On Representation Formulas for Long Run Averaging Optimal Control Problem

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Deterministic control problem :

+ A - compact metric space ;

+ $\mathcal{A} := \{\alpha : R \to A,\ \alpha$ Borel measurable $\}$;

+ $f : R^d \times A \to R^d$ bounded, continuous,
  $f(.,a)$ Lipschitz, uniformly w.r.t. $a \in A$;

Dynamics of the control problem : For given $\alpha \in \mathcal{A}$,

$$x'(t) = f(x(t), \alpha(t)),\ \ t \in R;\ x(0) = x_0 \in R^d;$$

unique solution : $x(t, x_0, \alpha) := x(t),\ t \in R$.

+ $\ell : R^d \times A \to R$ bounded, continuous ;

Cost functionals : two ways of averaging the cost along the trajectery $x(t, x_0, \alpha),\ t \geq 0$:

+ Cesàro Mean,

+ Abel Mean ;

The Problem

Cesàro Mean : For $T > 0$, $\frac{1}{T}\int_0^T \ell(x(s,x_0,\alpha),\alpha(s))ds$;

Abel Mean : For $\lambda > 0$, $\lambda \int_0^{+\infty} e^{-\lambda s}\ell(x(s,x_0,\alpha),\alpha(s))ds$.

This leads to the definition of the value functions :

$$v_T(x_0) := \inf_{\alpha\in\mathcal{A}} \frac{1}{T}\int_0^T \ell(x(s,x_0,\alpha),\alpha(s))ds;$$

$$u_\lambda(x_0) := \inf_{\alpha\in\mathcal{A}} \lambda \int_0^{+\infty} e^{-\lambda s}\ell(x(s,x_0,\alpha),\alpha(s))ds.$$

The existence of the limits, as $T \to +\infty$ and $\lambda \to 0^+$, resp., constitutes a crucial problem well studied in the literature :
+ Arisawa (Ergodic problem, 1997,1998), Arisawa, Lions (Ergodic Stochastic Control, 1998), Bardi, Capuzzo-Dolcetta (2007),
+ Oliu-Barton, Vigeral (Uniform Tauberian Theorem in Optimal Control, 2012),

+ Quincampoix, Renault (Limit values under non expansion condition for deterministic control, 2012),
+ Buckdahn, Goreac, Quincampoix (Limit values under non expansion condition for stochastic control, 2014).

• Works in Ergodic Control : use conditions guaranteeing existence of the limit in topology of uniform convergence ; limit is a constant (independent of x_0) ; usual assumption : coercitivity of Hamiltonian.
• Here : No assumption for ergodicity ; limit can depend on x_0 ;
• Main result : representation formulas for the accumulation points w.r.t. toplogy of uniform convergence of $v_T, T \to +\infty$, and $u_\lambda, \lambda \to 0^+$;
• Remark : A byproduct : There is at most one accumulation point ; Oliu-Barton, Vigeral (Uniform Tauberian Theorem in Optimal Control, 2012) : $(v_T)_{T>0}$ converges iff $(u_\lambda)_{\lambda>0}$ converges, and the limits (for $T \uparrow +\infty$ and $\lambda \downarrow 0^+$) are the same.

The Problem

Here : Two different representation formulas :
+ One for the case of $\ell(x, a) = \ell(x)$ independent of $a \in A$; based on invariant measures for differential inclusions ;
+ One for the general case, based on occupational measures and their limits ; approach more involved.

Commun structure of both representation formulas :
The only possible occupational point $u^*(\cdot)$ is the supremum of all bounded continuous functions $\omega(\cdot)$ satisfying :
i) $R_+ \ni t \to \omega(x(t, x_0, \alpha))$ nondecreasing, for all (x_0, α) ;
ii) $\int_X \omega(x) d\mu(x, a) \leq \int_{X \times A} \ell(x, a) d\mu(x, a)$, for all $\mu \in W$,
where
+ W - suitable set of probabibility measures on $X \times A$,
+ $X \subset R^d$ compact set, invariant w.r.t. the controlled system ;
+ for the case $\ell(x, a) = \ell(x)$ ii) reduces to $d\mu(x) \in \mathcal{M}$, $\mathcal{M}$ - set of invariant measures of the differential inclusion.

Assumption : • A compact metric space ;
• $\ell : R^d \times A \to [0,1]$ continuous ; $\ell(\cdot, a)$ uniformly Lipschitz ;
• $f : R^d \times A \to R^d$ continuous, bounded (by some $M \in R_+$) ; $f(\cdot, a)$ uniformly Lipschitz ;
• $f(x, A) := \{f(x,a),\ a \in A\}$ convex, for all $x \in R^d$;
• $\exists X \subset R^d$ compact subset, invariant by the dynamics of the control problem.

Recall : + Invariance means : $\forall (x_0, \alpha) \in X \times \mathcal{A},\ x(\cdot, x_0, \alpha) \subset X$.
+ Invariance, iff, in (viscosity sense)

$$\langle \nabla d_X(x), f(x,a) \rangle \leq 0, \text{ for all } (x,a) \in \partial X \times A.$$

Observe : Under the above assumptions, the control problem is well posed, $v_T(\cdot)$ and $u_\lambda(\cdot)$ are $[0,1]$-valued, continuous on X.

Occupational measures : $\mu_T^{x_0,\alpha} \in \Delta(X)$, for $T > 0,\ x_0 \in X$:

$$\mu_T^{x_0,\alpha}(Q) := \frac{1}{T} \int_0^T I_Q(x(s, x_0, \alpha))ds,\ Q \in \mathcal{B}(X).$$

First Representation Formula

$F(x) := \{f(x,a),\ a \in A\},\ x \in X$; note :

+ For all $x(\cdot) := x(\cdot, x_0, \alpha),\ \alpha \in \mathcal{A}$, we have :

$$x'(t) \in F(x(t)),\ t \in R, x(0) = x_0.$$

+ Conversely, for every absolutely continuous solution $x(\cdot)$ of the above differential inclusion : $\exists \alpha \in \mathcal{A}$ s.t. $x(\cdot) = x(\cdot, x_0, \alpha)$.

+ $\mathcal{S}(x_0)$ - set of absolutely continuous solutions of the above differential inclusion ; then :

$$v_T(x_0) = \inf_{x(.) \in \mathcal{S}(x_0)} \frac{1}{T} \int_0^T \ell(x(s))ds,$$

$$u_\lambda(x_0) = \inf_{x(.) \in \mathcal{S}(x_0)} \lambda \int_0^T e^{-\lambda s} \ell(x(s))ds.$$

Advantage of differential inclusion : its nice topological structure :

+ $\mathcal{S} := \bigcup_{x_0 \in X} \mathcal{S}(x_0)$ - set of all solution of the differential inclusion, endowed with the topology

$$|x|_\infty := \sup_{t \in R} \left(|x(t)| e^{-M|t|} \right);\ (M \in R \text{ bound of } f).$$

+ $(\mathcal{S}, |\cdot|_\infty)$ compact metric space (Aubin, 1992) ;

Artstein (1999) : flow on $\mathcal{S}$: for all $t \in R$,

$$\Phi_t : \mathcal{S} \to \mathcal{S},\ x(\cdot) \mapsto x(\cdot + t).$$

Observe :

+ Φ is continuous ;
+ $\Phi(0, x(\cdot)) = x(\cdot),\ x(\cdot) \in \mathcal{S}$;
+ $\Phi(t+s, x(\cdot)) = \Phi(t, \Phi(s, x(\cdot))),\ x(\cdot) \in \mathcal{S}$.

Recall : $p \in \Delta(\mathcal{S})$ is invariant for Φ, iff

$$p(\Phi_t(\Gamma)) = p(\Gamma),\ \forall \Gamma \in \mathcal{B}(\mathcal{S}),\ t \in R.$$

Also recall : Weak limits of occupational measures are invariant for Φ. More precisely :

Lemma

(See, e.g., Artstein, 1999) For any fixed $y(\cdot) \in \mathcal{S}$, $T > 0$, *define the occupational measure* $\mu_T^y \in \Delta\mathcal{S}$ *:*

$$\int_{\mathcal{S}} \varphi(x) d\mu_T^y(x) := \frac{1}{T}\int_0^T \varphi(\Phi_s(y))ds, \; \varphi \in C(\mathcal{S}).$$

If for some sequence $T_n \to +\infty$, *we have* $\mu_{T_n}^{y_n} \Rightarrow \mu$, *for some* $\mu \in \Delta(\mathcal{S})$, *then* μ *is invariant for the flow* Φ.

Remark. The lemma holds also for $\mathcal{S}$ replaced by a general compact metric space B endowed with a flow $\Phi = \{\Phi_t(x), (t,x) \in R \times B\}$.

Definition

Given $p \in \Delta(\mathcal{S})$, associated projected measure $\mu \in \Delta(X)$:
$\mu(B) := p(< B >),\; B \in \mathcal{B}(X),$ with $< B > := \{x(\cdot) \in \mathcal{S},\; x(0) \in B\}$,
i.e., $\int_X \varphi(x)\mu(dx) = \int_{\mathcal{S}} \varphi(x(0))p(dx(\cdot)), \; \varphi \in C(X).$

Definition

(Sequel).
+ If p invariant measure for Φ, μ is called projected invariant measure on X ;
+ $\mathcal{M}$ = set of all projected invariant measures on X ; $\mathcal{M}$ closed convex subset of $\Delta(X)$.

Definition

$\mathcal{H} := \{\omega : X \to [0,1]$: ω satisfies
i) $R_+ \ni t \to \omega(x(t, x_0, \alpha))$ non decreasing, for all (x_0, α) ;
ii) $\int_X \omega(x)\mu(dx) \leq \int_X \ell(x)\mu(dx)$, for all $\mu \in \mathcal{M}$. We also define :
$v^*(x_0) := \sup\{\omega(x_0), \omega \in \mathcal{H}\}$, $x_0 \in X$.

Example 1. X =unit cercle in R^2 ; uncontrolled dynamics is given by :

$$x'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x(t), \;\; t \in R;$$

+ Unique projected invariant measure : the uniform (Haar) measure on the unit cercle X ;
+ $\mathcal{H}$ = the set of functions $\omega : X \to [0,1]$ which are constant (by (i)) and not greater than $\frac{1}{2\pi}\int_0^{2\pi} \ell(e^{i\theta})d\theta$. (by (ii)). Thus,

$$v^*(x_0) = \frac{1}{2\pi}\int_0^{2\pi} \ell(e^{i\theta})d\theta, \; x_0 \in X.$$

Example 2. Two-dimensional dynamics $x(t) = (x_1(t), x_2(t))$, $t \in R$, with

$$\begin{cases} x_1'(t) &= a(t)(1 - x_1(t)) \\ x_2'(t) &= a^2(t)(1 - x_1(t)) \end{cases}, \; a(t) \in [0,1],\; x(0) = x_0 \in R^2,$$

and with cost function

$$\ell(x) = 1 - x_1(1 - x_2),\; x = (x_1, x_2) \in R^2.$$

Computation shows :

+ $X = \{x = (x_1, x_2) \in [0,1]^2,\; x_1 \geq x_2\}$ is an invariant set ;

+ As $x_1'(t) \geq x_2'(t) \geq 0$, using the properties of the projected invariant measures, one shows :

- $\forall \omega \in \mathcal{H} : \omega(x_1, x_2) \leq x_2$;
- $\omega(x_1, x_2) = x_2$ is in $\mathcal{H}$.

Consequently, $v^*(x_1, x_2) := x_2$, $x = (x_1, x_2) \in X$, depends on x_2, i.e., the problem cannot be reduced to an ergodic one.

Theorem

Under our assumptions, for $\ell(x,a) = \ell(x)$, any accumulation point (in the uniform convergence topology on X) of $(v_T(\cdot))_{T>0}$, as $T \to +\infty$, equals to $v^*(.)$.

Analogously, for Abel Means we have :

Theorem

Under our assumptions, for $\ell(x,a) = \ell(x)$, any accumulation point (in the uniform convergence topology on X) of $(u_\lambda(\cdot))_{\lambda>0}$, as $\lambda \to 0^+$, equals to $v^*(.)$.

Sketch of the proof of the 1st Theorem. Let v be an accumulation point of $(v_T(\cdot))_{T>0}$, as $T \to +\infty$. Then, up to a subsequence, v_T converges uniformly to v as $T \to +\infty$. For simplicity of notation, we suppose $v_T \to v$.

Step 1 : $v^*(x_0) \geq v(x_0)$

It suffices to show that $v \in \mathcal{H}$.

1) Monotonicity property : Let $r < t$, $x_0 \in X$ and $\alpha \in \mathcal{A}$, and prove $v(x(r, x_0, \alpha)) \leq v(x(t, x_0, \alpha))$.

W.l.o.g. $r = 0$. Then,

$$v_T(x(t, x_0, \alpha)) = \frac{1}{T} \inf_{\widetilde{\alpha} \in \mathcal{A}} \int_0^T \ell(x(s, x(t, x_0, \alpha), \widetilde{\alpha}))ds$$

$$= \frac{1}{T} \inf_{\widetilde{\alpha} \in \mathcal{A}} \int_0^T \ell(x(s + t, x_0, \alpha \odot \widetilde{\alpha}(\cdot - t)))ds, \text{ where}$$

$$\alpha \odot \widetilde{\alpha}(\cdot - t)(s) = \begin{cases} \alpha(s), & \text{if} \quad s \leq t, \\ \widetilde{\alpha}(s - t), & \text{if} \quad s > t. \end{cases}$$

Thus,

$$\begin{aligned} v_T(x(t,x_0,\alpha)) &= \frac{1}{T}\inf_{\widetilde{\alpha}}\int_t^{T+t} \ell(x(\sigma,x_0,\alpha\odot\widetilde{\alpha}(\cdot-t)))d\sigma \\ &\geq \frac{1}{T}\inf_{\widetilde{\alpha}}\int_0^{T} \ell(x(\sigma,x_0,\alpha\odot\widetilde{\alpha}(\cdot-t)))d\sigma - \frac{t}{T} \\ &\geq v_T(x_0) - \frac{t}{T}. \end{aligned}$$

Thus, as $T \to \infty$: $v(x(t,x_0,\alpha)) \geq v(x_0)$.

2) Verification of property ii) of the definition of $\mathcal{H}$: For $\mu \in \mathcal{M}$ (projected invariant measure) to show : $\int_X v d\mu \leq \int_X \ell d\mu$.

Let $T > 0$. For all $\alpha \in \mathcal{A}$,

$$v_T(x_0) \leq \frac{1}{T}\int_0^T \ell(x(s,x_0,\alpha))ds.$$

Hence, for all $x(\cdot) \in \mathcal{S}$,

$$v_T(x(0)) \leq \frac{1}{T}\int_0^T \ell(x(s))ds = \frac{1}{T}\int_0^T \ell([\Phi_s(x(\cdot)](0))ds.$$

Now, integrating the above inequality w.r.t. an invariant probability measure $p \in \Delta(\mathcal{S})$ which projection is μ, from the invariance of p,

$$\begin{aligned}
\int_X v_T d\mu &= \int_{\mathcal{S}} v_T(x(0))dp(x(\cdot)) \\
&\leq \frac{1}{T}\int_{\mathcal{S}}\int_0^T \ell([\Phi_s(x(\cdot)](0))dsdp(x(\cdot)) \\
&= \frac{1}{T}\int_0^T\int_{\mathcal{S}} \ell([\Phi_s(x(\cdot)](0))dp(x(\cdot))ds \\
&= \frac{1}{T}\int_0^T\int_{\mathcal{S}} \ell(x(0))dp(x(\cdot))ds \\
&= \int_{\mathcal{S}} \ell(x(0))dp(x(\cdot)) = \int_X \ell d\mu.
\end{aligned}$$

Taking $T \to \infty$, we get $\int_X v d\mu \leq \int_X \ell d\mu$. Hence, $v \in \mathcal{H}$.

Step 2 : $v^*(x_0) \leq v(x_0)$ By definition of v^* it is enough to show that $\omega(x_0) \leq v(x_0)$, for all $\omega \in \mathcal{H}$. Let us fix arbitrarily $\omega \in \mathcal{H}$. For any $T > 0$, $\varepsilon > 0$, $\exists\varepsilon$-optimal control $\alpha_\varepsilon \in \mathcal{A}$ s.t., for the occupational measure $\mu_T^{x_0,\alpha_\varepsilon}$,

$$\begin{aligned} v_T(x_0) + \varepsilon &\geq \int_X \ell d\mu_T^{x_0,\alpha_\varepsilon} := \frac{1}{T}\int_0^T \ell(x(s, x_0, \alpha_\varepsilon))ds \\ &= \frac{1}{T}\int_0^T \ell(\Phi_s x)(0, x_0, \alpha_\varepsilon))ds; \end{aligned}$$

$\mu_T^{x_0,\alpha_\varepsilon} \in \Delta(X)$ is the projection of the occupational measure $p_T \in \Delta(\mathcal{S})$ defined by

$$\int_{\mathcal{S}} \varphi dp_T = \frac{1}{T}\int_0^T \varphi(\Phi_s(x(\cdot, x_0, \alpha_\varepsilon)))ds, \ \varphi \in C(\mathcal{S}).$$

The compactness of $\mathcal{S}$, X implies that of $\Delta(\mathcal{S})$, $\Delta(X)$. Hence, there is $(p, \mu) \in \Delta(\mathcal{S}) \times \Delta(X)$ s.t. $(p_T, \mu_T^{x_0,\alpha_\varepsilon}) \Rightarrow (p, \mu)$ along a subsequence, as $T \to +\infty$.

First Representation Formula : Cesàro Means

Due to the above lemma (see Artstein) :
+ p is an invariant probability measure for Φ and, thus,
+ μ is the associated projected invariant, i.e., $\mu \in \mathcal{M}$.
Consequently, for $\omega \in \mathcal{H}$ (by definition of $\mathcal{H}$) :
+ $\int_X \omega d\mu \leq \int_X \ell d\mu$.
+ $\int_X \omega d\mu_T^{x_0,\alpha_\varepsilon} = \frac{1}{T}\int_0^T \omega(x(s,x_0,\alpha_\varepsilon))ds \geq \omega(x_0)$.
$T \to +\infty$ yields

$$v(x_0) + \varepsilon \geq \int_X \ell d\mu \geq \int_X \omega d\mu \geq \omega(x_0).$$

The result follows from the arbitrariness of $\varepsilon > 0$.

Discounted Occupational Measures on $X \times A$: Now $\ell = \ell(x, a)$:

For $\lambda > 0$, $x_0 \in X$, $\alpha \in \mathcal{A}$ we define $\nu_\lambda^{x_0,\alpha} \in \Delta(X \times A)$: For all $\varphi \in C(X \times A)$,

$$\int_{X\times A} \varphi d\nu_\lambda^{x_0,\alpha}(x,a) := \lambda \int_0^{+\infty} e^{-\lambda s}\varphi(x(s,x_0,\alpha),\alpha(s))ds,$$

$$\Gamma_\lambda(x_0) := \{\nu_\lambda^{x_0,\alpha} \in \Delta(X \times A),\ \alpha \in \mathcal{A}\}.$$

Observe that, for all $\nu_\lambda^{x_0,\alpha} \in \Gamma_\lambda(x_0)$ and $\varphi \in C^1(X)$,

$$\int_{X\times A} (\nabla\varphi(x) \cdot f(x,a) + \lambda(\varphi(x_0) - \varphi(x)))\, d\nu_\lambda^{x_0,\alpha}(x,a) = 0.$$

Indeed

$$\int_{X\times A} \nabla\varphi(x) \cdot f(x,a) d\nu_\lambda^{x_0,\alpha}(x,a)$$

$$= \lambda \int_0^{+\infty} e^{-\lambda s}\nabla\varphi(x(s,x_0,\alpha))f(x(s,x_0,\alpha),\alpha(s))ds$$

$$= \lambda \int_0^{+\infty} \left(\frac{d}{ds}[e^{-\lambda s}\varphi(x(s,x_0,\alpha))] + \lambda e^{-\lambda s}\varphi(x(s,x_0,\alpha)) \right) ds$$
$$= -\lambda\varphi(x_0) + \lambda^2 \int_0^{+\infty} e^{-\lambda s}\varphi(x(s,x_0,\alpha))ds$$
$$= -\lambda \int_{X\times A} (\varphi(x_0) - \varphi(x))d\nu_\lambda^{x_0,\alpha}(x,a).$$

This means that

$$\Gamma_\lambda(x_0) \subset W_\lambda(x_0),$$

where

$$W_\lambda(x_0) := \Big\{\nu \in \Delta(X \times A) : \text{ for all } \varphi \in C^1(X),$$
$$\int_{X\times A} (\nabla\varphi(x)\cdot f(x,a) + \lambda(\varphi(x_0) - \varphi(x)))\, d\nu(x,a) = 0\Big\}.$$

Observe : $W_\lambda(x_0)$ it is convex, compact ($X \times A$ is compact).

$\Rightarrow$ Consequently, $co(\Gamma_\lambda(x_0)) \subset W_\lambda(x_0)$ (co = closed convex hull).

Now, if $\nu_n \in W_{\lambda_n}(x_0)$ s.t. $\nu_n \Rightarrow \nu$ as $\lambda_n \to 0^+$, then

$\nu \in W :=$

$$\left\{\nu \in \Delta(X \times A) : \int_{X\times A} \nabla\varphi(x)\cdot f(x,a)d\nu(x,a) = 0,\ \varphi \in C^1(X)\right\}.$$

This shows :

$W \supset \overline{\lim\limits_{\lambda\to 0^+}}\ co(\Gamma_\lambda(x_0)) :=$ set of the accumulation points of all sequences $\nu_n \in co(\Gamma_{\lambda_n}(x_0))$, $n \geq 1$, with $\lambda_n \to 0^+$.

Moreover, from Gaitsgory, Quincampoix (2009) :

Lemma

$$\lim_{\lambda\to 0^+} d_H(co(\Gamma_\lambda(X)), W) = 0, \textit{ with :}$$

$+\ \Gamma_\lambda(X) = \bigcup_{x_0\in X}\Gamma_\lambda(x_0)$ *;*

$+\ d_H =$ *Haussdorff distance associated with any distance d consistent with the weak convergence of measures in* $\Delta(X\times A)$.

Recall : For $M_i \subset \Delta(X \times A)$, $i = 1, 2$,

$$d_H(M_1, M_2) = \max\{ \sup_{\mu \in M_1} d(\mu, M_2), \sup_{\mu \in M_2} d(\mu, M_1)\}.$$

Representation formula for Abel Means

Definition

For all x_0 in X,

$$u^*(x_0) := \sup\{\omega(x_0), \, \omega \in \mathcal{K}\},$$

where $\mathcal{K} := \{\omega : X \to [0,1]$ continuous, s.t. :

i) $[0, +\infty) \ni t \mapsto \omega(x(t, x_0, \alpha))$ is nondecreasing, $\forall (x_0, \alpha) \in X \times \mathcal{A}$;

ii) $\displaystyle\int_{X \times A} \omega(x) d\mu(x,a) \leq \int_{X \times A} \ell(x,a) d\mu(x,a), \, \forall \mu \in W.$

Second Representation Formula for Abel Means

Theorem

Any accumulation point - in the uniform convergence topology - of $(u_\lambda(\cdot))_{\lambda>0}$ as $\lambda \to 0^+$, is equal to $u^*(\cdot)$.

Remark : $u^*(\cdot)$ is viscosity solution of H-J equation :

$$\inf_{a\in A}\langle \nabla u^*(x), f(x,a)\rangle = 0,\ x \in X.$$

+ Indeed : $u^*(.)$ is uniform limit of a subsequence of $u_\lambda(.)$, $\lambda \downarrow 0^+$; take limit H-J equ. satisfied by u_λ.
+ Observe : H-J equ. for u^* justifies monotonicity assumption in the definition of $\mathcal{K}$.

As for the Abel Means we begin with discussing the Occupational Measures on $X \times A$

For $T > 0$, $(x_0, \alpha) \in X \times \mathcal{A}$, definition of $\mu_T^{x_0,\alpha} \in \Delta(X \times A)$:

$$\int_{X\times A} \varphi d\mu_T^{x_0,\alpha} := \frac{1}{T}\int_0^T \varphi(x(s,x_0,\alpha),\alpha(s))ds,\ \varphi \in C(X\times A),$$

$$\Gamma_T(x_0) := \{\mu_T^{x_0,\alpha} \in \Delta(X \times A),\ \alpha \in \mathcal{A}\}.$$

Observe, from Itô's formula, for all $\varphi \in C^1(X)$:

$$\int_{X\times A} < \nabla\varphi(x), f(x,a) > d\mu_T^{x_0,\alpha}(x,a) = \frac{1}{T}(\varphi(x(T,x_0,\alpha)) - \varphi(x_0)).$$

Thus, for $\Gamma_{T_n}(x_0) \ni \mu_n \Rightarrow \mu$, as $T_n \to +\infty$, we have :

$$\mu \in W = \left\{\nu \in \Delta(X\times A) : \int_{X\times A} \nabla\varphi(x)\cdot f(x,a)d\nu(x,a) = 0,\ \varphi \in C^1(X)\right\}.$$

As in the case of Abel Means, W can be understood as limit of the set of occupational measures.

Indeed, for $\Gamma_T(X) := \bigcup_{x_0 \in X} \Gamma_T(x_0)$:

Lemma

(See Gaitsgory, 2004)

$$\lim_{T \to +\infty} d_H(co(\Gamma_T(X)), W) = 0.$$

Recall : d_H = Haussdorff distance associated with any distance d consistent with the weak convergence of measures in $\Delta(X \times A)$.

Representation formula for Cesàro Means :

Theorem

Any accumulation point -w.r.t. the topology of uniform convergence- of $(v_T(\cdot))_{T>0}$, as $T \to +\infty$, coincides with $u^*(\cdot)$ (defined in the sub-section "Abel Means").

Comparison between both representation formulas

We've got two representations of different nature :
+ the first one : based on invariant measures, $\ell(x,a) = \ell(x)$,
+ the second one : $\ell(x,a)$; use of measures being limits of occupational measures on $X \times A$.
A natural question : possibility to use an invariant measure approach also in the 2nd case ? Answer : This is possible to a certain extent. Consider :

$$\mathcal{G} := \Big\{(x,\alpha) : \; x'(t) = f(x(t),\alpha(t)), \; t \in R, \alpha \in \mathcal{A}, x(0) \in X\Big\}$$

equipped with the product topology defined by :
+ the uniform topology for the $x(.)$-component :

$$||x(.)||_\infty := \sup_{t \in R} \left(|x(t)|e^{-M|t|}\right) \; (M \text{ bound of } f) \, ;$$

+ for the $\alpha(\cdot)$ component the L^2_{weak}-topology associated with the L^2-norm

$$||\alpha(\cdot)|||_{L^2} := \left(\int_{-\infty}^{+\infty} |\alpha(t)|^2 e^{-M|t|} dt\right)^{\frac{1}{2}} .$$

Comparison between both representation formulas

+ $\mathcal{G}$ is sequentially compact (Every sequence has a convergent subsequence) ; hence, as $\mathcal{G}$ is metrisable, $\mathcal{G}$ can be considered as compact metric space.

+ Definition of a continuous flow $\varphi = (\varphi_t)_{t\in R}$ on $\mathcal{G}$:

$$\varphi_t \left\{ \begin{array}{lcl} \mathcal{G} & \to & \mathcal{G} \\ (x(\cdot), \alpha(\cdot)) & \mapsto & (x(\cdot + t), \alpha(\cdot + t)). \end{array} \right.$$

Main difficulty for $\ell = \ell(x, a)$: To work with the measures, we have to take the limit in integrals $\int_{\mathcal{G}} \ell dp$ for $p \in \Delta(\mathcal{G})$. But for this we have to consider ℓ as continuous function on $\mathcal{G}$ endowed with the weak topology defined above.

Observe : For all φ continuous in x and affine in a, the following mapping is continuous :

$$G_s \left\{ \begin{array}{lcl} \mathcal{G} & \to & \mathbb{R} \\ (x(\cdot), \alpha(\cdot)) & \mapsto & \int_0^s \varphi(x(r), \alpha(r)) dr \end{array} \right.$$

For all $p \in \Delta(\mathcal{G})$ we define $\nu_s \in \Delta(X \times A)$, $s > 0$, s.t., for all $\varphi \in C(X \times A)$ affine in a :

$$\int_{X\times A} \varphi(x,a)d\nu_s(x,a) := \frac{1}{s}\int_{\mathcal{G}}\int_0^s \varphi(x(r),\alpha(r))drdp(x,\alpha).$$

This allows to proceed similarly to the case $\ell(x,a) = \ell(x)$:
$\widehat{\mathcal{M}}$:= set of all $\mu \in \Delta(X \times A)$ accumulation points -w.r.t. the weak convergence of measures - for $(\nu_s)_{s>0}$ associated with all possible invariant probability measures $p \in \Delta(\mathcal{G})$ by the above relation.

Definition

$$v^*(x_0) := \sup\{\omega(x_0),\ \omega \in \widehat{\mathcal{H}}\},\ x_0 \in X,$$

where $\widehat{\mathcal{H}} := \Big\{\omega \in C(X \to [0,1])$:
i) $t \in [0,+\infty) \mapsto \omega(x(t,x_0,\alpha))$ nondecreasing, $(x_0,\alpha) \in X \times \mathcal{A}$;
ii) $\displaystyle\int_{X\times A}\omega(x)d\mu(x,a) \le \int_{X\times A}\ell(x,a)d\mu(x,a), \quad \mu \in \widehat{\mathcal{M}}\Big\}$.

Proposition

Suppose that $a \mapsto \ell(x,a)$ is affine, for all $x \in X$. Then any accumulation point -w.r.t. the topology of the uniform convergence- of $(v_T(\cdot))_{T>0}$, as $T \to \infty$, is equal to v^*.

We also have the corresponding result $(u_\lambda(\cdot))_{\lambda>0}$.

Proposition

If $a \mapsto \ell(x,a)$ is affine, for all $x \in X$, then any accumulation point -w.r.t. the topology of the uniform convergence- of $(u_\lambda(\cdot))_{\lambda>0}$, as $\lambda \to 0^+$, coincides with v^*.

Remark. Proof rather technical because of the topology on $\mathcal{G}$; however, the above results are weaker than those gotten with using occupational measures.

Deduction of the results for $\ell(x)$ from the general case $\ell = \ell(x, a)$:

Recall : $\lim_{T \to +\infty} d_H(co(\Gamma_T(X)), W) = 0,$

where

+ $\Gamma_T(x_0) := \{\mu_T^{x_0,\alpha} \in \Delta(X \times A),\ \alpha \in \mathcal{A}\};$

+ $\Gamma_T(X) := \bigcup_{x_0 \in X} \Gamma_T(x_0).$

Using the lemma (Artstein, 1999) we get :

$$\Pi_X(W) \subset \mathcal{M},$$

where $\Pi_X(m) := m(\cdot \times A) \in \Delta(X)$ denotes the projection on X of a measure $m \in \Delta(X \times A)$.

We even have :

Theorem

$$\Pi_X(W) = \mathcal{M}.$$

One of the main advantages of the representation formulas is that convergence of averaging values is deduced from equicontinuity.

Corollary

If $(v_T(\cdot))_{T>0}$ is equicontinuous, it converges uniformly to $u^*(\cdot)$, as $T \to \infty$. The same holds true for $(u_\lambda(\cdot))_{\lambda>0}$: Its equicontinuity implies its uniform convergence to $u^*(\cdot)$, as $\lambda \to 0^+$.

Proof. Indeed, from Arzelà-Ascoli : $(v_T(\cdot))_{T>0}$, $(u_\lambda(\cdot))_{\lambda>0}$ are relatively compact, and by the the above theorems we obtain their uniform convergence to $u^*(\cdot)$.

We illustrate this observation :

Nonexpansivity

Corollary

(Nonexpansivity : Quincampoix, Renault, 2012 ; extension to stochastic control : Buckdahn, Goreac, Quincampoix, 2014). Let us suppose the following nonexpansivity condition : There is some $c \in \mathbb{R}_+$ *such that, for all* $x_1, x_2 \in X$*,*
for all $a_1 \in A$ *there exists* $a_2 \in A$ *s.t. :*
i) $\langle x_1 - x_2, f(x_1, a_1) - f(x_2, a_2)\rangle \leq 0$,
ii) $|\ell(x_1, a_1) - \ell(x_2, a_2)| \leq c|x_1 - x_2|\}$.
Then $v_T(\cdot)$ *and* $u_\lambda(.)$ *converge uniformly to* u^**, as* $T \to \infty$ *and* $\lambda \to 0^+$*, respectively.*

Remark. 1) If $\ell(x, a) = \ell(x)$, ii) means that ℓ is Lipschitz.
2) Nonexpansivity implies equicontinuity, and then the above corollary applies.

3) The interest of the nonexpansivity property consists in the fact that it allows $u^*(\cdot)$ to depend on the initial condition x_0.
4) Unlike the nonexpansivity property, the following stronger condition - called dissipativity condition - implies that the limit $u^*(\cdot)$ is a constant independent of x_0 (See Artstein, Gaitsgory, 2000, $\ell(x,a) = \ell(x)$ Lipschitz) :

There exists a constant $C > 0$ such that
for all $x_1, x_2 \in X$, for all $a_1 \in A$ there is $a_2 \in A$ s.t.

$$< x_1 - x_2, f(x_1, a_1) - f(x_2, a_2) > \leq -C||x_1 - x_2||^2.$$

Last slide

Thank you very much for your attention !

Good luck and good continuation, Vlad !