On the binomial approximation of the American put

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Outline

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In the Black-Scholes model, the stock price at time t is given by

$$S_t = S_0 e^{(r-\delta-\frac{\sigma^2}{2})t+\sigma B_t},$$

where $r > 0$ is the interest rate, $\delta \geq 0$ the dividend rate, and, under the risk neutral, measure, $(B_t)_{t\geq 0}$ is a standard Brownian motion. The price at time t of the American put with maturity T and strike price K, is given by $P(t, S_t)$, where

$$P(t,s) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}_s \left(e^{-r\tau} f(S_\tau) \right),$$

with $f(x) = (K-x)^+$, and $\mathbb{E}_s = \mathbb{E}\left(\cdot \mid S_0 = s\right)$. Here $\mathcal{T}_{0,t}$ denotes the set of all stopping times with respect to the Brownian filtration $\mathbb{F}$.

Note that we also have $P(t,s) = U(T-t, \ln s)$, if we define

$$U(T,x) = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left(e^{-r\tau}\varphi(X_\tau^x)\right),$$

where

$$X_t^x = x + \mu t + \sigma B_t, \quad \text{with } \mu = r - \delta - \frac{\sigma^2}{2}, \text{ and } \varphi(x) = (K - e^x)^+.$$

We know that U solves the variational inequality

$$\max\left(-\frac{\partial U}{\partial t}(t,x) + AU(t,x) - rU(t,x), \varphi - U(t,x)\right) = 0,$$

with initial condition $U(0,.) = \varphi$ where

$$AU = \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2} + \mu\frac{\partial U}{\partial x}, \text{ with } \mu = r - \delta - \frac{\sigma^2}{2}.$$

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With these notations, the exercise boundary (parameterized by time until maturity) is given by

$$b(t) = \inf\{x > 0 \mid U(t,x) > \varphi(x)\}, \quad t > 0,$$

and we have $\lim_{t\downarrow 0} b(t) = \ln K \wedge \ln(rK/\delta)(=: b(0))$. We have, for $t \downarrow 0$,

$$b(0) - b(t) \sim \begin{cases} C\sqrt{t} \text{ if } r < \delta, \\ C\sqrt{t|\ln t|} \text{ if } r \geq \delta. \end{cases}$$

The binomial approximation

We now introduce the random walk approximation of Brownian motion. To be more precise, assume $(X_n)_{n\geq 1}$ is a sequence of i.i.d. real random variables satisfying $\mathbb{E}X_n^2 = 1$ and $\mathbb{E}X_n = 0$, and define, for any positive integer n, the process $B^{(n)}$ by

$$B_t^{(n)} = \sqrt{T/n} \sum_{k=1}^{[nt/T]} X_k, \quad 0 \leq t \leq T,$$

where $[nt/T]$ denotes the greatest integer in nt/T. We will make the additional assumptions that X_1 is bounded and $\mathbb{E}(X_1^3) = 0$. Note that X will denote a random variable with the same distribution as X_1, independent of the sequence $(X_n)_{n\geq 1}$.

In the following, we fix S_0 and set

$$P_0 = P(0, S_0) = U(T, \ln S_0).$$

Note that, if we introduce the notation $g(x) = (K - S_0 e^{\sigma x})^+$, we have

$$P_0 = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left(e^{-r\tau} g(\mu_0 \tau + B_\tau)\right)$$

with $\mu_0 = \mu/\sigma$. We now have a natural approximation of P_0, given by

$$P_0^{(n)} = \sup_{\tau \in \mathcal{T}_{0,T}^{(n)}} \mathbb{E}\left(e^{-r\tau} g(\mu_0 \tau + B_\tau^{(n)})\right),$$

where $\mathcal{T}_{0,T}^{(n)}$ denotes the set of all stopping times (with respect to the natural filtration of $B^{(n)}$), with values in $\{0, T/n, 2T/n, \dots, (n-1)T/n, T\}$.

Theorem

There exists a positive constant C, such that for any positive integer n,

$$-C\frac{(\ln n)^{\beta}}{n} \leq P_0^{(n)} - P_0 \leq C\frac{(\ln n)^{\alpha}}{n},$$

with

$$\alpha = \begin{cases} 5/4, & \text{if } \delta \leq r, \\ 1, & \text{if } \delta > r. \end{cases}$$

and

$$\beta = \begin{cases} 3/2, & \text{if } \delta \leq r, \\ 1, & \text{if } \delta > r. \end{cases}$$

Upper bound for $P_0^{(n)} - P_0$

Introduce the modified value function

$$u(t,x) = e^{-rt}U(T-t, \ln(S_0) + \mu t + \sigma x), \quad t \geq 0, x \in \mathbb{R}.$$

We have $P_0 = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left(e^{-r\tau} g(\mu_0 \tau + B_\tau)\right) = u(0,0)$ and, for $t \in [0,T]$,

$$u(t,x) \geq e^{-rt}(K - S_0 e^{\mu t + \sigma x})^+ = e^{-rt} g(\mu_0 t + x),$$

so that

$$P_0^{(n)} - P_0 \leq \sup_{\tau \in \mathcal{T}_{0,T}^{(n)}} \mathbb{E}\left(u(\tau, B_\tau^{(n)}) - u(0,0)\right)$$

We will also use the notation:

$$h = \frac{T}{n} \; .$$

With this notation, we have

$$B_t^{(n)} = \sqrt{h} \sum_{k=1}^{[t/h]} X_k, \quad 0 \leq t \leq T.$$

We have , for all $t \in \{0, h, 2h, \dots, (n-1)h, nh = T\}$,

$$u(t, B_t^{(n)}) = u(0,0) + M_t + \sum_{j=1}^{t/h} \mathcal{D}u((j-1)h, B_{(j-1)h}^{(n)}),$$

where $(M_t)_{0 \leq t \leq T}$ is a martingale (with respect to the natural filtration of $B^{(n)}$), such that $M_0 = 0$, and

$$\mathcal{D}u(t,x) = \mathbb{E}\left(u\left(t+h, x+\sqrt{h}X\right)\right) - u(t,x), \quad 0 \leq t \leq T-h, \quad x \in \mathbb{R}.$$

Note that, if v is smooth,

$$\mathcal{D}v(t,x) = h\delta v(t,x) + hO(h),$$

where

$$\delta v = \frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}.$$

On the other hand, we have and

$$\begin{aligned} \delta u(t,x) &= e^{-rt}\left(-\frac{\partial U}{\partial t} + (A-r)U\right)(T-t, \ln(S_0) + \mu t + \sigma x) \\ &= e^{-rt}(A-r)\varphi(\ln(S_0) + \mu t + \sigma x)\mathbf{1}_{\{x \le b_0(t)\}}, \end{aligned}$$

where $b_0(t) = (\tilde{b}(T-t) - \mu t - \ln(S_0))/\sigma$. In particular, we have

$$\delta u \le 0$$

and $\delta u = 0$ on the set $\mathcal{C} = \{(t,x) \in (0,T) \times \mathbb{R} \mid x > b_0(t)\}$.

A representation for the operator $\mathcal{D}$

Proposition

Assume that v is a function of class C^3 on $[0,T]\times\mathbb{R}$. We have, for $0\leq t\leq T-h$ and $x\in\mathbb{R}$,

$$\mathcal{D}v(t,x)=\tilde{\mathcal{D}}v(t,x)+2\int_0^{\sqrt{h}}d\xi\int_0^{\xi}dz\mathbb{E}\left(X^2\delta v(t+\xi^2,x+zX)\right),$$

where

$$\tilde{\mathcal{D}}v(t,x) \;=\; 2\int_0^{\sqrt{h}}d\xi\int_0^{\xi}dz(\xi-z)Rv(t,x,\xi,z),$$

with

$$Rv(t,x,\xi,z)=\mathbb{E}\left[X^2\left(\xi-X^2\frac{(\xi-z)}{2}\right)\frac{\partial^3 v}{\partial t\partial x^2}(t+\xi^2,x+zX)\right].$$

From the definition of $\tilde{\mathcal{D}}$, using the boundedness of X, we derive the following estimates.

$$
\begin{aligned}
\left|\tilde{\mathcal{D}}v(t,x)\right| &\leq C\int_0^{\sqrt{h}}\xi^2 d\xi \mathbb{E}\left[\int_0^{\xi} dz|X|\left|\frac{\partial^3 v}{\partial t\partial x^2}(t+\xi^2, x+zX)\right|\right] \\
&\leq C\sqrt{h}\int_t^{t+h} ds\int dy \mathbb{E}\left(\mathbf{1}_{\{|y-x|\leq\sqrt{h}|X|\}}\right)\left|\frac{\partial^3 v}{\partial t\partial x^2}(s,y)\right|,
\end{aligned}
$$

where we have set $s=t+\xi^2$ and $y=x+zX$.

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where we have set $s=t+\xi^2$ and $y=x+zX$.
It can be proved, using classical Berry-Esseen estimates, that, for $k\in(1,3]$,

$$\mathbb{P}\left(\left|B_{jh}^{(n)}-y\right|\leq\sqrt{h}|X|\right) \leq \frac{C_k}{\sqrt{j}\,(1+|y|^k)}.$$

Hence

$$\sum_{j=1}^{n-2}\mathbb{E}\left(\left|\tilde{\mathcal{D}}v(jh,B_{jh}^{(n)})\right|\right) \leq C_k h\sqrt{2}\int_0^{T-h}\frac{ds}{\sqrt{s}}\int\frac{dy}{1+|y|^k}\left|\frac{\partial^3 v}{\partial t\partial x^2}(s,y)\right|.$$

Quadratic estimates for the second order time derivative

We now introduce the difference $\tilde{U} = U - \bar{U}$, where $\bar{U}$ is defined by

$$\bar{U}(t,x) = e^{-rt}\mathbb{E}(\varphi(X_t^x)), \quad t \geq 0,\ x \in \mathbb{R}.$$

We have the following L_2-estimate for the second time derivative of $\tilde{U} = U - \bar{U}$. Here $\nu_k(dx) = dx/(1+x^2)^{k/2}$

Theorem

Fix $T > 0$ and $k > 1$. There exists a constant $C > 0$ such that, for all $\xi \in (0, T]$,

$$\int_{\xi}^{T} (t-\xi) \left\| \frac{\partial^2 \tilde{U}}{\partial t^2}(t,.) \right\|_{L_2(\nu_k)}^2 dt \leq C\left(1 + |\ln \xi|^{\alpha}\right),$$

with

$$\alpha = \begin{cases} 3/2, & \text{if } \delta \leq r, \\ 1, & \text{if } \delta > r. \end{cases}$$

Lower bound for $P_0^{(n)} - P_0$

To derive a lower bound for $P^{(n)} - P$, we introduce the following stopping time:

$$\tau = \tau_1 \mathbf{1}_{\{\tau_1 < T-h\}} + T\mathbf{1}_{\{\tau_1 = T-h\}},$$

where

$$\tau_1 = \inf\left\{ t \in [0, T-h] \mid t/h \in \mathbb{N} \text{ and } d(B_t^{(n)}, I_{t+h}) \leq \kappa\sqrt{h} \right\},$$

and $I_{t+h} = (-\infty, b_0(t+h)]$ (and $I_T = \mathbb{R}$). The positive constant κ is chosen so that

$$x > b_0(t+h) + \kappa\sqrt{h} \Rightarrow [t, t+h] \times [x - \sqrt{h}||X||_\infty, x + \sqrt{h}||X||_\infty] \subset \mathcal{C},$$

which implies $\mathcal{D}u(t,x) = \tilde{\mathcal{D}}u(t,x)$.

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which implies $\mathcal{D}u(t,x) = \tilde{\mathcal{D}}u(t,x)$.
We have

$$\begin{aligned} P^{(n)} - P &\geq \mathbb{E}\left(e^{-r\tau} g(\mu_0 \tau + B_\tau^{(n)}) - u(0,0)\right) \\ &= \mathbb{E}\left(e^{-r\tau} g(\mu_0 \tau + B_\tau^{(n)}) - u(\tau, B_\tau^{(n)})\right) \\ &\quad + \mathbb{E}\left(u(\tau, B_\tau^{(n)}) - u(0,0)\right). \end{aligned}$$

Note that $\{\tau \geq T-h\} = \{\tau = T\}$, and, on $\{\tau = T\}$, $u(\tau, B_\tau^{(n)}) - e^{-r\tau} g(\mu_0 \tau + B_\tau^{(n)})$. On the other hand, on $\{\tau < T-h\}$, one can prove that

$$\left| u(\tau, B_\tau^{(n)}) - e^{-r\tau} g(\mu_0 \tau + B_\tau^{(n)}) \right| \leq C \frac{h}{\sqrt{T-\tau-h}}.$$

$\diamond$

Lemma

There exists a positive constant C such that

$$\mathbb{E}\left(\frac{1}{\sqrt{T-\tau-h}} \mathbf{1}_{\{\tau \leq T-2h\}} \right) \leq C \left(\log n\right)^{\alpha},$$

with

$$\alpha = \begin{cases} 3/2, & \text{if } \delta \leq r, \\ 1, & \text{if } \delta > r. \end{cases}$$