Sorbonne Université
Ecole Polytechnique

Master Probabilités et Finance
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Volatilité locale et stochastique, calibration de modèle

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Exercise 1. [A preliminary exercise: in view of the derivation of call and put prices with respect to strike.] Let X be an integrable random variable with values in $\mathbb{R}_+ = [0, \infty)$.

1. Show that, for every $a \geq 0$,

$$\mathbb{E}[(a - X)^+] = \int_0^a \mathbb{P}(X \leq y) dy, \qquad \mathbb{E}[(X - a)^+] = \int_a^\infty \mathbb{P}(X > y) dy.$$

2. Deduce that the right and left derivatives ∂_a^+ and ∂_a^- of the expectations above exist for all $a > 0$ and satisfy

$$\begin{aligned} &\partial_a^+ \mathbb{E}[(a - X)^+] = \mathbb{P}(X \leq a), \qquad &&\partial_a^- \mathbb{E}[(a - X)^+] = \mathbb{P}(X < a) \\ &\partial_a^+ \mathbb{E}[(X - a)^+] = -\mathbb{P}(X > a), \qquad &&\partial_a^- \mathbb{E}[(a - X)^+] = -\mathbb{P}(X \geq a) \end{aligned}$$

   Conclude that, if $x \mapsto \mathbb{P}(X \leq x)$ is continuous at the point a, then $\frac{d}{da}\mathbb{E}[(a - X)^+] = \mathbb{P}(X \leq a)$ and $\frac{d}{da}\mathbb{E}[(X - a)^+] = -\mathbb{P}(X > a)$.

Exercise 2. [Another proof of the Markovian projection theorem (or Gyöngy's theorem) for stochastic volatility.] We consider a process S_t that satisfies

$$S_t = S_0 + \int_0^t b\, S_u du + \int_0^t \eta_u S_u dW_u, \qquad t \geq 0, \tag{1}$$

where $(W_t)_{t \geq 0}$ is a Brownian motion on a probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$, $b \in \mathbb{R}$ is a constant, and $(\eta_t)_{t \geq 0}$ is a positive and bounded process (in the sense: there exists a constant $\overline{\eta} > 0$ such that $0 \leq \eta_t \leq \overline{\eta}$, $\forall t \geq 0$, a.s.) adapted to $\mathcal{F}_t$.

1. Write the solution of (1) as $S_t = S_0 e^{Y_t}$, where Y_t is stochastic process that you will write down explicitly.

2. Show that $\forall p \geq 1$, $\forall t \geq 0$, $\mathbb{E}\big[S_t^p\big] \leq S_0^p \exp(Ct)$, where C is a constant depending on b, $\overline{\eta}$ and p.

We assume that there exists a continuous function $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\mathbb{E}\big[\eta_t^2 \big| S_t\big] = \sigma(t, S_t)^2 \qquad \forall t > 0, \tag{2}$$

and moreover, σ satisfies the hypothesis (H) given in the lectures. We consider a function $v \in C^{1,2}([0, T) \times \mathbb{R}_+, \mathbb{R})$ satisfying

$$\begin{cases} \partial_t v(t, x) + b\, x\, \partial_x v(t, x) + \dfrac{1}{2}\sigma(t, x)^2 x^2 \partial_{xx} v(t, x) - b\, v(t, x) = 0 \qquad (t, x) \in [0, T) \times [0, \infty) \\ v(T, x) = f(x) \qquad x \in [0, \infty)\,, \end{cases}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function with polynomial growth: $\exists\, C, q > 0$ s.t. $\forall x \geq 0$, $f(x) \leq C(1 + x^q)$.
We assume that $\partial_x v$ and $\partial_{xx} v$ also have polynomial growth: $|\partial_x v(t, x)| + |\partial_{xx} v(t, x)| \leq C_1(1 + x^{q_1})$, $\forall\, 0 \leq t \leq T$, $\forall x \geq 0$, for some positive constants C_1 and q_1.

Now consider the process V_t defined by $V_0 = v(0, S_0)$ and

$$dV_t = (b\, V_t - b\, \delta_t S_t) dt + \delta_t dS_t \qquad t \leq T,$$

where $\delta_t = \partial_x v(t, S_t)$.

3. Is the process $(e^{-bt}V_t)_{0\le t\le T}$ a martingale?

4. Compute $dv(t,S_t)$, then $d\left(e^{-bt}(V_t - v(t,S_t))\right)$.

Let X be such that

$$X_t = S_0 + \int_0^t b\,X_u du + \int_0^t \sigma(u,X_u)X_u dW_u, \qquad t\ge 0$$

5. Show that $\mathbb{E}[V_T] = \mathbb{E}[f(S_T)]$, then that $\mathbb{E}[f(X_T)] = \mathbb{E}[f(S_T)]$. Conclude that X_T and S_T have the same law , for every T.

Exercise 3. Consider a market containing a tradable asset S, a constant risk-free rate r and a constant repo rate q. Let $C(T,K)$ be a surface of call prices on the asset S, parametrized by maturity T and strike price K. It is possible (and classical) to show that, if the market is free of arbitrage at time $t=0$, the function $C(\cdot)$ satisfies the conditions

(0) $\left(S_0e^{-qT} - Ke^{-rT}\right)^+ \le C(T,K) \le S_0e^{-qT}$, for every $T, K \ge 0$.

(i) The function $K \mapsto C(T,K)$ is convex, for every $T\ge 0$.

(ii) *A condition about the dependence of C with respect to T.*

(iii) $C(T,K)\to 0$ as $K\to\infty$, for every $T\ge 0$.

1. Give the missing condition (ii).

The goal of this exercise is to show the following result, that holds for fixed maturities T

> *Proposition*: Let $K\mapsto C(K)$ be a function that satisfies conditions (0), (i) and (iii), for some fixed $T>0$. Then, there exist a probability space $(\Omega,\mathcal{F},\mathbb{P})$ supporting a positive random variable X_T such that $C(K) = \mathbb{E}[e^{-rT}(X_T-K)^+]$, for all $K\ge 0$.

Let $C(\cdot)$ be as in the Proposition above. We set

$$c(K) := e^{qT}C\left(Ke^{(r-q)T}\right) \qquad \text{for all } K\ge 0.$$

Note that the function c will satisfy

$$(S_0-K)^+ \le c(K) \le S_0; \qquad \text{c is convex;} \qquad c(K)\to 0 \ \text{ as } K\to\infty.$$

Recall that the following properties follow from the convexity of c: the right derivative $\partial_K^+ c(K)$ exists for all $K\ge 0$, and $\partial_K^+ c$ is a right-continuous and increasing function.[1]

2. Show that $\partial_K^+ c(0) \ge -1$.

3. Show that $\partial_K^+ c(K)\to 0$ as $K\to\infty$.

4. Define the function $F(K) = \begin{cases} 1+\partial_K^+ c(K) & \text{if } K\ge 0 \\ 0 & \text{if } K<0. \end{cases}$ Show that F is a cumulative distribution function on $\mathbb{R}$.

5. Let γ be a probability measure on $\mathbb{R}$ such that $F_\gamma(K) = \gamma((-\infty,K]) = F(K)$ for all K. Show that $\int_{\mathbb{R}}(y-K)^+\gamma(dy) = c(K)$ for all K.

6. Conclude on the proof of the Proposition above.

[1] Actually, a convex function is locally Lipschitz, so that the derivative $c'(K)$ exists for almost every K.