Restoring Uniqueness in Mean-Field Games by Randomizing the Equilibria

Vlad Bally 60th Birthday

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Le Mans

1. Background

Controlled dynamics

- N interacting players (state in $\mathbb{R}$)
  - controlled players with mean-field interaction
  - deterministic dynamics of player number $i \in \{1, \dots, N\}$

$$dX_t^i = \Big(b(X_t^i, \bar{\mu}_t^N) + \alpha_t^i\Big)dt, \ t \in [0, T]$$

  - i.i.d. initial conditions $X_0^1, \dots, X_0^N$, $\quad \bar{\mu}_t^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{X_t^j}$
  - choose $\underbrace{\alpha_t^i}_{\text{at any } t}$

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  - choose $\underbrace{\alpha_t^i}_{\text{at any } t}$
- Willing to minimize cost/energy $J^i(\alpha^1, \ldots, \alpha^N)$

$$J^i(\ldots) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T \Big(f(X_t^i, \bar{\mu}_t^N) + \tfrac{1}{2}|\alpha_t^i|^2\Big)dt\Big]$$

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- $(\alpha^{1,\star}, \ldots, \alpha^{N,\star})$ Nash equilibrium if

$$J^i(\ldots, \alpha^{i-1,\star}, \alpha^i, \alpha^{i+1,\star}, \ldots) \geq J^i(\ldots, \alpha^{i-1,\star}, \alpha^{i,\star}, \alpha^{i+1,\star}, \ldots)$$

Principle of MFG

• Define the asymptotic equilibrium state of the population as the solution of a fixed point problem

(1) fix a flow of probability measures $(\mu_t)_{0 \le t \le T}$ (in $\mathcal{P}_2(\mathbb{R})$)

(2) solve the deterministic optimal control problem in the environment $(\mu_t)_{0 \le t \le T}$

$$dX_t = \Big(b(X_t, \mu_t) + \alpha_t\Big)dt$$

- with X_0 random
- with cost $J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T \Big(f(X_t, \mu_t) + \frac{1}{2}|\alpha_t|^2\Big)dt\Big]$

(3) let $(X_t^{\star,\mu})_{0 \le t \le T}$ be the unique optimizer (under nice assumptions) $\rightsquigarrow$ find $(\mu_t)_{0 \le t \le T}$ such that

$$\mu_t = \mathcal{L}(X_t^{\star,\mu}), \quad t \in [0, T]$$

• Proof of convergence is non-trivial $\rightsquigarrow$ recent works only

Program

- Existence of equilibria

  ∘ proved by compactness arguments using PDE or probabilistic description of the optimal control problem

- Uniqueness of equilibria

  ∘ difficult question $\rightsquigarrow$ known when $b \equiv 0$ and f and g are monotonous in the direction of the measure

$$\int_{\mathbb{R}} \Big(f(x,\mu) - f(x,\mu') \Big) d(\mu - \mu')(x) \geq 0$$

  ∘ same condition on $g \rightsquigarrow$ monotonicity condition of the same type as for solving Burgers equation

  ∘ known example $\rightsquigarrow$ local cost $f(x,\mu)$ increases when local mass at x increases

- Goal of the talk: remove monotonicity

  ∘ strategy is to randomize equilibria $\rightsquigarrow$ inspired from restoration of uniqueness for ODEs/SDEs

2. Forward-backward system

McKean-Vlasov forward backward

- Characterize MFG equilibrium through McKean-Vlasov forward-backward system
  - use the Pontryagin principle (when $b(x,\mu) \equiv b(\mu)$)
- When $(\mu_t)_{0 \le t \le T}$ is frozen, solve

$$
\begin{aligned}
dX_t &= (b(\mu_t) - Y_t)dt \\
dY_t &= -\partial_x f(X_t, \mu_t)dt, \quad Y_T = \partial_x g(X_T, \mu_T)
\end{aligned}
$$

  - when $\partial_x f$ and $\partial_x g$ non-decreasing and Lipschitz in $x \rightsquigarrow$ unique solution
  - forward path is optimal for control problem in $(\mu_t)_{0 \le t \le T}$
- Implement the MFG condition
  - solve forward-backward system with $\mu_t = \mathcal{L}(X_t) \rightsquigarrow$ McKean-Vlasov system
  - law is upon the randomness in the initial condition $\rightsquigarrow$ understand monotonicity in μ as a parallel with monotonicity in x

Randomizing the solution

- Aim is to get rid of monotonicity in μ
  - ∘ strategy is to randomize the state variable $\rightsquigarrow \mathcal{L}(X_t)$!
  - ∘ force the dynamics so that smoothing effect in the direction of the measure
  - ∘ instead of forcing the law $\rightsquigarrow$ force the random variable itself seen as an element of L^2 space
- Construct the initial condition on $L^2(\mathbb{S}^1)$ with $\mathbb{S}^1$ = circle
  - ∘ random variables $X_t, Y_t : \mathbb{S}^1 \to \mathbb{R}$ and $\mathcal{L}(X_t) = \text{Leb}_{\mathbb{S}^1} \circ X_t^{-1}$
- Dynamics rewrite

$$dX_t(x) = \Big(b(\text{Leb}_{\mathbb{S}^1} \circ X_t^{-1}) - Y_t(x)\Big)dt$$

$$dY_t(x) = -\partial_x f(X_t(x), \text{Leb}_{\mathbb{S}^1} \circ X_t^{-1})dt$$

$$Y_T(x) = \partial_x g(X_T(x), \text{Leb}_{\mathbb{S}^1} \circ X_T^{-1}), \quad x \in \mathbb{S}^1$$

  - ∘ force the dynamics with infinite dimensional white noise!

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\begin{aligned}
dX_t(x) &= \Big(b(\mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1}) - Y_t(x)\Big)dt \qquad + dB_t(x) \\
dY_t(x) &= -\partial_x f(X_t(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1})dt \\
Y_T(x) &= \partial_x g(X_T(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_T^{-1}), \quad x \in \mathbb{S}^1
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\begin{aligned}
dX_t(x) &= \Big(b(\mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1}) - Y_t(x)\Big)dt + \partial_x^2 X_t(x)dt + dB_t(x) \\
dY_t(x) &= -\partial_x f(X_t(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1})dt \\
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dY_t(x) &= -\partial_x f(X_t(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1})dt + ??? \\
Y_T(x) &= \partial_x g(X_T(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_T^{-1}), \quad x \in \mathbb{S}^1
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  ◦ force the dynamics with infinite dimensional white noise!

Infinite dimensional forward-backward

- Look at the system

$$dX_t(x) = \Big(b(\mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1}) - Y_t(x)\Big)dt + \partial_x^2 X_t(x)dt + dB_t(x)$$
$$dY_t(x) = -\partial_x f(X_t(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1})dt + ???$$
$$Y_T(x) = \partial_x g(X_T(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_T^{-1}), \quad x \in \mathbb{S}^1$$

  - ∘ B_t $L^2(\mathbb{S}^1)$-valued white noise
  - ∘ X_t random element of $L^2(\mathbb{S}^1) \rightsquigarrow \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1}$ random measure
  - ∘ ??? $= dM_t$ martingale w.r.t filtration generated by B

- Formal stochastic Pontryagin for the optimization of

$$\mathbb{E}\Big[\int_{\mathbb{S}^1} g(U_T(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_T^{-1})dx + \int_0^T \int_{\mathbb{S}^1} \Big[f(U_t(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1}) + \tfrac{1}{2}|\alpha_t(x)|^2\Big]dxdt\Big]$$

  - ∘ over $dU_t(x) = b(\mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1})dt + \alpha_t(x)dt + \partial^2 X_t(x)dt + dB_t(x)$

Solvability results

- Assumptions

  ◦ $\partial_x f$, $\partial_x g$ non-decreasing in x $\rightsquigarrow$ use of Pontryagin principle

  ◦ b, $\partial_x f$, $\partial_x g$ bounded and Lipschitz $\rightsquigarrow$ use the 2-Wasserstein distance to make it compatible with the L^2 framework:

$$W_2^2(\mu,\nu) = \inf \mathbb{E}[|X - Y|^2], \quad X \sim \mu, \; Y \sim \nu$$

- Theorem: Existence and uniqueness for any initial condition

  ◦ $Y_t = \mathcal{U}(t, X_t)$ where $\mathcal{U}$ mild solution of infinite dimensional system of PDEs on $L^2(\mathbb{S}^1)$ (Zabczyk, Fuhrman et al...)

$$\partial_t \mathcal{U}(t,X) + D\mathcal{U}(t,X)\cdot b\Big(\mathrm{Leb}_{\mathbb{S}^1} \circ X^{-1}\Big) - D\mathcal{U}(t,X)\cdot \mathcal{U}(t,X)$$

$$+ \partial_x f\Big(X, \mathrm{Leb}_{\mathbb{S}^1} \circ X^{-1}\Big) + L\mathcal{U}(t,X) = 0$$

$$\mathcal{U}(T,X) = \partial_x g\Big(X, \mathrm{Leb}_{\mathbb{S}^1} \circ X^{-1}\Big)$$

  ◦ where D is Fréchet derivative and L is Ornstein-Uhlenbeck operator on $L^2(\mathbb{S}^1)$ $\rightsquigarrow$ viscous mollification of MFG master equation

$$LU(t,X) = \frac{1}{2}\mathrm{Trace}(D^2 U(t,X)) + \langle DU(t,X), \partial^2 X\rangle_{L^2(\mathbb{S}^1)}$$

Sketch of proof

- Cauchy Lipschitz theory works in small time

  ∘ small time $\rightsquigarrow$ depends upon Lipschitz constant of terminal condition $\mathcal{U}(T,\cdot)$

- Aim at propagating

  ∘ need a priori bound for Lipschitz constant of $\mathcal{U}(t,\cdot)$

  ∘ given by the smoothing property of Ornstein-Ulhenbeck operator

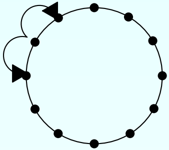
$$\sup_{h\in L^2(\mathbb{S}^1)} \left| D(e^{tL}\varphi)(h) \right| \leq C t^{-1/2} \sup_{h\in L^2(\mathbb{S}^1)} \left| \varphi(h) \right|$$

  ∘ control the Lipschitz constant away from the boundary using mild formulation

- Next: Connection with games? Zero-noise limit?

3. Connection with games

Approximating particle system



- Consider N particles
  - particle k located at $\exp(i2\pi k/N)$ on $\mathbb{S}^1$
  - $X_t^k \rightsquigarrow$ state of particle number k

- Mean-field plus local interactions to nearest neighbors

$$dX_t^k = \Big(b(\bar{\mu}_t^N) - Y_t^k + \underbrace{N^2(X_t^{k+1} + X_t^{k-1} - 2X_t^k)}_{\text{discrete Laplace}}\Big)dt + \sqrt{N}dB_t^k$$

$$dY_t^k = -\partial_x f(X_t^k, \bar{\mu}_t^N)dt + d\text{martingale}_t, \quad Y_T^k = \partial_x g(X_T^k, \bar{\mu}_T^N)$$

  - $B^1, \ldots, B^N$ independent Brownian motions

$$\sqrt{N}dB_t^k = N \int_{k/N}^{(k+1)/N} dB_t(x)$$

- Expect $\underbrace{X_t^k}_{\text{discrete state}} \approx N \int_{k/N}^{(k+1)/N} \underbrace{X_t(x)}_{\text{limiting state}} dx$
  -

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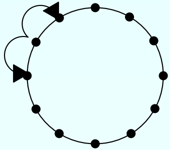

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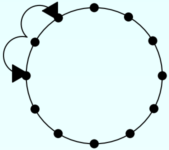
  - $B^1, \dots, B^N$ independent Brownian motions

$$\sqrt{N}dB_t^k = N \int_{k/N}^{(k+1)/N} dB_t(x)$$

- Expect $\displaystyle\sum_{k=0}^{N-1} X_t^k 1_{[k/N,(k+1)/N)} \approx X_t$
  - Claim: $\bar{\mu}_t^N \overset{\text{u.c.p}}{\longrightarrow} \text{Leb}_{\mathbb{S}^1} \circ X_t^{-1}$ limit law of mollified model

Approximating particle system



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  - $B^1, \dots, B^N$ independent Brownian motions

$$\sqrt{N} dB_t^k = N \int_{k/N}^{(k+1)/N} dB_t(x)$$

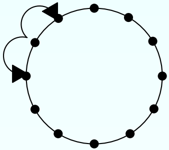
- Expect $\displaystyle\sum_{k=0}^{N-1} X_t^k \mathbf{1}_{[k/N,(k+1)/N)} \approx X_t$
  - Proof: Expand master PDE along discrete the dynamics $\rightsquigarrow$ nearly solution of forward-backward

Interpretation as a game

- Interpret the particle system as a game?
  - cannot be Nash over

$$dX_t^i = \big(b(\bar{\mu}_t^N) + \alpha_t^i + N^2(X_t^{i+1} + X_t^{i-1} - 2X_t^i)\big)dt + \sqrt{N}dB_t^i$$

  - local interaction too sensitive to variation of α^i
- Strategy $\rightsquigarrow$ combine local and mean-field
  - consider N^2 particles instead of N
  - N particles per site
  - $X_t^{k,j} \rightsquigarrow$ state of particle nb. k at site j



- Consider Nash system for

$$dX_t^{k,j} = \Big(b(\bar{\mu}_t^N) + \alpha_t^{k,j} + N\sum_{j=1}^{N}(X_t^{k+1,j} + X_t^{k-1,j} - 2X_t^{k,j})\Big)dt + \sqrt{N}dB_t^k$$

  - $B^1, \ldots, B^N \perp\!\!\!\perp$ Brownian motions and $\bar{\mu}_t^N = N^{-2}\sum_{k,j}\bar{X}_t^{k,j}$
  - Claim: Nash equilibrium $\underset{N\to\infty}{\longrightarrow}$ mollified solution

4. Zero noise limit

Small noise system

- Consider small viscosity $\varepsilon > 0$

$$dX_t(x) = \Big(b(\mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1}) - Y_t(x)\Big)dt + \varepsilon^2 \partial_x^2 X_t(x)dt + \varepsilon dB_t(x)$$
$$dY_t(x) = -\partial_x f(X_t(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1})dt + d\mathrm{martingale}_t$$
$$Y_T(x) = \partial_x g(X_T(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_T^{-1}), \quad x \in \mathbb{S}^1$$

  - $(X_t, Y_t)_{0 \le t \le T} \rightsquigarrow (X_t^\varepsilon, Y_t^\varepsilon)_{0 \le t \le T}$

- Limits as $\varepsilon \searrow 0$? (initial law of X_0 being fixed)
  - $\big((\mu_t^\varepsilon = \mathrm{Leb}_{\mathbb{S}^1} \circ (X_t^\varepsilon)^{-1})_{0 \le t \le T}\big)_{\varepsilon \in (0,1)}$ tight on $\mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}))$

- Claim: Weak limits $(\mu_t)_{0 \le t \le T}$ are random equilibria of original MFG
  - $(\mu_t)_{0 \le t \le T}$ random process $\perp\!\!\!\perp X_0 \sim \mu_0$, $\mathbb{F} \rightsquigarrow$ canonical filtration

$$dX_t = \Big(b(X_t, \mu_t) + \alpha_t\Big)dt, \quad X_0 \sim \mu_0$$

  - with cost $J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T \Big(f(X_t, \mu_t) + \frac{1}{2}|\alpha_t|^2\Big)dt\Big]$

$$\mu_t = \mathcal{L}(X_t^{\star,\mu} | (\mu_s)_{0 \le s \le t}), \quad t \in [0, T]$$

Toy example in $d = 1$

- Choose the coefficients
  - $b(\mu) = b(\int_{\mathbb{R}} x' d\mu(x'))$
  - $f(x,\mu) = xf\Big(\int_{\mathbb{R}} x' d\mu(x')\Big)$
  - $g(x,\mu) = xg\Big(\int_{\mathbb{R}} x' d\mu(x')\Big)$
- Equilibria must be Gaussian! $\rightsquigarrow$ characterized by mean only
  - forward path of

$$\dot{x}_t = b(x_t) - y_t, \quad \dot{y}_t = -f(x_t), \quad y_T = g(x_T)$$

- characteristics system of inviscid Burgers PDE

$$-\partial_t v(t,x) = \partial_x v(t,x)\Big(b(x) - v(t,x)\Big) + f(x), \quad v(T,x) = g(x)$$

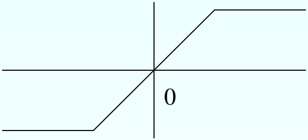
  - well-posed if $f, g \nearrow \Rightarrow$! of characteristics
  - if not $\Rightarrow$ shocks may emerge in finite time...

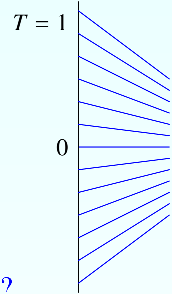
Plots of the characteristics

- Consider the simple example $b \equiv 0, f \equiv 0$

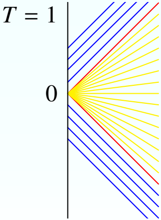
- Plots of the characteristics

if $g(x) = (-1 \vee x \wedge 1)$





- Which limit to select when no uniqueness?
  - When starting from 0, select extremal characteristics with probability 1/2 (generalization of Bafico-Baldi...)



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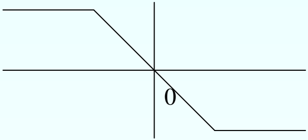

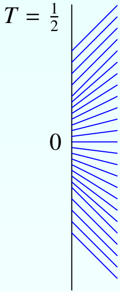

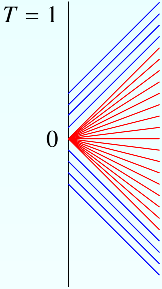

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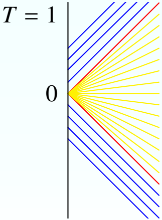

Bon anniversaire !
La mulți ani !