Exercise 1. Let $X$ be a positive integrable random variable. Denote $F(x) = P(X \leq x)$ the cumulative distribution function (cdf) of $X$, and $\overline{F}(x) = 1 - F(x)$. Show that, for every $a \geq 0$,

$$
E[(a - X)^+] = \int_0^a F(x)dx,
$$

$$
E[(X - a)^+] = \int_a^{\infty} \overline{F}(x)dx.
$$

If $F$ is continuous at the point $a$, deduce that

$$
\frac{d}{da} E[(a - X)^+] = F(a),
$$

$$
\frac{d}{da} E[(X - a)^+] = -\overline{F}(a).
$$

Exercise 2. We consider a market with risky asset $S_t$ and a riskless asset $S^0_t = e^{rt}$ with interest rate $r \geq 0$. We consider the following local volatility model for $S_t$, given on a space $(\Omega, \mathcal{F})$ and directly under its risk-neutral probability measure $\mathbb{P}$:

$$
S_t = S_0 + \int_0^t rS_u du + \int_0^t \sigma(u, S_u)S_u dW_u, \quad t \geq 0,
$$

where $(W_t)_{t \geq 0}$ is a Brownian motion under $\mathbb{P}$. We assume that the function $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following conditions:

- $(t, x) \mapsto \sigma(t, x)$ is continuous and such that $\sigma(t, x) > 0, \forall t, x \geq 0$.
- $\exists C > 0$ such that $\forall t, x, y \geq 0, |x\sigma(t, x) - y\sigma(t, y)| \leq C|x - y|$.

Consider a continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\exists C, q > 0$ s.t. $\forall x \geq 0, f(x) \leq C(1 + x^q)$, and set

$$
v(t, x) = e^{-r(T-t)}E[f(S_{T}^{x,l})], \quad 0 \leq t \leq T, \ x \geq 0,
$$

where $S_{T}^{x,l}$ denotes the solution $(S_{u}^{l,x})_{u \geq t}$ of the SDE (1) that starts from the point $x$ at time $t$, $S_{t}^{x,x} = x$.

1. Why is the expectation defining $v(t, x)$ finite?

2. We admit that $v(t, x)$ is $C^{1,2}([0, T] \times \mathbb{R}_+, \mathbb{R})$. Recall that $v(t, x)$ satisfies a certain PDE, that you will write down explicitly.

We now consider an asset $\tilde{S}_t$ that follows the dynamics

$$
\tilde{S}_t = S_0 + \int_0^t r\tilde{S}_u du + \int_0^t \eta_u \tilde{S}_u dW_u, \quad t \geq 0,
$$

where $(\eta_t)_{t \geq 0}$ is an $\mathcal{F}_t$-adapted positive and bounded process (that is, there exists a constant $\bar{\eta} > 0$ s.t. $0 \leq \eta_t \leq \bar{\eta}, \forall t \geq 0$, a.s.) such that

$$
E[\eta_t^2 | \mathcal{F}_t] = \sigma(t, \tilde{S}_t)^2, \quad \forall t \geq 0, \ a.s.
$$
3. We consider a portfolio $V_t$ given by $\delta^0_t$ riskless assets and $\delta_t$ assets $S_t$. Denote $V_t = \delta^0_t S_t + \delta_t \tilde{S}_t$. Give the self-financing condition for this portfolio.

4. Show that $\forall q \geq 1$, $\forall t \geq 0$, $E[\tilde{S}_t^q] \leq S_0^q \exp(Ct)$, where $C$ is a constant depending on $r$, $\eta$ and $q$. Hint: one can start by showing that $\tilde{S}_t^q \leq S_0^q \exp(Ct) \exp(q \int_0^t \eta_u dW_u - \frac{1}{2} \int_0^t \eta_u^2 du)$.

From now on, we will consider the portfolio with initial value $V_0 = v(0, S_0)$ and with number of risky assets given by

$$\delta_t = \partial_x v(t, \tilde{S}_t).$$

We assume that $|\partial_x v(t, x)| + |\partial_{xx} v(t, x)| \leq C(1 + x^q)$, $\forall 0 \leq t \leq T$, $\forall x \geq 0$.

5. Show that $E[V_T] = E[f(S_T)]$. (NB: on the right hand side, we consider the process $S$, and not $\tilde{S}$.)

6. Compute $dV_t$ and $dv(t, \tilde{S}_t)$, then $d\left(e^{-rt}(V_t - v(t, \tilde{S}_t))\right)$. Show that $\forall t \in [0, T]$, $E[V_t] = E[v(t, \tilde{S}_t)]$. Interpret this result in terms of properties of the hedging error for the portfolio $V$.

7. Deduce that $E[f(S_T)] = E[f(\tilde{S}_T)]$. Conclude that for every $T \geq 0$, $S_T$ and $\tilde{S}_T$ have the same law.

8. Reobtain the result in question 7 using a theorem from the lectures.

**Exercise 3.** Recall that an arbitrage-free Call price surface $C(T, K)$ at time $t = 0$ satisfies the following conditions:

i) $(S_0 e^{-dT} - Ke^{-rT})^+ \leq C(T, K) \leq S_0 e^{-dT}$, for all $T, K \geq 0$;

ii) the function $K \mapsto C(T, K)$ is convex, for every $T \geq 0$;

where $r, d \geq 0$ are the continuous interest and dividend rates. Therefore, i) and ii) are **necessary** conditions for a function $(T, K) \mapsto C(T, K)$ to be an arbitrage-free Call price surface.

The goal of this exercise is to show that, for a fixed maturity $T$, the conditions i) and ii) above, together with the additional property

- $C(T, K) \to 0$ as $K \to \infty$,

are **sufficient** conditions for $C(T, \cdot)$ to be an arbitrage-free Call price for the maturity $T$, in the sense of the following definition:

**Definition:** a function $K \mapsto c(K)$ is an arbitrage-free Call price for some fixed maturity if there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a positive random variable $S_T$ such that $c(K) = \mathbb{E}[e^{-rT}(S_T - K)^+]$, for all $K \geq 0$.

Given a function $C(\cdot, \cdot)$ satisfying i), ii) and $\bullet$, and given $T \geq 0$, we set

$$c(K) := e^{dT} C(T, Ke^{(r-d)T}) \quad \text{for all } K \geq 0.$$ 

Note that the function $c$ will satisfy

$$(S_0 - K)^+ \leq c(K) \leq S_0; \quad c \text{ is convex; } c(K) \to 0 \text{ as } K \to \infty.$$ 

We recall that the following properties follow from the convexity of $c$, the right derivative $\partial^+_{K} c(K)$ exists for all $K \geq 0$, and $\partial^+_{K} c$ is a right-continuous and increasing function.$^1$

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$^1$Actually, a convex function is locally Lipschitz, so that the derivative $c'(K)$ exists for a.a. $K$. 

1. Show that $\partial_K^+ c(0) \geq -1$.

2. Show that $\partial_K^+ c(K) \to 0$ as $K \to \infty$.

3. Define the function $F(K) = \begin{cases} 1 + \partial_K^+ c(K) & \text{if } K \geq 0 \\ 0 & \text{if } K < 0 \end{cases}$. Show that $F$ is a cumulative distribution function on $\mathbb{R}$.

4. Let $\gamma$ be a probability measure on $\mathbb{R}$ such that $F_\gamma(K) = \gamma((-\infty, K]) = F(K)$ for all $K$. Show that $\int_\mathbb{R} (y - K)^+ \gamma(dy) = c(K)$ for all $K$.

**Conclusion:** we have shown that, if the function $K \mapsto C(T, K)$ satisfies i), ii) and •, then

$$C(T, K) = e^{-dT} c(K e^{-(r-d)T}) = e^{-dT} \int_\mathbb{R} (y - K e^{-(r-d)T})^+ \gamma(dy)$$

$$= \int_\mathbb{R} e^{-rT} (e^{(r-d)T} y - K)^+ \gamma(dy) = \mathbb{E}[e^{-rT} (e^{(r-d)T} X_T - K)^+]$$

where $X_T$ is a random variable defined on a (any) probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with law $\gamma$. 