ds \overset{d}{=} X_t, \quad \forall t
\]

Consequence: even if we calibrate the marginal laws, the prices and hedging strategies of path-dependent options (Asian, Barrier) can deviate from the market.
iii) Dynamic behavior of the implied volatility surface: define the impl. vol \( \sigma_t^{BS}(T, K) \) at time \( t \leq T \) by imposing

\[
C_{t,S_t}(T, K) = C_{t,S_t}^{BS}(T, K; \sigma_t^{BS}(T, K))
\]
iii) Dynamic behavior of the implied volatility surface: define the impl. vol $\sigma_t^{BS}(T, K)$ at time $t \leq T$ by imposing

$$C_{t,S_t}(T, K) = C_{t,S_t}^{BS}(T, K; \sigma_t^{BS}(T, K))$$

- for every $(T, K)$, $(\sigma_t^{BS}(T, K))_{t \leq T}$ is a stochastic process that can be observed on the market.
iii) Dynamic behavior of the implied volatility surface: define the impl. vol \( \sigma^\text{BS}_t(T,K) \) at time \( t \leq T \) by imposing

\[
C_{t,S_t}(T,K) = C^\text{BS}_{t,S_t}(T,K; \sigma^\text{BS}_t(T,K))
\]

- for every \( (T,K), (\sigma^\text{BS}_t(T,K))_{t \leq T} \) is a stochastic process that can be observed on the market.
- The behavior of \( \sigma^\text{BS}_t \) predicted by the LV model often does not correspond to the observed behavior.
iii) Dynamic behavior of the implied volatility surface: define the impl. vol \( \sigma_{t}^{BS}(T, K) \) at time \( t \leq T \) by imposing

\[
C_{t,S_{t}}(T, K) = C_{t,S_{t}}^{BS}(T, K; \sigma_{t}^{BS}(T, K))
\]

- for every \( (T, K) \), \( (\sigma_{t}^{BS}(T, K))_{t \leq T} \) is a stochastic process that can be observed on the market.

- The behavior of \( \sigma_{t}^{BS} \) predicted by the LV model often does not correspond to the observed behavior.

Practitioners often prefer to use a *stochastic volatility* model in order to hedge options:

\[
\begin{align*}
    dS_{t} &= (r - d)S_{t}dt + \sqrt{v_{t}}S_{t}dW_{t} \\
    dv_{t} &= \text{some SDE}
\end{align*}
\]
iii) Dynamic behavior of the implied volatility surface: define the implied volatility \( \sigma_{t}^{BS}(T,K) \) at time \( t \leq T \) by imposing

\[
C_{t,S_{t}}(T,K) = C_{t,S_{t}}^{BS}(T,K;\sigma_{t}^{BS}(T,K))
\]

- for every \((T,K)\), \((\sigma_{t}^{BS}(T,K))_{t \leq T}\) is a stochastic process that can be observed on the market.

- The behavior of \( \sigma_{t}^{BS} \) predicted by the LV model often does not correspond to the observed behavior.

- Practitioners often prefer to use a stochastic volatility model in order to hedge options:

\[
\begin{align*}
    dS_{t} &= (r - d)S_{t}dt + \sqrt{v_{t}}S_{t}dW_{t} \\
    dv_{t} &= b(v_{t})dt + \gamma(v_{t})dZ_{t}
\end{align*}
\]
iii) Dynamic behavior of the implied volatility surface: define the impl. vol $\sigma_{t}^{BS}(T,K)$ at time $t \leq T$ by imposing

$$C_{t,S_{t}}(T,K) = C_{t,S_{t}}^{BS}(T,K;\sigma_{t}^{BS}(T,K))$$

- for every $(T,K), (\sigma_{t}^{BS}(T,K))_{t \leq T}$ is a stochastic process that can be observed on the market.
- The behavior of $\sigma_{t}^{BS}$ predicted by the LV model often does not correspond to the observed behavior.

- Practitioners oftener prefer to use a stochastic volatility model in order to hedge options:

$$dS_{t} = (r - d)S_{t}dt + \sqrt{\nu_{t}}S_{t}dW_{t}$$
$$dv_{t} = b(\nu_{t})dt + \gamma(\nu_{t})dZ_{t}$$

where $(Z_{t})_{t \geq 0}$ is another Brownian motion (in general, not independent of $W_{t}$).
As in the Black-Scholes setting, the LV model

\[ dS_t = (r - d)S_t dt + \sigma(t, S_t)S_t dW_t \]

\[ S_t^0 = e^{rt} \]

is also complete in the sense: a portfolio \( V_t = \delta_t S_t + \delta_t^0 S_t^0 \) is sufficient to perfectly hedge an option \( g(S_T) \).
Retour sur la vol. locale et la formule de Dupire

- As in the Black-Scholes setting, the LV model

\[ dS_t = (r - d)S_t \, dt + \sigma(t, S_t)S_t \, dW_t \]
\[ S^0_t = e^{rt} \]

is also complete in the sense: a portfolio \( V_t = \delta_t S_t + \delta^0 S^0_t \) is sufficient to perfectly hedge an option \( g(S_T) \).

- This is no more the case with stochastic volatility
Retour sur la vol. locale et la formule de Dupire

- As in the Black-Scholes setting, the LV model

\[
dS_t = (r - d)S_t dt + \sigma(t, S_t)S_t dW_t
\]

\[
S^0_t = e^{rt}
\]

is also complete in the sense: a portfolio \( V_t = \delta_t S_t + \delta^0_t S^0_t \) is sufficient to perfectly hedge an option \( g(S_T) \).

- This is no more the case with stochastic volatility

\( \leadsto \) Heuristically, we cannot eliminate the risk associated to the additional random component \( Z_t \) contained in the volatility of \( S \) only using the value \( S_t \).

\( \leadsto \) This is what is called “volatility risk”.
Retour sur la vol. locale et la formule de Dupire

As in the Black-Scholes setting, the LV model

\[
\begin{align*}
    dS_t &= (r - d)S_t dt + \sigma(t, S_t)S_t dW_t \\
    S^0_t &= e^{rt}
\end{align*}
\]

is also complete \(\leadsto\) in the sense: a portfolio \(V_t = \delta_t S_t + \delta^0_t S^0_t\) is sufficient to perfectly hedge an option \(g(S_T)\).

This is no more the case with stochastic volatility

\(\leadsto\) Heuristically, we cannot eliminate the risk associated to the additional random component \(Z_t\) contained in the volatility of \(S\) only using the value \(S_t\).

\(\leadsto\) This is what is called “volatility risk”.

In order to hedge this additional risk, other derivatives have been introduced. The most important is the Variance Swap.