Model Calibration

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**Exercise 1.** Let $X$ be a positive integrable random variable. Denote $F(x) = \mathbb{P}(X \leq x)$ the cumulative distribution function of $X$, and $\bar{F}(x) = 1 - F(x) = \mathbb{P}(X > x)$. Show that, for every $a \geq 0$

$$\mathbb{E}[(a - X)^+] = \int_{0}^{a} F(x) dx,$$

$$\mathbb{E}[(X - a)^+] = \int_{a}^{\infty} \bar{F}(x) dx.$$

If $F$ is continuous at the point $a$, deduce that $\frac{d}{da} \mathbb{E}[(a - X)^+] = F(a)$, $\frac{d}{da} \mathbb{E}[(X - a)^+] = -\bar{F}(a)$.

**Exercise 2.** Put-Call duality in the Black–Scholes model.

Consider $x, r, d, \sigma \geq 0$. Show the Put-Call duality (not to be confused with Put-Call parity) in the Black-Scholes model: for every $T, K > 0$

$$\mathbb{E}[e^{-rT} (S_{T} - K)^{+}] = \mathbb{E}[e^{-dT} (x - \tilde{S}_{T})^{+}]$$

where $S_{T} = x \exp \left( (r - d)T - \frac{1}{2} \sigma^{2}T + \sigma W_{T} \right)$ and $\tilde{S}_{T} = K \exp \left( (d - r)T - \frac{1}{2} \sigma^{2}T + \sigma W_{T} \right)$. Note that the roles of $K \leftrightarrow x$ and $r \leftrightarrow d$ are exchanged.

*Hint:* this is a consequence of Girsanov’s theorem.

**Exercise 3.** We consider a market with risky asset $S_{t}$ and a riskless asset $S_{t}^{0} = e^{rt}$ with interest rate $r \geq 0$. We place ourselves directly on a risk-neutral probability space equipped with a Brownian motion $(W_{t})_{t \geq 0}$, and denote $(\mathcal{F}_{t})_{t \geq 0}$ the filtration generated by $W$. We consider the following model for $S_{t}$:

$$S_{t} = S_{0} + \int_{0}^{t} rS_{u} du + \int_{0}^{t} \sigma(u, S_{u}) dW_{u}, \quad t \geq 0$$

where the function $\sigma : \mathbb{R}_{+} \times \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ satisfies the following conditions:

- $(t, x) \mapsto \sigma(t, x)$ is continuous and such that $\sigma(t, x) > 0$, $\forall t, x \geq 0$.
- $\exists C > 0$ such that $\forall t, x, y \geq 0$, $|x \sigma(t, x) - y \sigma(t, y)| \leq C |x - y|$.

Consider a continuous function $f : \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ such that $\exists C, q > 0$ s.t. $\forall x \geq 0$, $f(x) \leq C (1 + x^{q})$, and set

$$v(t, x) = e^{-r(T-t)} \mathbb{E}[f(S_{T}^{LX})], \quad 0 \leq t \leq T, x \geq 0.$$

where $S_{T}^{LX}$ denotes the solution $(S_{u}^{LX})_{u \geq t}$ of the SDE that starts from the point $x$ at time $t$, $S_{t}^{LX} = x$. 

1. Why is the expectation defining \( v(t,x) \) finite?

2. We admit that \( v(t,x) \) is \( C^{1,2}([0,T] \times \mathbb{R}_+,R) \). Show that \( v(t,x) \) satisfies a certain PDE, that you will write down explicitly.

We now consider an asset \( \tilde{S}_t \) that follows the dynamics

\[
\tilde{S}_t = S_0 + \int_0^t r \tilde{S}_u du + \int_0^t \eta_u \tilde{S}_u dW_u, \quad t \geq 0
\]

where \( (\eta_t)_{t \geq 0} \) is an \( \mathcal{F}_t \)-adapted positive and bounded process (that is, \( \exists \bar{\eta} > 0 \) s.t. \( 0 \leq \eta_t \leq \bar{\eta}, \forall t \geq 0, \text{a.s.} \)) such that

\[
\mathbb{E}[\eta_t^2 | \mathcal{F}_t] = \sigma(t, \tilde{S}_t)^2, \quad \forall t \geq 0, \text{a.s.}
\]

3. We consider a portfolio \( V_t \) given by \( \delta^0 \) riskless assets and \( \delta \) assets \( \tilde{S}_t \). Denote \( V_t = \delta^0 S^0_t + \delta \tilde{S}_t \). Write down the self-financing condition for this portfolio.

4. Show that \( \forall \eta \geq 1, \forall t \geq 0, \mathbb{E}[\tilde{S}^1_t] \leq S^0_0 \exp(Ct) \), where \( C \) is a constant depending on \( r, \bar{\eta} \) and \( \eta \). Hint: one can start by showing that \( \tilde{S}^1_t \leq S^0_0 \exp(Ct) \exp(q \int_0^t \eta_u dW_u - \frac{\eta_t^2}{2} \int_0^t \eta_u^2 du) \).

From now on, we will consider the portfolio with initial value \( V_0 = v(0, S_0) \) and with number of risky assets given by

\[
\delta = \partial_x v(t, \tilde{S}_t).
\]

We assume that \( |\partial_x v(t,x)| \leq C(1 + x^q), \forall 0 \leq t \leq T, \forall x \geq 0. \)

5. Show that \( \mathbb{E}[V_T] = \mathbb{E}[f(S_T)] \).

6. Compute \( dV_t \) and \( dv(t, \tilde{S}_t) \), then \( d(e^{-rt}(V_t - v(t, \tilde{S}_t))) \). Show that \( \forall t \in [0,T], \mathbb{E}[V_t] = \mathbb{E}[v(t, \tilde{S}_t)] \).

7. Deduce that \( \mathbb{E}[f(S_T)] = \mathbb{E}[f(\tilde{S}_T)] \), hence that for every \( T \geq 0 \), \( S_T \) and \( \tilde{S}_T \) have the same law.

8. Obtain the result in question 7 using a theorem given in the lectures.

**Exercice 4. Explicit computation of the characteristic function in the Heston model, and explosion of moments.**

Consider the Heston model

\[
dS_t = \alpha(\beta - v_t) dt + \xi \sqrt{v_t} dW_t, \quad S_0 > 0
\]

\[
dv_t = \alpha(\beta - v_t) dt + \xi \sqrt{v_t} dB_t, \quad v_0 > 0
\]

with zero interest rate \( r \). The model parameters satisfy \( \alpha, \beta, \xi, v_0 > 0 \), and \( W_t \) and \( Z_t \) are two Brownian motions with correlation \( \rho \in [-1,1] \). One can show that the SDE above admits a unique solution \( (S_t, v_t)_{t \geq 0} \) such that \( v_t \geq 0, \forall t \geq 0, \text{a.s.} \) Set \( X_t = \log S_t \) and denote

\[
f(t,x,v) := \mathbb{E}[e^{iuX_T} | X_t = x, v_t = v],
\]
Hereafter we will assume a priori the function $f$ to be a suitably regular and we will try to determine its explicit form.

1. Write the dynamics of $(X_t, v_t)$ and apply Itô’s formula to $f(t, X_t, v_t)$.
2. Show that $f$ satisfies a certain PDE.
3. Find an explicit solution of this PDE of the form $f(t, x, v) = e^{iux}e^{A(T-t)+vB(T-t)}$ by inferring the system of ODEs that the functions $A$ and $B$ have to solve.
4. Find the explicit form of the function $B$.
5. Let $p > 1$ and assume $\rho \neq -1$. In light of the previous points, and depending on the sign of the determinant $\Delta = (\alpha - \rho \xi)^2 - p \xi^2 (p - 1)$ and of $\gamma = \alpha - \rho \xi p$, infer the existence of a maturity $T^*$ such that

$$
\begin{align*}
&\begin{cases}
\mathbb{E}[S_T^+] < \infty & \text{if } t < T^*, \\
\mathbb{E}[S_T^+] = \infty & \text{if } t \geq T^*.
\end{cases}
\end{align*}
$$

**Exercice 5. Short-time behavior of the Black-Scholes pricing formula.**

We are interested in studying the asymptotic behavior of the Black-Scholes price of a Call option as the time-to-maturity goes to 0. In particular, we want to obtain a short-time asymptotic expansion for the price, the latter being the cornerstone for obtaining a short-time expansion of the implied volatility. Let us denote

$$
C_{BS}(K, T; S, \sigma) = \mathbb{E}[(S_T - K)^+] = S N(d_1) - K N(d_2),
$$

$$
d_1 = \frac{\ln(S/K) + \sigma \sqrt{T}}{\sqrt{2}} + \frac{1}{2} \sigma \sqrt{T}, \quad d_2 = d_1 - \sigma \sqrt{T},
$$

the price of a Call option with strike $K$ and maturity $T$ under the Black-Scholes model with volatility $\sigma$ and zero interest rate. $N(x) = \int_{-\infty}^{x} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}$ denotes the cdf of a standard normal distribution. We set $v(t) = \sigma \sqrt{T}$.

1. Show that $\lim_{T \to 0} d_{1,2} = \infty$ (resp. $-\infty$) if $S > K$ (resp. $S < K$), and that for all $K \neq S$

$$
(d_{1,2})^{-5} \sim \frac{\sigma(T)^5}{(\log(S/K))^5}, \quad \text{as } T \to 0
$$

2. Using the expansion of the normal cdf

$$
N(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(-\frac{1}{x} + \frac{1}{x^3} + O(x^{-5})\right), \quad \text{as } x \to -\infty
$$

and

$$
1 - N(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(\frac{1}{x} - \frac{1}{x^3} + O(x^{-5})\right), \quad \text{as } x \to \infty,
$$

prove that the following expansion holds for every $K \neq S$:

$$
C_{BS}(K, T; S, \sigma) - (S - K)^+ \sim \frac{1}{\sqrt{2\pi}} \frac{\sqrt{KS}}{(\ln(K/S))^3} \exp\left(-\frac{(\ln(K/S))^2}{2(\sigma^2 T)}\right) \left(1 + R\left(\alpha \sqrt{T}, K\right)\right),
$$
where the function $R(v, K)$ satisfies the estimate

$$|R(v, K)| = O(v^2) \quad \text{as } v \to 0,$$

for all $K \neq S$. 