**Model Calibration**

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**Exercice 1.** Let $X$ be a positive integrable random variable. Denote $F(x) = \mathbb{P}(X \leq x)$ the cumulative distribution function of $X$, and $\bar{F}(x) = 1 - F(x) = \mathbb{P}(X > x)$. Show that, for every $a \geq 0$

$$
\mathbb{E}[(a - X)^+] = \int_0^a F(x)dx,
\mathbb{E}[(X - a)^+] = \int_a^\infty \bar{F}(x)dx.
$$

If $F$ is continuous at the point $a$, deduce that $\frac{d}{da}\mathbb{E}[(a - X)^+] = F(a)$, $\frac{d}{da}\mathbb{E}[(X - a)^+] = -\bar{F}(a)$.

**Exercice 2.** Put-Call duality in the Black–Scholes model.
Consider $x, r, d, \sigma \geq 0$. Show the Put-Call duality (not to be confused with Put-Call parity) in the Black–Scholes model: for every $T, K > 0$

$$
\mathbb{E}[e^{-rT} (S_T - K)^+] = \mathbb{E}[e^{-dT} (x - \tilde{S}_T)^+] 
$$

where $S_T = x \exp\left((r - d)T - \frac{1}{2} \sigma^2 T + \sigma W_T\right)$ and $\tilde{S}_T = K \exp\left((d - r)T - \frac{1}{2} \sigma^2 T + \sigma W_T\right)$. Note that the roles of $K \leftrightarrow x$ and $r \leftrightarrow d$ are exchanged.

*Hint:* this is a consequence of Girsanov’s theorem.

**Exercice 3.** We consider a market with risky asset $S_t$ and a riskless asset $S^0_t = e^{rt}$ with interest rate $r \geq 0$. We place ourselves directly on a risk-neutral probability space equipped with a Brownian motion $(W_t)_{t \geq 0}$, and denote $(\mathcal{F}_t)_{t \geq 0}$ the filtration generated by $W$. We consider the following model for $S_t$:

$$
S_t = S_0 + \int_0^t rS_u du + \int_0^t \sigma(u, S_u)S_u dW_u, \quad t \geq 0
$$

where the function $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following conditions:

- $(t, x) \mapsto \sigma(t, x)$ is continuous and such that $\sigma(t, x) > 0$, $\forall t, x \geq 0$.
- $\exists C > 0$ such that $\forall t, x, y \geq 0, |x\sigma(t, x) - y\sigma(t, y)| \leq C|x - y|$.

Consider a continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\exists C, q > 0$ s.t. $\forall x \geq 0, f(x) \leq C(1 + x^q)$, and set

$$
v(t, x) = e^{-r(T-t)}\mathbb{E}[f(S^{LX}_T)], \quad 0 \leq t \leq T, x \geq 0,
$$

where $S^{LX}_T$ denotes the solution $(S^{LX}_u)_{u \geq t}$ of the SDE that starts from the point $x$ at time $t$, $S^{LX}_t = x$. 

1. Why is the expectation defining \( v(t, x) \) finite?

2. We admit that \( v(t, x) \) is \( C^{1,2}([0, T] \times \mathbb{R}_+, \mathbb{R}) \). Show that \( v(t, x) \) satisfies a certain PDE, that you will write down explicitly.

We now consider an asset \( \tilde{S}_t \) that follows the dynamics

\[
\tilde{S}_t = S_0 + \int_0^t r\tilde{S}_u du + \int_0^t \eta_u \tilde{S}_u dW_u, \quad t \geq 0
\]

where \( (\eta_t)_{t \geq 0} \) is a \( \mathcal{F}_t \)-adapted positive and bounded process (that is, \( \exists \tilde{\eta} > 0 \) s.t. \( 0 \leq \eta_t \leq \tilde{\eta} \), \( \forall t \geq 0 \), a.s.) such that

\[
\mathbb{E}[\eta_t^2 | \mathcal{F}_t] = \sigma(t, \tilde{S}_t)^2, \quad \forall t \geq 0, \text{ a.s.}
\]

3. We consider a portfolio \( V_t \) given by \( \delta^0 \) riskless assets and \( \delta_t \) assets \( \tilde{S}_t \). Denote \( V_t = \delta^0 S^0_t + \delta_t \tilde{S}_t \). Write down the self-financing condition for this portfolio.

From now on, we will consider the portfolio with initial value \( V = v(0, S_0) \) and with number of risky assets given by

\[
\delta_t = \partial_x v(t, \tilde{S}_t).
\]

We assume that \( |\partial_x v(t, x)| \leq C(1 + x^q) \), \( \forall 0 \leq t \leq T \), \( \forall x \geq 0 \).

4. Show that \( \forall q \geq 1, \forall t \geq 0, \mathbb{E}[\tilde{S}_t^q] \leq S^0_t \exp(Ct) \), where \( C \) is a constant depending on \( r, \tilde{\eta} \) and \( q \). Hint: one can start by showing that \( \tilde{S}_t^q \leq S^0_t \exp(Ct) \exp\left(q \int_0^t \eta_u dW_u - \frac{\sigma^2}{2} \int_0^t \eta_u^2 du\right) \).

5. Show that \( \mathbb{E}[V_T] = \mathbb{E}[f(S_T)] \).

6. Compute \( dV_t \) and \( dv(t, \tilde{S}_t) \), then \( d\left(e^{-rt}(V_t - v(t, \tilde{S}_t))\right) \). Show that \( \forall t \in [0, T], \mathbb{E}[V_t] = \mathbb{E}[v(t, \tilde{S}_t)] \).

7. Deduce that \( \mathbb{E}[f(S_T)] = \mathbb{E}[f(\tilde{S}_T)] \), hence that for every \( T \geq 0 \), \( S_T \) and \( \tilde{S}_T \) have the same law.

8. Obtain the result in question 7 using a theorem given in the lectures.

**Exercise 4. Explicit computation of the characteristic function in the Heston model, and explosion of moments.**

Consider the Heston model

\[
\begin{align*}
\text{d}S_t &= S_t \sqrt{\nu_t} \text{d}W_t, \quad S_0 > 0, \\
\text{d}\nu_t &= \alpha (\beta - \nu_t) \text{dt} + \xi \sqrt{\nu_t} \text{d}B_t, \quad \nu_0 > 0
\end{align*}
\]

with zero interest rate \( r \). The model parameters satisfy \( \alpha, \beta, \xi, v_0 > 0 \), and \( W_t \) and \( Z_t \) are two Brownian motions with correlation \( \rho \in [-1, 1] \). One can show that the SDE above admits a unique solution \((S_t, \nu_t)_{t \geq 0}\) such that \( \nu_t \geq 0, \forall t \geq 0 \), a.s. Set \( X_t = \log S_t \) and denote

\[
f(t, x, v) := \mathbb{E}[e^{iuX_T} | X_t = x, \nu_t = v],
\]
Hereafter we will assume a priori the function $f$ to be a suitably regular and we will try to determine its explicit form.

1. Write the dynamics of $(X_t, v_t)$ and apply Itô's formula to $f(t, X_t, v_t)$.
2. Show that $f$ satisfies a certain PDE.
3. Find an explicit solution of this PDE of the form $f(t, x, v) = e^{iux}e^{A(T-t)+vB(T-t)}$ by inferring the system of ODEs that the functions $A$ and $B$ have to solve.
4. Find the explicit form of the function $B$.
5. Let $p > 1$ and assume $\rho \neq -1$. In light of the previous points, and depending on the sign of the determinant $\Delta = (\alpha - \rho \xi p)^2 - p \xi^2(p - 1)$ and of $\gamma = \alpha - \rho \xi p$, infer the existence of a maturity $T^*$ such that

\[
\begin{cases}
\mathbb{E}[S_T^+] < \infty & \text{if } t < T^*, \\
\mathbb{E}[S_T^+] = \infty & \text{if } t \geq T^*.
\end{cases}
\]

**Exercise 5. Short-time behavior of the Black-Scholes pricing formula.**

We are interested in studying the asymptotic behavior of the Black-Scholes price of a Call option as the time-to-maturity goes to 0. In particular, we want to obtain a short-time asymptotic expansion for the price, the latter being the cornerstone for obtaining a short-time expansion of the implied volatility. Let us denote

$$C_{BS}(K, T; S, \sigma) = \mathbb{E}[(S_T - K)^+] = S N(d_1) - K N(d_2),$$

$$d_1 = \frac{\ln(S/K)}{\sigma \sqrt{T}} + \frac{1}{2} \sigma \sqrt{T}, \quad d_2 = d_1 - \sigma \sqrt{T},$$

the price of a Call option with strike $K$ and maturity $T$ under the Black-Scholes model with volatility $\sigma$ and zero interest rate. $N(x) = \int_{-\infty}^{x} e^{-y^2/2} dy \sqrt{2\pi}$ denotes the cdf of a standard normal distribution. We set $\nu(T) = \sigma \sqrt{T}$.

1. Show that $\lim_{T \to 0} d_{1,2} = \infty$ (resp. $-\infty$) if $S > K$ (resp. $S < K$), and that for all $K \neq S$

$$N(x) \sim \frac{1}{(\log(S/K))^5}, \quad \text{as } T \to 0$$

2. Using the expansion of the normal cdf

$$N(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(-\frac{1}{x} + \frac{1}{x^3} + O(x^{-5})\right), \quad \text{as } x \to -\infty$$

and

$$1 - N(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(\frac{1}{x} - \frac{1}{x^3} + O(x^{-5})\right), \quad \text{as } x \to \infty,$$

prove that the following expansion holds for every $K \neq S$:

$$C_{BS}(K, T; S, \sigma) - (S - K)^+ \sim \frac{1}{\sqrt{2\pi}} \sqrt{KS} \left(\frac{\sigma \sqrt{T}}{(\ln(K/S))^2}\right) \exp\left(-\frac{(\ln(K/S))^2}{2\sigma^2 T}\right) \left(1 + R \left(\sigma \sqrt{T}, K\right)\right),$$
where the function $R(v, K)$ satisfies the estimate

$$|R(v, K)| = O(v^2) \quad \text{as } v \to 0,$$

for all $K \neq S$. 