Gradient bounds for the solution of the filtering equation

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Stochastic calculus, Monte-Carlo methods and Mathematical Finance
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- Kusuoka-Stroock results for diffusion semigroups
- Framework
  - Ramdomly perturbed semigroups
  - The stochastic PDE
  - Applications to nonlinear filtering
  - The UFG condition (Kusuoka-Stroock)
- Main results
- Building blocks of the proofs
- Final remarks

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D Crisan, C Litterer, T Lyons, Kusuoka-Stroock gradient bounds for the solution of the filtering equation, Journal of Functional Analysis 268 (2015), pp. 1928-1971. http://arxiv.org/abs/1311.0480

A classical result

In the eighties, Kusuoka and Stroock analyzed the smoothness properties of the (perturbed) semigroup associated to a diffusion process:

$$(P_t^c\varphi)(x) = \mathbb{E}\left[\varphi(X_t^x)\exp\left(\int_0^t c\left(X_s^x\right)ds\right)\right], \quad t \geq 0, \quad x \in \mathbb{R}^{d_1},$$

where

$$X_t^x = x + \int_0^t V_0(X_s^x)ds + \sum_{i=1}^N \int_0^t V_i(X_s^x) \circ dB_s^i, \quad t \geq 0. \tag{1}$$

- $\{V_i \quad i = 0, ..., N\}$ are smooth and bounded satisfying the UFG condition
- the stochastic integrals in (1) are of Stratonovich type
- B is an N-dimensional standard Brownian motion
- $c : \mathbb{R}^{d_1} \to \mathbb{R}$ are smooth and bounded functions
- $\varphi : \mathbb{R}^{d_1} \to \mathbb{R}$ is an arbitrary bounded measurable function.

The UFG condition

- states that the $C_b^\infty\left(\mathbb{R}^{d_1}\right)$ −module $\mathcal{W}$ generated by the vector fields $\{V_i \quad i=1,...,N\}$ within the Lie algebra generated by $\{V_i, \quad i=0,...,N\}$ is finite dimensional.
- the condition does not require that the vector space $\{W(x)|W\in\mathcal{W}\}$ is homeomorphic to $\mathbb{R}^{d_1}$.
- in this sense, the UFG condition is weaker than the Hörmander condition.

Theorem (Kusuoka and Stroock, 1987)

$P_t^c\varphi$ is differentiable in the direction of any vector field W belonging to $\mathcal{W}$.

Theorem (Kusuoka and Stroock, 1987)

For any smooth compactly supported function $\varphi:\mathbb{R}^{d_1}\to\mathbb{R}$,

$$\|W_1\dots W_m P_t^c(W_{m+1}\dots W_{m+n}\varphi)\|_p\le C^{m,n}t^{-l}\|\varphi\|_p,\quad p\in[1,\infty],\tag{2}$$

where $l=l(W_i\in\mathcal{W}, i=1,...,m+n)$.

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Define the randomly perturbed semigroup

$$\rho_t^{Y(\omega)}(\varphi)(x) = \mathbb{E}\left[\varphi(X_t^x) Z_t(X^x, Y) \middle| \mathcal{Y}_t\right](\omega), \ t \geq 0, \ x \in \mathbb{R}^{d_1}, \tag{3}$$

where

$$Z_t(X^x, Y) = \exp\left(\sum_{i=1}^{d_2} \int_0^t h^i(X_s^x)\, dY_s^i - \frac{1}{2}\sum_{i=1}^{d_2} \int_0^t h^i(X_s^x)^2\, ds\right).$$

- $Y = \left\{ \left(Y_t^i\right)_{i=1}^{d_2}, t \geq 0 \right\}$ is a d_2-dim Bm independent of X, $\mathcal{Y}_t = \sigma\{Y_s, s \in [0, t]\}$.
- $h^i : \mathbb{R}^{d_1} \to \mathbb{R}$, $i = 1, ..., d_2$ are smooth bounded functions with bounded derivatives of all orders.

Theorem (Crisan, Litterer and Lyons)

There exists a random variable $\omega \to C^{m,n}(\omega)$ such that for any smooth compactly supported $\varphi : \mathbb{R}^{d_1} \to \mathbb{R}$

$$\| W_1 \ldots W_m \rho_t^{Y(\omega)}(W_{m+1} \ldots W_{m+n}\varphi)\|_p \leq C^{m,n}(\omega) t^{-l} \|\varphi\|_p, \quad p \in [1, \infty], \tag{4}$$

where $l = l(W_i \in \mathcal{W}, i = 1, ..., m + n)$.

$$\rho_t^{Y(\omega)}(\varphi)(x) = \int \varphi(y)\rho_t^x(dy) = \rho_t^x(\varphi)$$

The measure valued process $\{\rho_t^x, t \geq 0\}$ solves the following linear parabolic stochastic PDE (written in its weak form), called the called the Duncan-Mortensen-Zakai equation:

$$\begin{aligned} d\rho_t^x(\varphi) &= \rho_t^x(A\varphi)dt + \sum_{k=1}^{d_2} \rho_t^x(h_k\varphi)dY_t^k, \\ \rho_0^x &= \delta_x. \end{aligned} \tag{5}$$

Here A is the differential operator

$$A\varphi = V_0\varphi + \frac{1}{2}\sum_{i=1}^{N} V_i^2\varphi$$

and φ is a suitably chosen test function. The normalized solution of (5) gives the conditional distribution of a partially observed stochastic process. Particular case:

$$d\rho_t^x(y) = \frac{1}{2}\Delta\rho_t^x(y)dt + h(y)\rho_t^x(y)dY_t.$$

$(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ probability space, $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions.

• the *signal* process:

$$dX_t = V_0(X_t)dt + \sum_{i=1}^{N} V_i(X_t) \circ dB_t^i, \quad X_0 = x, \quad t \geq 0, \tag{6}$$

W an $\mathcal{F}_t$-adapted d_2-dimensional Brownian motion independent of X.
• the *observation* process:

$$Y_t = \int_0^t h(X_s)ds + W_t, \tag{7}$$

$\mathcal{Y}_t = \sigma(Y_s,\ s \in [0,t]) \vee \mathcal{N}$, $\mathcal{N}$ comprises all $\tilde{\mathbb{P}}$-null sets.

The filtering problem. Determine π_t, the conditional distribution of the signal X at time t given Y in the interval $[0, t]$.

$$\pi_t(\varphi) = \tilde{\mathbb{E}}[\varphi(X_t) \mid \mathcal{Y}_t] \quad , \quad \varphi \text{ Borel bounded function.} \tag{8}$$

Let $\mathbb{P}$ be absolutely continuous with respect to $\tilde{\mathbb{P}}$ such that

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t(X,Y).$$

$$Z_t(X,Y) = \exp\left(\sum_{i=1}^{d_2}\int_0^t h^i(X_s)\,dY_s^i - \frac{1}{2}\sum_{i=1}^{d_2}\int_0^t h^i(X_s)^2\,ds\right).$$

By Girsanov's theorem, under $\mathbb{P}$, Y is a Brownian motion independent of X; additionally the law of X under $\tilde{\mathbb{P}}$ is the same as its law under $\mathbb{P}$.

Kallianpur-Striebel formula

$$\pi_t(\varphi) = \frac{\rho_t^{Y(\omega)}(\varphi)}{\rho_t^{Y(\omega)}(\mathbf{1})} \quad \tilde{\mathbb{P}}(\mathbb{P}) - \text{a.s.}, \tag{9}$$

$$\rho_t^{Y(\omega)}(\varphi) = \mathbb{E}\left[\varphi(X_t)Z_t(X,Y)\,|\,\mathcal{Y}_t\right](\omega),\ t \geq 0, \tag{10}$$

- $\mathbf{1}$ is the constant function $\mathbf{1}(x) = 1$ for any $x \in \mathbb{R}^{d_1}$.
- ρ_t^X the *unnormalised* conditional distribution the signal.

Let $(V_i)_{0\le i\le N} \in \mathcal{C}_b^K(\mathbb{R}^{d_1}, \mathbb{R}^{d_1})$. Notation

$$V_{[i]} = V_i, \; V_{[\alpha\star i]} = \left[V_{[\alpha]}, V_i\right], \quad i \in \{0, \ldots, N\},$$

where $[\cdot,\cdot]$ is the commutator and $\alpha \star i = (\alpha_1, \ldots, \alpha_n, i)$ when α is given by $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_j \in \{0, \ldots, N\}$, $j = 1, \ldots, n$.

The following “lengths” of a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ are used:

$$|\alpha| = |(\alpha_1, \ldots, \alpha_n)| = n,$$

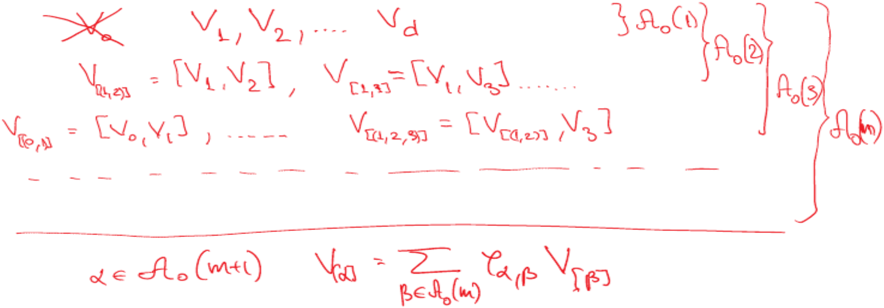
$$\|\alpha\| = \|(\alpha_1, \ldots, \alpha_n)\| = n + \sharp\{i : \alpha_i = 0\}.$$

$\mathcal{A}_0(m) =$ the set of multi-indices α different from (0) for which $\|\alpha\| \le m$.

Definition

Let $m \in \mathbb{N}^*$ such that $K \geq m+3$. The vector fields $\{V_i, 0 \leq i \leq N\}$ satisfy *the UFG condition of order m* if for any $\alpha \in \mathcal{A}_0(m+1)$ there exists $\varphi_{\alpha,\beta} \in \mathcal{C}_b^{K+1-|\alpha|}(\mathbb{R}^{d_1})$, with $\beta \in \mathcal{A}_0(m)$ such that

$$V_{[\alpha]}(x) = \sum_{\beta \in \mathcal{A}_0(m)} \varphi_{\alpha,\beta}(x) V_{[\beta]}(x), \quad x \in \mathbb{R}^{d_1}.$$



- The vector fields $\{V_i, 0 \le i \le N\}$ satisfy the *uniform* Hörmander condition if there exists $m > 0$ such that

$$\inf_{\{x,\xi \in \mathbb{R}^{d_1} \,|\, \|\xi\|=1\}} \sum_{\beta \in \mathcal{A}_0(m)} (V_{[\beta]}(x), \xi)^2 > 0.$$

  If the vector fields $\{V_i, 0 \le i \le N\}$ satisfy the uniform Hörmander condition then they satisfy the UFG condition.
- In particular if the vector fields $\{V_i, 1 \le i \le N\}$ satisfy the strict ellipticity condition then they satisfy the UFG condition.
- The following

$$V_0(x_1, x_2) = \sin x_1 \frac{\partial}{\partial x_1} \qquad V_1(x_1, x_2) = \sin x_1 \frac{\partial}{\partial x_2}$$

  satisfy the UFG condition of order $m = 4$, but not the Hörmander condition.
- For smooth vector fields, let $\mathcal{W}$ be the $C_b^\infty(\mathbb{R}^{d_1})$-module generated by the vector fields $\{V_i, i = 1, ..., N\}$ within the Lie algebra generated by $\{V_i, i = 1, ..., N\}$. Then, the UFG condition means that $\mathcal{W}$ is finitely generated *as a vector space* and $\{V_{[\alpha]}, \alpha \in \mathcal{A}_0(m)\}$ is a finite set of generators for $\mathcal{W}$.

V_i, $i = 0, \ldots, N$ satisfy UFG condition, $\alpha_1, \ldots, \alpha_j, \ldots, \alpha_m \in \mathcal{A}_0(m)$ $m \geq j \geq 0$.

Theorem (Crisan, Litterer and Lyons)

Let $h \in C_b^\infty\left(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}\right)$. There exists $c_\infty(\omega)$ a.s. finite such that

$$\left\|\left(V_{[\alpha_1]} \cdots V_{[\alpha_j]} \rho_t^{Y(\omega)}\left(V_{[\alpha_{j+1}]} \cdots V_{[\alpha_m]} \varphi\right)\right)\right\|_\infty \leq c_\infty(\omega)\, t^{-(\|\alpha_1\| + \cdots + \|\alpha_m\|)/2} \|\varphi\|_\infty$$

for any $\varphi \in C_b^\infty\left(\mathbb{R}^N\right)$.
Let $h \in C_0^\infty\left(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}\right)$. There exists $c_p(\omega)$ a.s. finite such that

$$\left\|\left(V_{[\alpha_1]} \cdots V_{[\alpha_j]} \rho_t^{Y(\omega)}\left(V_{[\alpha_{j+1}]} \cdots V_{[\alpha_m]} \varphi\right)\right)\right\|_p \leq c_p(\omega)\, t^{-(\|\alpha_1\| + \cdots + \|\alpha_m\|)/2} \|\varphi\|_p$$

for all $\varphi \in C_0^\infty\left(\mathbb{R}^N\right)$, $p \in (1, \infty]$.

Corollary

Let $h \in C_b^\infty\left(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}\right)$. There exists $\bar{c}_\infty(\omega)$ a.s. finite such that

$$\left\| \left(V_{[\alpha_1]} \cdots V_{[\alpha_j]} \pi_t^{Y(\omega)} \left(V_{[\alpha_{j+1}]} \cdots V_{[\alpha_m]} \varphi \right) \right) \right\|_\infty \leq \bar{c}_\infty(\omega)\, t^{-(\|\alpha_1\| + \cdots + \|\alpha_m\|)/2} \|\varphi\|_\infty$$

for any $\varphi \in C_b^\infty\left(\mathbb{R}^N\right)$.

Proof: Assume $j = 1$, and $m = 2$. We have

$$\begin{aligned} V_{[\alpha_1]} \pi_t \left(V_{[\alpha_2]} \varphi \right) &= V_{[\alpha_1]} \left[\rho_t^{Y(\omega)} \left(V_{[\alpha_2]} \varphi \right) / \rho_t^{Y(\omega)}(1) \right] \\ &= V_{[\alpha_1]} \rho_t^{Y(\omega)} \left(V_{[\alpha_2]} \varphi \right) \left(\rho_t^{Y(\omega)}(1) \right)^{-1} \\ &\quad + \rho_t^{Y(\omega)} \left(V_{[\alpha_2]} \varphi \right) V_{[\alpha_1]}((\rho_t^{Y(\omega)}(1))^{-1}). \end{aligned}$$

Then

$$\left| V_{[\alpha_1]} \pi_t \left(V_{[\alpha_2]} \varphi \right)(x) \right| \leq \bar{C}_T(\omega)\, t^{-(\|\alpha_1\| + \|\alpha_2\|)/2} \max \|\varphi\|_\infty ,$$

where $\bar{C}_T(\omega) = C_T(\omega)\left((\rho_t^{Y(\omega)}(1))^{-1}(x) + (\rho_t^{Y(\omega)}(1))^{-2}(x)\right)$.

- If $V_i, i = 0, ..., d_1$ satisfy the UFG condition, we cannot guarantee the existence of a density of $\bar{\rho}_t^{Y(\omega)}$ w.r.t. the Lebesgue measure given any starting point.
- $\rho_t^{Y(\omega)}$ will have a density $y \to \bar{\rho}_t^{x,Y(\omega)}(y)$ if for *any* vector field V with coefficients in $C_b^\infty(\mathbb{R}^N)$, there exist $u_{V,\beta} \in C_b^\infty(\mathbb{R}^N)$ satisfying

$$V = \sum_{\beta \in A_1(k)} u_{\alpha,\beta} V_{[\beta]}. \tag{11}$$

- (11) is equivalent to the existence of a positive integer k such that for $i = 1, ..., N$, there exist $u_{i,\beta} \in C_b^\infty(\mathbb{R}^N)$ satisfying

$$\partial_i = \sum_{\beta \in A_1(k)} u_{i,\beta} V_{[\beta]}. \tag{12}$$

  In particular this means that

$$\text{Span}\{V_{[\alpha]}(x) : \alpha \in \mathcal{A}(k)\} = \mathbb{R}^N$$

  holds for all $x \in \mathbb{R}^N$.

Corollary

Assume that $V_i, i = 0, ..., d_1$ *satisfy condition (11),* $\pi_0 = \delta_x$ *and* $h \in C_b^\infty(\mathbb{R}^N)$*. Then, for all* $t > 0$*, the unnormalised conditional distribution of the signal* $\rho_t^{Y(\omega)}$ *has a smooth density* $y \to \bar{\rho}_t^{x,Y(\omega)}(y)$*.*
Moreover for any $T > 0$*, and any multi-index* $\iota = (i_1, ..., i_n) \in \mathbb{A}$ *there exists a random variable* $\bar{C}_{T,\iota}(\omega)$ *almost surely finite such that*

$$\left\| \partial_{i_1} ... \partial_{i_n} \bar{\rho}_t^{x,Y(\omega)} \right\|_1 \leq \bar{C}_{T,\iota}(\omega)\, t^{-\frac{kn}{2}}, \quad t \in (0, T]. \tag{13}$$

If, in addition, $h \in C_0^\infty(\mathbb{R}^N)$ *then, or any* $T > 0$*, any multi-index* $\iota = (i_1, ..., i_n) \in \mathbb{A}$ *there exists a random variable* $\bar{C}_{T,\iota}(\omega)$ *almost surely finite such that for any* $p \in [1, \infty]$*, we have*

$$\left\| \partial_{i_1} ... \partial_{i_n} \bar{\rho}_t^{x,Y(\omega)} \right\|_p \leq \bar{C}_{T,\iota}(\omega)\, t^{-\frac{kn}{2}}, \quad t \in (0, T]. \tag{14}$$

Step 1. Chaos expansion of $\rho_t^{Y(\omega)}$

Introduce the set of operators: $R_{q,\bar{q}}$ where $q = (t_1, t_2, \dots, t_k)$ is a non-empty multi-index with entries $t_1, t_2, \dots, t_k \in [0, \infty)$ that have increasing values $t_1 < t_2 < ... < t_k$ and $\bar{q} = (i_1, ..., i_{k-1})$ is a multi-index with entries $i_1, ..., i_{k-1} \in \{1, 2, ..., m\}$ defined

$$R_{(s,t),\varnothing}(\varphi) = P_{t-s}(\varphi)$$

and, inductively, for $k > 1$,

$$\begin{aligned} R_{(s,t_1,t_2,\dots,t_k),(i_1,\dots,i_{k-1})}(\varphi) &= R_{(s,t_1,t_2,\dots,t_{k-1})}\left(h_{i_{k-1}} P_{t_k - t_{k-1}}(\varphi)\right) \\ &= P_{t_1 - s}\left(h_{i_1} P_{t_2 - t_1} \dots \left(h_{i_{k-1}} P_{t_k - t_{k-1}}(\varphi)\right)\right) \\ &= P_{t_1 - s}\left(h_{i_1} R_{(t_2 - t_1, t_3 - t_1, \dots, t_k - t_1)}(\varphi)\right) \end{aligned}$$

Note that the length of the multi-index $\bar{q}$ is always one unit less that q. $S(m)$ all multi-indices $\bar{q}$ with entries in the set $\{1, ..., m\}$.

Lemma

We have almost surely that

$$\rho_t^x(\varphi) = P_t(\varphi)(x) + \sum_{m=1}^{\infty} \sum_{\bar{q} \in S(m)} R_{0,t}^{m,\bar{q}}(\varphi) \tag{15}$$

where, for $\bar{q} = (i_1, ..., i_m)$*,*

$$R_{0,t}^{m,\bar{q}}(\varphi) = \underbrace{\int_0^t \int_0^{t_m} \dots \int_0^{t_2}}_{m \text{ times}} R_{(0,t_1,\dots,t_m,t),\bar{q}}(\varphi)(x) dY_{t_1}^{i_1} \dots dY_{t_m}^{i_m}.$$

Proof.

Induction. □

Step 2. Pathwise representation of the iterated integrals $R_t^m(\varphi)$.

$$q_{s,t}^k(Y) = \underbrace{\int_s^t \int_s^{t_k} \dots \int_s^{t_2}}_{k \quad \text{times}} dY_{t_1}^{i_1} \dots dY_{t_k}^{i_k}$$

and $q_{\bar{t}}^{\bar{k}}(Y), \bar{k} = (k_1, \dots, k_r)\ t = (t_1, \dots t_r)$ be the products of iterated integrals

$$q_{s,\bar{t}}^{\bar{k}}(Y) = \prod_{i=1}^{r} q_{s,t_i}^{k_i}(Y).$$

We define a formal degree on these products of iterated integrals by letting

$$\deg\left(q_{s,\bar{t}}^{\bar{k}}(Y)\right) = \sum_{i=1}^{r} k_i.$$

Next define the sets Θ_k

$$\Theta_k = \mathrm{sp}\left\{ q_{\bar{t}}^{\bar{k}}(Y),\ \ \bar{k} = (k_1, \dots, k_r), \sum_{i=1}^{r} k_i \le k \right\}.$$

Also for $\bar{q} \in S(k)$ let $\bar{q} \in (i_1, ..., i_k)$ define $\Phi_{\bar{q}}$, $\Psi_{\bar{q}}$, be the following operators

$$\begin{aligned} \Phi_{\bar{q}}\varphi &= h^{i_1}...h^{i_k}\varphi \\ \Psi_{\bar{q}}\varphi &= [\Phi_{\bar{q}}, A](\varphi) = A(h^{i_1}...h^{i_k})\varphi + \sum_{i=1}^{d} V_i(h^{i_1}...h^{i_k})V_i\varphi. \end{aligned}$$

and Γ be the set of operators

$$\Gamma = \{\Phi_{\bar{q}_1}, \Psi_{\bar{q}_2}, \Psi_{\bar{q}_1}\Phi_{\bar{q}_2}, \quad \bar{q}_1, \bar{q}_2 \in S(k),\ k \geq 1\}.$$

Theorem

$$R^m_{s,t}(\varphi) = P_{t-s}(h^m\varphi)(x)\underbrace{\int_s^t\int_s^{t_m}\cdots\int_s^{t_2}}_{m\ \text{times}} dY_{t_1}\ldots dY_{t_m}$$

$$+\sum_{k=1}^{m-1} q^{k,m}_{s,t}(Y)\underbrace{\int_s^t\int_s^{t_k}\cdots\int_s^{t_2}}_{k\ \text{times}} q^{k,m}_{(s,t_1,\ldots,t_k)}(Y)\,\bar{R}^k_{(s,t_1,\ldots,t_k,t)}(\varphi)(x)dt_1\ldots dt_k$$

$$+\sum_{k=1}^{m}\underbrace{\int_s^t\int_s^{t_k}\cdots\int_s^{t_2}}_{k\ \text{times}} \bar{q}^{k,m}_{(s,t_1,\ldots,t_k)}(Y)\,\bar{R}^k_{(s,t_1,\ldots,t_k,t)}(\varphi)(x)dt_1\ldots dt_k, \qquad (16)$$

and $q^{k,m}_{(s,t_1,\ldots,t_k)}(Y)$, $\bar{q}^{k,m}_{(s,t_1,\ldots,t_k)}(Y) \in \Theta_m$ *are linear combinations of iterated integrals of* Y *and* $\bar{R}^k_{(t_1,\ldots,t_k,t)}(\varphi)$ *are given by*

$$\bar{R}^k_{(s,t_1,\ldots,t_k,t)}(\varphi) = P_{t_1-s}\left(\bar{\Phi}_1 P_{t_2-t_1}\ldots\left(\bar{\Phi}_k P_{t-t_k}(\varphi)\right)\right)$$

and $\bar{\Phi}_i, \tilde{\Phi}_i \in \Gamma$, $i = 1, .., k$.

Moreover we have

$$\deg\left(q_{s,t}^{k,m}(Y)\right)+\deg\left(q_{(s,t_1,\ldots,t_k)}^{k,m}(Y)\right)=\deg\left(\bar{q}_{(s,t_1,\ldots,t_k)}^{k,m}(Y)\right)=m. \qquad (17)$$

Proof.

Integration by parts. □

Step 3. Pathwise control of the iterated integrals $q_{s,t}^{k}(Y(\omega))$.

Lemma

For any $1/3<\gamma<1/2$ *there exists a positive random variable* $c=c(\omega,k,\gamma,\|h\|_\infty)$ *independent of* x *and some* $\beta>0$ *such that, almost surely*

$$\left|q_{s,t}^{k}(Y(\omega))\right|\leq\frac{\left(c(\omega,\gamma,\|h\|_\infty)\,|s-t|\right)^{k\gamma}}{\beta(k\gamma)!}$$

for all $0\leq s\leq t\leq 1$.

Step 4. Pathwise control of the terms $R^m_{s,t}$

Theorem

Let $h \in C^\infty_b\left(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}\right)$. There exists $c^m_\infty(\omega)$ such that

$$\left\| \left(V_{[\alpha_1]} \cdots V_{[\alpha_j]} R^m_{s,t} \left(V_{[\alpha_{j+1}]} \cdots V_{[\alpha_m]} \varphi \right) \right) \right\|_\infty \leq c^m_\infty(\omega)\, t^{-(\|\alpha_1\| + \cdots + \|\alpha_m\|)/2} \|\varphi\|_\infty$$

for any $\varphi \in C^\infty_b\left(\mathbb{R}^N\right)$ and $\sum_m c^m_\infty(\omega) < \infty$ a.s.
Let $h \in C^\infty_0\left(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}\right)$. There exists $c^m(\omega)$ such that

$$\left\| \left(V_{[\alpha_1]} \cdots V_{[\alpha_j]} R^m_{s,t} \left(V_{[\alpha_{j+1}]} \cdots V_{[\alpha_m]} \varphi \right) \right) \right\|_p \leq c^m(\omega)\, t^{-(\|\alpha_1\| + \cdots + \|\alpha_m\|)/2} \|\varphi\|_p$$

for all $\varphi \in C^\infty_0\left(\mathbb{R}^N\right)$, $p \in [1, \infty]$ and $\sum_m c^m(\omega) < \infty$ a.s.

Proof.

Kusuoka-Stroock estimates and rough paths techniques. □

- We obtain sharp gradient bounds for perturbed diffusion semigroups. In contrast with existing results, the perturbation is here random and the bounds obtained are pathwise.
- The results build on the classical work of Kusuoka and Stroock and extend their program developed for the heat semi-group to solutions of stochastic partial differential equations.
- The analysis allows us to derive pathwise gradient bounds for the un-normalised conditional distribution of a partially observed signal. We use a pathwise representation of the perturbed semigroup in the spirit of classical work by Ocone.
- The estimates we derive have sharp small time asymptotics.

Vlad attended my undergraduate course in Probability at the Faculty of Mathematics in Bucharest and, in the following year, my course in Stochastic processes for Master students. He graduated in 1979 with distinction. Vlad stood out among his peers through his commitment for high level achievements, never being happy with the easy ones. I recall that he always said that if a lecture course is not a challenge in what concerns the effort of understanding it, then it is boring and it is not worthwhile taking. This attitude continued after his graduation: He enjoyed the challenge of reading the chapter on Local Time in the famous Itô and McKean book. This led him to his Ph.D. thesis in 1985 entitled ”The structure of a class of Markov processes” (I was the supervisor). By its contents, the thesis is remarkable in that it tackles most of the essential problems of the Markov processes theory. Indeed, Vlad's thesis is an honour for the Romanian Doctoral School in Probability Theory.

Recently, Vlad gave a talk at the annual conference of our ”Romanian Society of Probability, Statistics and Operational Research” and I was happy to learn that he continues to carry the Probability Theory flag as strong as ever.

Professor Vlad Bally, I wish you Happy Birthday, long life and continuous mathematical satisfactions!

La multi ani !

Ioan Cuculescu