Sorbonne Université
Ecole Polytechnique

Master Probabilités et Finance
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Volatilité locale et stochastique, calibration de modèle

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Exercise 1. [A preliminary exercise: in view of the derivation of call and put prices with respect to strike.] Let X be an integrable random variable with values in $\mathbb{R}_+ = [0, \infty)$.

1. Show that, for every $a \geq 0$,

$$\mathbb{E}[(a - X)^+] = \int_0^a \mathbb{P}(X \leq y)dy, \qquad \mathbb{E}[(X - a)^+] = \int_a^\infty \mathbb{P}(X > y)dy.$$

Answer. It follows from Fubini's theorem that

$$\mathbb{E}[(a - X)^+] = \mathbb{E}[(a - X)1_{X \leq a}] = \mathbb{E}\left[\int_0^a 1_{X \leq x} dx \, 1_{X \leq a}\right] = \int_0^a \mathbb{E}[1_{X \leq x}]dx = \int_0^a \mathbb{P}(X \leq x)dx.$$

The same type of argument yields $\mathbb{E}[(X - a)^+] = \int_a^\infty \mathbb{P}(X > x)dx$. □

2. Deduce that the right and left derivatives ∂_a^+ and ∂_a^- of the expectations above exist for all $a > 0$ and satisfy

$$\begin{aligned} \partial_a^+ \mathbb{E}[(a - X)^+] &= \mathbb{P}(X \leq a), \qquad & \partial_a^- \mathbb{E}[(a - X)^+] &= \mathbb{P}(X < a) \\ \partial_a^+ \mathbb{E}[(X - a)^+] &= -\mathbb{P}(X > a), \qquad & \partial_a^- \mathbb{E}[(a - X)^+] &= -\mathbb{P}(X \geq a) \end{aligned}$$

Conclude that, if $x \mapsto \mathbb{P}(X \leq x)$ is continuous at the point a, then $\frac{d}{da}\mathbb{E}[(a - X)^+] = \mathbb{P}(X \leq a)$ and $\frac{d}{da}\mathbb{E}[(X - a)^+] = -\mathbb{P}(X > a)$.

Exercise 2. [Another proof of the Markovian projection theorem (or Gyöngy's theorem) for stochastic volatility.] We consider a process S_t that satisfies

$$S_t = S_0 + \int_0^t b \, S_u du + \int_0^t \eta_u S_u dW_u, \qquad t \geq 0, \tag{1}$$

where $(W_t)_{t \geq 0}$ is a Brownian motion on a probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$, $b \in \mathbb{R}$ is a constant, and $(\eta_t)_{t \geq 0}$ is a positive and bounded process (in the sense: there exists a constant $\overline{\eta} > 0$ such that $0 \leq \eta_t \leq \overline{\eta}$, $\forall t \geq 0$, a.s.) adapted to $\mathcal{F}_t$.

1. Write the solution of (1) as $S_t = S_0 e^{Y_t}$, where Y_t is stochastic process that you will write down explicitly.

Answer. Let $Y_t = bt - \frac{1}{2}\int_0^t \eta_u^2 du + \int_0^t \eta_u dW_u$. An application of Itô's formula allows to show that the process $S_t = S_0 e^{Y_t}$ is a solution of (1). This is also the unique solution: why? □

2. Show that $\forall p \geq 1$, $\forall t \geq 0$, $\mathbb{E}\big[S_t^p\big] \leq S_0^p \exp(Ct)$, where C is a constant depending on b, $\overline{\eta}$ and p.

Answer. Using the explicit expression for S_t and the boundedness of the process η_t, we get

$$\begin{aligned}
S_t^p &= S_0^p \exp\left(pbt - \frac{p}{2}\int_0^t \eta_u^2 du + p\int_0^t \eta_u dW_u\right) \\
&= S_0^p \exp\left(pbt + \frac{1}{2}p(p-1)\int_0^t \eta_u^2 du + p\int_0^t \eta_u dW_u - \frac{p^2}{2}\int_0^t \eta_u^2 du\right) \\
&\le S_0^p \exp(Ct)\exp\left(p\int_0^t \eta_u dW_u - \frac{p^2}{2}\int_0^t \eta_u^2 du\right)
\end{aligned}$$

where $C = pb + \frac{1}{2}p(p-1)\overline{\eta}^2$. Since η is bounded, the process $Z_t = \exp\left(p\int_0^t \eta_u dW_u - \frac{p^2}{2}\int_0^t \eta_u^2 du\right)$ is a martingale (this fact is a consequence of Novikov's condition, but it can also be proven directly using standard moment controls for SDEs: how?). In particular, $\mathbb{E}[Z_t] = Z_0 = 1$. We conclude that $\mathbb{E}[S_t^p] \le S_0^p \exp(Ct)$. □

We assume that there exists a continuous function $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\mathbb{E}\left[\eta_t^2 \middle| S_t\right] = \sigma(t, S_t)^2 \qquad \forall t > 0, \tag{2}$$

and moreover, σ satisfies the hypothesis (H) given in the lectures. We consider a function $v \in C^{1,2}([0,T) \times \mathbb{R}_+, \mathbb{R})$ satisfying

$$\begin{cases} \partial_t v(t,x) + b\,x\,\partial_x v(t,x) + \dfrac{1}{2}\sigma(t,x)^2 x^2 \partial_{xx} v(t,x) - b\,v(t,x) = 0 & (t,x) \in [0,T) \times [0,\infty) \\ v(T,x) = f(x) \qquad x \in [0,\infty), \end{cases}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function with polynomial growth: $\exists\, C, q > 0$ s.t. $\forall x \ge 0$, $f(x) \le C(1 + x^q)$.
We assume that $\partial_x v$ and $\partial_{xx} v$ also have polynomial growth: $|\partial_x v(t,x)| + |\partial_{xx} v(t,x)| \le C_1(1 + x^{q_1})$, $\forall\, 0 \le t \le T$, $\forall x \ge 0$, for some positive constants C_1 and q_1.

Now consider the process V_t defined by $V_0 = v(0, S_0)$ and

$$dV_t = (b\,V_t - b\,\delta_t S_t)dt + \delta_t dS_t \qquad t \le T,$$

where $\delta_t = \partial_x v(t, S_t)$.

3. Is the process $(e^{-bt}V_t)_{0 \le t \le T}$ a martingale?

   Answer. Using the expression of dV_t,

   $$\begin{aligned}
   d(e^{-bt}V_t) = e^{-bt}(-b\delta_t S_t)dt + e^{-bt}\delta_t dS_t &= e^{-bt}(b\delta_t S_t - b\delta_t S_t)dt + e^{-bt}\delta_t\,\eta_t S_t dW_t \\
   &= e^{-rt}\delta_t\,\eta_t S_t dW_t,
   \end{aligned}$$

   Therefore,

   $$V_t = V_0 + \int_0^t e^{-bu}\partial_x v(u, S_u)\eta_u S_u dW_u$$

   is a local martingale, and here it is a martingale because of: the boundedness of the process η, the assumption of polynomial growth for $\partial_x v$, and the control of moments of S obtained in question 2. □

4. Compute $dv(t, S_t)$, then $d\left(e^{-bt}(V_t - v(t, S_t))\right)$.

   Answer. dV_t was given above. We have

   $$dv(t, S_t) = \left[\partial_t v(t, S_t) + \frac{1}{2}\eta_t^2 S_t^2 \partial_{xx} v(t, S_t)\right]dt + \partial_x v(t, S_t)dS_t$$

   therefore

   $$\begin{aligned}
   dV_t - dv(t, S_t) &= \left[b\,V_t - b\,S_t\partial_x v(t, S_t) - \partial_t v(t, S_t) - \frac{1}{2}\eta_t^2 S_t^2 \partial_{xx} v(t, S_t)\right]dt \\
   &= \left[b\,V_t - b\,v(t, S_t) + \frac{1}{2}S_t^2\left(\sigma(t, S_t)^2 - \eta_t^2\right)\partial_{xx} v(t, S_t)\right]dt.
   \end{aligned}$$

where we have used the PDE satisfied by v in the second identity. By Itô's formula,

$$d\left(e^{-bt}(V_t - v(t,S_t))\right) = e^{-bt}\frac{1}{2}S_t^2\partial_{xx}v(t,S_t)\Big(\sigma(t,S_t)^2 - \eta_t^2\Big)dt$$

so that , for all $t \in [0,T]$

$$e^{-bt}(V_t - v(t,S_t)) = \frac{1}{2}\int_0^t e^{-bu}S_u^2\partial_{xx}v(u,S_u)\Big(\sigma(u,S_u)^2 - \eta_u^2\Big)du$$

□

Let X be such that

$$X_t = S_0 + \int_0^t b\,X_u du + \int_0^t \sigma(u,X_u)X_u dW_u, \qquad t \geq 0$$

5. Show that $\mathbb{E}[V_T] = \mathbb{E}[f(S_T)]$, then that $\mathbb{E}[f(X_T)] = \mathbb{E}[f(S_T)]$. Conclude that X_T and S_T have the same law , for every T.

   Answer. Taking expectations in the previous questions, we obtain

$$\begin{aligned}
\mathbb{E}[e^{-bt}(V_t - v(t,S_t))] &= \frac{1}{2}\mathbb{E}\Big[\int_0^t e^{-bu}S_u^2\partial_{xx}v(u,S_u)\Big(\sigma(u,S_u)^2 - \eta_u^2\Big)du\Big] \\
&= \frac{1}{2}\int_0^t e^{-bu}\mathbb{E}\Big[S_u^2\partial_{xx}v(u,S_u)\Big(\sigma(u,S_u)^2 - \eta_u^2\Big)\Big]du \\
&= \frac{1}{2}\int_0^t e^{-bu}\mathbb{E}\Big[S_u^2\partial_{xx}v(u,S_u)\mathbb{E}\Big[\sigma(u,S_u)^2 - \eta_u^2\Big|S_u\Big]\Big]du = 0
\end{aligned}$$

   where we have used the polynomial growth estimate on $\partial_{xx}v(u,S_u)$ in order to apply Fubini's Theorem in the second step, and the fact that $\mathbb{E}[\eta_u^2|S_u] = \sigma(u,S_u)^2$ for all u in the last step. When $t = T$, we obtain

$$\mathbb{E}[e^{-bT}V_T] = \mathbb{E}[e^{-bT}v(T,S_T)] = \mathbb{E}[e^{-bT}f(S_T)]$$

   Now, $\mathbb{E}[e^{-bT}V_T] = V_0 = v(0,S_0)$ by question 3, and by the Feynman-Kac theorem $v(0,S_0) = \mathbb{E}[e^{-bT}f(X_T)]$, where the process X satisfies the equation above. Putting things together, we have obtained

$$\mathbb{E}[f(X_T)] = \mathbb{E}[f(S_T)]$$

   Since this identity holds for any continuous and bounded function f, we conclude on the required identity in law. □

Exercise 3. Consider a market containing a tradable asset S, a constant risk-free rate r and a constant repo rate q. Let $C(T,K)$ be a surface of call prices on the asset S, parametrized by maturity T and strike price K. It is possible (and classical) to show that, if the market is free of arbitrage at time $t = 0$, the function $C(\cdot)$ satisfies the conditions

(0) $\left(S_0e^{-qT} - Ke^{-rT}\right)^+ \leq C(T,K) \leq S_0e^{-qT}$, for every $T, K \geq 0$.

(i) The function $K \mapsto C(T,K)$ is convex, for every $T \geq 0$.

(ii) *A condition about the dependence of C with respect to T.*

(iii) $C(T,K) \to 0$ as $K \to \infty$, for every $T \geq 0$.

1. Give the missing condition (ii).

   Answer. See lecture 4 on Feb 12, 2020. □

The goal of this exercise is to show the following result, that holds for fixed maturities T

> *Proposition*: Let $K \mapsto C(K)$ be a function that satisfies conditions (0), (i) and (iii), for some fixed $T > 0$. Then, there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a positive random variable X_T such that $C(K) = \mathbb{E}[e^{-rT}(X_T - K)^+]$, for all $K \geq 0$.

Let $C(\cdot)$ be as in the Proposition above. We set

$$c(K) := e^{qT} C\big(K e^{(r-q)T}\big) \qquad \text{for all } K \geq 0.$$

Note that the function c will satisfy

$$(S_0 - K)^+ \leq c(K) \leq S_0; \quad \text{c is convex;} \quad c(K) \to 0 \text{ as } K \to \infty.$$

Recall that the following properties follow from the convexity of c: the right derivative $\partial_K^+ c(K)$ exists for all $K \geq 0$, and $\partial_K^+ c$ is a right-continuous and increasing function.[1]

2. Show that $\partial_K^+ c(0) \geq -1$.

   Answer. From the bounds on c, we have $c(0) = S_0$. Therefore, $\frac{c(K)-c(0)}{K} \geq \frac{(S_0-K)^+ - S_0}{K} = -1$ for all $K < S_0$. In particular then, $\partial_K^+ c(0) \geq -1$. □

3. Show that $\partial_K^+ c(K) \to 0$ as $K \to \infty$.

   Answer. First of all, note that $\lim_{K\to\infty} \partial_K^+ c(K)$ exists because the fct is monotone. Denote l this limit.

   - We have $c(K) \geq c(K_0) + \partial_K^+ c(K_0)(K - K_0) \geq \partial_K^+ c(K_0)(K - K_0)$. Assume $l < 0$. Then, $\partial_K^+ c(K_0)(K - K_0) \to +\infty$ as $K_0 \to \infty$, which is a contradiction.
   - Assume $l > 0$. Then, there exists $\overline{K}$ such that $\partial_K^+ c(K) > l/2$ for all $K \geq \overline{K}$. Using the fact that c is locally Lipschitz, hence absolutely continuous (therefore: c is the integral of its derivative), we can write

   $$c(K) = c(\overline{K}) + \int_{\overline{K}}^K \partial_K^+ c(z)dz \geq c(\overline{K}) + \int_{\overline{K}}^K \frac{l}{2} dz = c(\overline{K}) + \frac{l}{2}(K - \overline{K}) \to \infty \quad \text{as } K \to \infty,$$

   which is a contradiction. Overall, we have shown $l = 0$.

   □

4. Define the function $F(K) = \begin{cases} 1 + \partial_K^+ c(K) & \text{if } K \geq 0 \\ 0 & \text{if } K < 0. \end{cases}$ Show that F is a cumulative distribution function on $\mathbb{R}$.

   Answer. By construction, F is right continuous, identically zero on $(-\infty, 0)$, increasing on $[0, \infty)$. By the previous question, we have $\lim_{K\to 0} F(K) = 1$. Finally, we just have to note that $F(0) = 1 + \partial_K^+ c(0) \geq 0$ by question 1, so that F is increasing on the whole $\mathbb{R}$. Overall, F is a cdf on $\mathbb{R}$. □

5. Let γ be a probability measure on $\mathbb{R}$ such that $F_\gamma(K) = \gamma((-\infty, K]) = F(K)$ for all K. Show that $\int_{\mathbb{R}} (y - K)^+ \gamma(dy) = c(K)$ for all K.

   Answer. Using the result in Exercise 1, we have

   $$\begin{aligned} \int_{\mathbb{R}} (y - K)^+ \gamma(dy) &= \int_K^\infty (1 - F_\gamma(x))dx \\ &= \int_K^\infty (-\partial_K^+ c(x))dx = \lim_{A\to\infty} \int_K^A (-\partial_K^+ c(x))dx = \lim_{A\to\infty} (c(K) - c(A)) = c(K), \end{aligned}$$

[1] Actually, a convex function is locally Lipschitz, so that the derivative $c'(K)$ exists for almost every K.

by condition $\bullet$. $\square$

6. Conclude on the proof of the Proposition above.

   Answer. Going back to the function C, we have shown that

$$C(K) = e^{-dT} c\big(Ke^{-(r-d)T}\big) = e^{-dT} \int_{\mathbb{R}} (y - Ke^{-(r-d)T})^{+} \gamma(dy)$$
$$= \int_{\mathbb{R}} e^{-rT} (e^{(r-d)T} y - K)^{+} \gamma(dy) = \mathbb{E}\big[e^{-rT} \big(e^{(r-d)T} X_T - K\big)^{+}\big],$$

   where X_T is a random variable defined on a (any) probability space $(\Omega, \mathcal{F}, \mathbb{P})$, distributed according to the law γ. $\square$