LAN property for sde's with additive fractional noise and continuous time observation

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The Ornstein-Uhlenbeck process

- $dX_t = -\theta X_t\, dt + dB_t, \qquad t \in [0, \tau], \quad \theta > 0.$
- B_t is a standard Brownian motion.
- Let $\hat{\theta}_\tau$ be the MLE of θ from the continuous observation of X in $[0, \tau]$.
- Then, it is well-known

  $$\lim_{\tau \to \infty} \hat{\theta}_\tau = \theta \quad \text{a.s.}$$

  and that

  $$\mathcal{L}(\mathbf{P}_\theta) - \lim_{\tau \to \infty} \sqrt{\tau}(\hat{\theta}_\tau - \theta) = \mathcal{N}(0, 2\theta).$$

- where $\mathbf{P}_\theta$ is the probability law of the solution in the space $\mathcal{C}(\mathbf{R}_+; \mathbf{R})$.

The LAN propety for the Ornstein-Uhlenbeck process

- The parametric statistical model $\{\mathbf{P}_\theta,\ \theta \in \Theta\}$ satisfies the LAN property at $\theta \in \Theta$ with rate $\sqrt{\tau}$ since for any $u \in \mathbf{R}$, as $\tau \to \infty$:

$$\log\left(\frac{d\mathbf{P}^{\tau}_{\theta+\frac{u}{\sqrt{\tau}}}}{d\mathbf{P}^{\tau}_{\theta}}\right) \xrightarrow{\mathcal{L}(\mathbf{P}_\theta)} u\mathcal{N}\left(0, \frac{1}{2\theta}\right) - \frac{u^2}{4\theta},$$

  where $\mathbf{P}^{\tau}_{\theta}$ is probability law of the solution in the space $\mathcal{C}([0,\tau];\mathbf{R})$.

- The local log likelihood ratio is asymptotically normal, with a locally constant covariance matrix and a mean equal to minus one half the variance.

Consequence of the LAN property

- **Minimax Theorem :** Let $(\hat{\theta}_\tau)_{\tau\geq 0}$ be a family of estimators of the parameter θ. Then

$$\lim_{\delta\to 0}\liminf_{\tau\to\infty}\sup_{|\theta'-\theta|<\delta}\mathbf{E}_{\theta'}\left[\tau(\hat{\theta}_\tau-\theta')^2\right]\geq 2\theta.$$

- In particular, the MLE is asymptotic minimax efficient.
- The LAN property is an important tool in order to quantify the identifiability of a system. Started by Le Cam'60. Parallel theory to Cramér-Rao bound.

LAN property for ergodic diffusions

- Consider a non-linear d-dimensional ergodic diffusion

$$dX_t = b(X_t;\theta)dt + \sigma(X_t)dB_t, \qquad t \in [0,\tau], \quad \theta \in \Theta \subset \mathbf{R}^q.$$

- Under regularity, ellipticity, and ergodic assumptions, for any $\theta \in \Theta$ and $u \in \mathbf{R}^q$, as $\tau \to \infty$:

$$\log\left(\frac{d\mathbf{P}^{\tau}_{\theta+\frac{u}{\sqrt{\tau}}}}{d\mathbf{P}^{\tau}_{\theta}}\right) \xrightarrow{\mathcal{L}(\mathbf{P}_\theta)} u^{\mathrm{T}}\mathcal{N}(0,\Gamma(\theta)) - \frac{1}{2}u^{\mathrm{T}}\Gamma(\theta)u,$$

where $\overline{X}$ is the ergodic limit of X, and

$$\Gamma(\theta) = \mathbf{E}_\theta[\partial_\theta b(\overline{X};\theta)^{\mathrm{T}}\sigma^{-1}(\overline{X})^{\mathrm{T}}\sigma^{-1}(\overline{X})\partial_\theta b(\overline{X};\theta)].$$

- **Proof :** Girsanov's theorem, CLT for martingales and ergodicity.
- **Consequence : Minimax theorem :**

$$\lim_{\delta\to 0}\liminf_{\tau\to\infty}\sup_{|\theta'-\theta|<\delta}\mathbf{E}_{\theta'}\left[\tau(\hat{\theta}_\tau-\theta')^2\right] \geq \Gamma(\theta)^{-1}.$$

The fractional Ornstein-Uhlenbeck process

- $X_t = -\theta \int_0^t X_s \, ds + B_t, \qquad t \in [0, \tau], \quad \theta > 0.$
- B_t fractional Brownian motion with Hurst parameter $H > 1/2$.
- $\mathbf{P}_\theta$ is the probability law of the solution in the space $\mathcal{C}^\lambda(\mathbf{R}_+; \mathbf{R})$, for any $\lambda < H$.
- Let $\hat{\theta}_\tau$ be the MLE of θ from the continuous observation of X in $[0, \tau]$.
- Then, it is well-known

  $$\lim_{\tau \to \infty} \hat{\theta}_\tau = \theta \quad \text{a.s.}$$

  and that

  $$\mathcal{L}(\mathbf{P}_\theta) - \lim_{\tau \to \infty} \sqrt{\tau}(\hat{\theta}_\tau - \theta) = \mathcal{N}(0, 2\theta).$$

- This suggests that the LAN property holds with the same rate $\sqrt{\tau}$.

Ergodic sde's with additive fractional noise

$$X_t = x_0 + \int_0^t b(X_s;\theta)\,ds + \sum_{j=1}^d \sigma_j B_t^j, \qquad t \in [0,\tau].$$

- $\theta \in \Theta$, where Θ is compactly embedded in $\mathbf{R}^q$.
- ergodicity condition : $\langle b(x;\theta) - b(y;\theta),\, x-y\rangle \le -\alpha\,|x-y|^2$.
- $\hat{b}$ is the Jacobian matrix $\partial_\theta b$.
- assumptions : $\partial_x b$, $\partial_{xx} b$, $\partial_x \hat{b}$, $\partial_{xx}\hat{b}$ bounded, $b, \hat{b}$ linear growth, $\hat{b}$ Lipschitz in θ and x, and σ invertible.
- The solution converges for $t \to \infty$ a.s. to a unique stationary process $(\overline{X}_t, t \ge 0)$.
- $\mathbf{P}_\theta^\tau$ is the probability laws of the solution in the spaces $\mathcal{C}^\lambda([0,\tau];\mathbf{R}^d)$, for any $\lambda < H$.

The LAN property

Theorem : For any $\theta \in \Theta$ and $u \in \mathbf{R}^q$, as $\tau \to \infty$,

$$\log\left(\frac{d\mathbf{P}^{\tau}_{\theta+\frac{u}{\sqrt{\tau}}}}{d\mathbf{P}^{\tau}_{\theta}}\right) \xrightarrow{\mathcal{L}(\mathbf{P}_\theta)} u^{\mathrm{T}}\mathcal{N}(0,\Gamma(\theta)) - \frac{1}{2}u^{\mathrm{T}}\Gamma(\theta)u,$$

where the matrix $\Gamma(\theta)$ is defined by

$$\Gamma(\theta) = \int_{\mathbf{R}^2_+} \frac{\mathbf{E}_\theta[(\hat{b}(\overline{X}_0;\theta) - \hat{b}(\overline{X}_{r_1};\theta))^{\mathrm{T}}(\sigma^{-1})^{\mathrm{T}}\sigma^{-1}(\hat{b}(\overline{X}_0;\theta) - \hat{b}(\overline{X}_{r_2};\theta))]}{r_1^{1/2+H} r_2^{1/2+H}}\, dr_1 dr_2.$$

Remark : The efficiency of the MLE in the fractional Ornstein-Uhlenbeck case remains open....

Steps of the proof of the LAN property

- Use the representation of the fBm given in Hairer'05 (introduced by Mandelbrot and Van Ness'68) which is suitable to get the desired ergodic properties.
- Apply Girsanov's theorem for the fBm following Moret and Nualart'02.
- Handle the singularities popping out the fractional derivatives in the Girsanov exponent.
- Get ergodic results in Hölder type norms for our process X.
- In order to apply a CLT for Brownian martingales, we use Malliavin calculus techniques : derive concentration properties for the Girsanov exponents by means of a Poincaré type inequality (Üstunel'95), which needs to conviniently upper bound some Malliavin derivatives.

Mendelbrot and Van Ness representation of fBm

Let W be a two sided Wiener process, then the following defines a two-sided fBm : for any $t \in \mathbf{R}$:

$$B_t = c_H \int_{\mathbf{R}_-} (-r)^{H-1/2} \left[dW_{t+r} - dW_r\right]$$
$$= \quad c_H \left\{ \int_{-\infty}^{0} \left[(-(r-t))^{H-1/2} - (-r)^{H-1/2} \right] dW_r - \int_0^t (-(r-t))^{H-1/2} dW_r \right\}.$$

<u>Abstract Wiener space :</u> $(\mathcal{B}, \bar{\mathcal{H}}, \mathbf{P})$, where

$$\mathcal{B} = \left\{ f \in \mathcal{C}(\mathbf{R};\ \mathbf{R}^d);\ \frac{|f_t|}{1+|t|} < \infty \right\},$$

P is the law of our fBm, and h is an element of the Cameron-Martin space $\bar{\mathcal{H}}$ iff there exists an element X_h in the first chaos such that

$$h_t = \mathbf{E}[B_t X_h], \quad \text{and} \quad \|h\|_{\bar{\mathcal{H}}} = \|X_h\|_{L^2(\Omega)}.$$

Properties of the SDE

- **Proposition :** There exists a unique continuous pathwise solution on any arbitrary interval $[0, \tau]$ such that :
- The map $X : (x_0, B) \in \mathbf{R}^d \times \mathcal{C}([0,\tau];\mathbf{R}^d) \to \mathcal{C}([0,\tau];\mathbf{R}^d)$ is locally Lipschitz continuous.
- For any $\theta \in \Theta$, $p \geq 1$, and $s, t \geq 0$,

$$\mathbf{E}\left[|X_t|^p\right] \leq c_p, \qquad \text{and} \quad \mathbf{E}\left[|\delta X_{st}|^p\right] \leq k_p|t-s|^{pH},$$

  where δ denotes the increment.
- For all $\varepsilon \in (0, H)$ there exists a random variable $Z_\varepsilon \in \cap_{p\geq 1} L^p(\Omega)$ such that a.s.

$$|X_t| \leq Z_\varepsilon \left(1+t\right)^{2\varepsilon}, \quad \text{and} \quad |\delta X_{st}| \leq Z_\varepsilon \left(1+t\right)^{2\varepsilon} |t-s|^{H-\varepsilon},$$

  uniformly for $0 \leq s \leq t$.

Ergodic properties of the SDE

- Garrido-Atienza, Kloeden and Neuenkirch'09 :
- Shift operators $\theta_t : \Omega \to \Omega : \theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \qquad t \in \mathbb{R}, \quad \omega \in \Omega.$
- The shifted process $(B_s(\theta_t \cdot))_{s\in\mathbb{R}}$ is still a d-dimensional fractional Brownian motion and for any integrable random variable $F : \Omega \to \mathbb{R}$

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau F(\theta_t(\omega))\, dt = \mathbf{E}[F],$$

  for $\mathbf{P}$-almost all $\omega \in \Omega$.
- **Theorem :** There exists a random variable $\overline{X} : \Omega \to \mathbb{R}^d$ such that

$$\lim_{t\to\infty} |X_t(\omega) - \overline{X}(\theta_t\omega)| = 0$$

  for $\mathbf{P}$-almost all $\omega \in \Omega$. Moreover, we have $\mathbf{E}[|\overline{X}|^p] < \infty$ for all $p \geq 1$.

Ergodic properties of the SDE

- **Theorem :** For any $\theta \in \Theta$ and any $f \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R})$ such that

$$|f(x)| + |\partial_x f(x)| \leq c\left(1 + |x|^N\right), \qquad x \in \mathbf{R}^d,$$

we have

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau f(X_t)\, dt = \mathbf{E}[f(\overline{X})], \qquad \mathbf{P}\text{-}a.s.$$

- **Proposition :** Let $\alpha \in (0, H)$. There exists a random variable Z admitting moments of any order such that for all $0 \leq s \leq t$

$$\left|X_t - \overline{X}_t\right| \leq Z\, e^{-cs} \quad \text{and} \quad \left|\delta\left[X - \overline{X}\right]_{st}\right| \leq Z\, e^{-cs}(t-s)^\alpha.$$

Operator that transforms W into B

Proposition : For $w \in \mathcal{C}_c^\infty(\mathbf{R})$ and $H \in (0,1)$, set

$$[K_H w]_t = c_H \int_{\mathbf{R}_-} (-r)^{H-1/2} \left[\dot{w}_{t+r} - \dot{w}_r\right] dr.$$

Then : (i) There exists a constant c_H such that

$$[K_H w]_t = \begin{cases} -c_H \left([I_+^{H-1/2} w]_t - [I_+^{H-1/2} w]_0 \right), & \text{for } H > \frac{1}{2} \\ -c_H \left([D_+^{1/2-H} w]_t - [D_+^{1/2-H} w]_0 \right), & \text{for } H < \frac{1}{2}, \end{cases}$$

where

$$[D_+^\alpha \varphi]_t = c_\alpha \int_{\mathbf{R}_+} \frac{\varphi_t - \varphi_{t-r}}{r^{1+\alpha}} \, dr, \quad \text{and} \quad [I_+^\alpha \varphi]_t = \tilde{c}_\alpha \int_{\mathbf{R}_+} \varphi_{t-r} \, r^{\alpha-1} \, dr.$$

(ii) For $H > 1/2$, K_H can be extended as an isometry from $L^2(\mathbf{R})$ to $I_+^{H-1/2}(L^2(\mathbf{R}))$.
(iii) There exists a constant c_H such that $K_H^{-1} = c_H K_{1-H}$.

Girsanov's transformation

Proposition : For a given $\theta \in \Theta$, onsider the d-dimensional process

$$Q_t = \int_0^t \sigma^{-1} b(X_s; \theta)\, ds + B_t.$$

Then Q is a d-dimensional fractional Brownian motion under the probability $\tilde{\mathbf{P}}_\theta$ defined by $\frac{d\tilde{\mathbf{P}}_\theta}{d\mathbf{P}_\theta}|_{[0,\tau]} = e^{-L}$, with

$$L = \int_0^\tau \langle \sigma^{-1}[D_+^{H-1/2} b(X;\theta)]_u, dW_u\rangle + \frac{1}{2}\int_0^\tau |\sigma^{-1}[D_+^{H-1/2} b(X;\theta)]_u|^2\, du.$$

Proof : show that $D_+^{H-1/2} b(X;\theta)$ is well defined on $[0,\tau]$, and Novikov's condition : there exists $\lambda > 0$ such that

$$\sup_{t\in[0,\tau]} \mathbf{E}_\theta \left[\exp\left(\lambda \int_0^t |\sigma^{-1}[D_+^{H-1/2} b(X;\theta)]_s|^2 ds \right)\right] < \infty.$$

Proof of the LAN property

- **Step 1 :** Apply Girsanov's theorem. Fix $\theta \in \Theta$, and set $\theta_\tau = \theta + \tau^{-1/2} u$. Then

$$\log\left(\frac{d\mathbf{P}^\tau_{\theta_\tau}}{d\mathbf{P}^\tau_\theta}\right) = -\int_0^\tau \langle \sigma^{-1}([D_+^{H-1/2} b(X;\theta_\tau)]_t - [D_+^{H-1/2} b(X;\theta)]_t), dW_t\rangle$$
$$- \frac{1}{2}\int_0^\tau |\sigma^{-1}([D_+^{H-1/2} b(X;\theta_\tau)]_t - [D_+^{H-1/2} b(X;\theta)]_t)|^2 dt.$$

- **Step 2 :** Linearize this relation : add and substract the d-dimensional vector

$$[D_+^{H-1/2}\hat{b}(X;\theta)]_t(\theta_\tau - \theta) = \frac{1}{\sqrt{\tau}}[D_+^{H-1/2}\hat{b}(X;\theta)]_t,$$

where $\hat{b} = \partial_\theta b$.

Step 2 : linearization

$$\log\left(\frac{d\mathbf{P}^{\tau}_{\theta_\tau}}{d\mathbf{P}^{\tau}_{\theta}}\right) = I_1 - I_2 - \frac{1}{2} I_3 - I_4,$$

where

$$I_1 = \frac{1}{\sqrt{\tau}} \int_0^{\tau} \langle \sigma^{-1} [D_+^{H-1/2} \hat{b}(X;\theta)]_t u, dW_t \rangle - \frac{1}{2\tau} \int_0^{\tau} |\sigma^{-1} [D_+^{H-1/2} \hat{b}(X;\theta)]_t u|^2 dt$$

$$I_2 = \int_0^{\tau} \langle \sigma^{-1} ([D_+^{H-1/2} b(X;\theta_\tau)]_t - [D_+^{H-1/2} b(X;\theta)]_t - [D_+^{H-1/2} \hat{b}(X;\theta)]_t (\theta_\tau - \theta)), dW_t \rangle$$

$$I_3 = \int_0^{\tau} |\sigma^{-1} ([D_+^{H-1/2} b(X;\theta_\tau)]_t - [D_+^{H-1/2} b(X;\theta)]_t - [D_+^{H-1/2} \hat{b}(X;\theta)]_t (\theta_\tau - \theta))|^2 dt$$

$$I_4 = \int_0^{\tau} \langle \sigma^{-1} ([D_+^{H-1/2} b(X;\theta_\tau)]_t - [D_+^{H-1/2} b(X;\theta)]_t - [D_+^{H-1/2} \hat{b}(X;\theta)]_t (\theta_\tau - \theta)), \sigma^{-1} [D_+^{H-1/2} \hat{b}(X;\theta)]_t (\theta_\tau - \theta) \rangle dt.$$

Remaining steps of the proof

- **Step 3 :** Main contribution to our log-likelihood : we show that as $\tau \to \infty$

$$\frac{1}{\tau}\int_0^\tau |\sigma^{-1}[D_+^{H-1/2}\hat{b}(X;\theta)]_t u|^2 dt \xrightarrow{\mathbf{P}_\theta} u^{\mathrm{T}}\Gamma(\theta)u.$$

- Together with multivariate central limit theorem for Brownian martingales implies that as $\tau \to \infty$

$$I_1 \xrightarrow{\mathcal{L}(\mathbf{P}_\theta)} u^{\mathrm{T}}\mathcal{N}(0,\Gamma(\theta)) - \frac{1}{2}u^{\mathrm{T}}\Gamma(\theta)u.$$

- **Step 4 :** Negligible contributions : We show that the terms I_2, I_3 and I_4 converge to zero in $\mathbf{P}_\theta$-probability as $\tau \to \infty$.
- For I_3 apply Taylor's expansion and some computations, I_3 is the quadratic variation of the martingale I_2, and by Cauchy-Schwarz inequality, I_4 is bounded by I_3.

Proof of Step 3

Step 3a : We have that

$$\frac{1}{\tau}\int_0^\tau |\sigma^{-1}[D_+^{H-1/2}\hat{b}(X;\theta)]_t u|^2 dt \equiv \frac{1}{\tau}J_\tau(X) = \frac{1}{\tau}\int_0^\tau \left|\sigma^{-1}N_t(X)\right|^2 dt,$$

where

$$N_t(X) = \int_{\mathbf{R}_+} \frac{(\hat{b}(X_t;\theta) - \hat{b}(X_{t-r};\theta))u}{r^{H+1/2}}\, dr = N_{1,t}(X) + N_{2,t}(X)$$

$$= \int_0^t \frac{(\hat{b}(X_t;\theta) - \hat{b}(X_{t-r};\theta))u}{r^{H+1/2}}\, dr + \int_t^\infty \frac{(\hat{b}(X_t;\theta) - \hat{b}(x_0;\theta))u}{r^{H+1/2}}\, dr$$

where we have set $X_t = x_0$ for all $t \leq 0$.

We denote by $J_\tau(\overline{X}), N_t(\overline{X}), N_{1,t}(\overline{X}), N_{2,t}(\overline{X})$ the same quantities with X replaced by $\overline{X}$.

Proof of Step 3

Step 3b : We show that

- $N_t(X) - N_t(\overline{X})$ is of order $t^{-\eta} Z$ with $\eta > 0$ and $Z \in \cap_{p\geq 1} L^p(\Omega)$.
- Hence $J_\tau(X) - J_\tau(\overline{X})$ is of order $\tau^{1-2\eta}$, which is a negligible term on the scale τ.
- This allows us to consider the limiting behavior of $J_\tau(\overline{X})$ instead of $J_\tau(X)$.

Proof of Step 3

Step 3c : This step is devoted to reduce our computations to an evaluation for the expected value.

- We show that

$$\lim_{\tau\to\infty} \frac{1}{\tau} \mathbf{E}_\theta \left[|J_\tau(X) - \mathbf{E}_\theta[J_\tau(X)]| \right] = 0,$$

- **Poincaré type inequality :** Let $F : \mathcal{B} \to \mathbf{R}$ be a functional in $\mathbb{D}^{1,2}$. Then,

$$\mathbf{E}\left[|F - \mathbf{E}[F]|\right] \leq \frac{\pi}{2} \mathbf{E}\left[\|DF\|_{\bar{\mathcal{H}}}\right]$$

- This reduces to show that

$$\lim_{\tau\to\infty} \frac{\mathbf{E}_\theta\left[\|DJ_\tau(X)\|_{\bar{\mathcal{H}}}\right]}{\tau} = 0.$$

Proof of Step 3

Proposition : For all $t > 0$, X_t belongs to $\mathbb{D}^{1,2}$, and the Malliavin derivative satisfies that

$$\|DX_t\|_{\bar{\mathcal{H}}} \leq c \exp\left(-\frac{\alpha\, t}{2}\right),$$

uniformly in $t \in \mathbf{R}_+$. Moreover, for $0 \leq u \leq v$,

$$\|D(\delta X_{uv})\|_{\bar{\mathcal{H}}} \leq c_1 \exp\left(-\frac{\alpha\, u}{2}\right) (v-u)^{H/2},$$

uniformly in u and v.

Idea of proof : Derive contraction properties of the map $h \to X^h$, $h \in \bar{\mathcal{H}}$, where X^h is the solution to our SDE driven by $B + h$:

$$|X_t^h - X_t| \leq c \exp\left(-\frac{\alpha t}{2}\right) \|h\|_{\bar{\mathcal{H}}},$$

uniformly in $t \in \mathbf{R}_+$.

Proof of Step 3

Step 3d : We are now reduced to the analysis of the quantity $\mathbf{E}_\theta[J_\tau(\overline{X})]$.

- This is equal to $u^{\mathrm{T}}\Psi u$ where Ψ equals the matrix

$$\int_0^\tau dt \int_{\mathbf{R}_+^2} \frac{\mathbf{E}_\theta[(\hat{b}(\overline{Y}_t;\theta)-\hat{b}(\overline{Y}_{t-r_1};\theta))^{\mathrm{T}}(\sigma^{-1})^{\mathrm{T}}\sigma^{-1}(\hat{b}(\overline{Y}_t;\theta)-\hat{b}(\overline{Y}_{t-r_2};\theta))]}{r_1^{1/2+H} r_2^{1/2+H}}\, dr_1\, dr_2$$

- By stationarity of $\overline{Y}$, the expected value does not depend on t and

$$|\mathbf{E}_\theta[(\hat{b}(\overline{Y}_0;\theta)-\hat{b}(\overline{Y}_{r_1};\theta))^{\mathrm{T}}(\sigma^{-1})^{\mathrm{T}}\sigma^{-1}(\hat{b}(\overline{Y}_0;\theta)-\hat{b}(\overline{Y}_{r_2};\theta))]| \lesssim \left(r_1^H \wedge 1\right)\left(r_2^H \wedge 1\right)$$

- We obtain that Ψ is a convergent integral and

$$\mathbf{E}_\theta[J_\tau(\overline{Y})] = \tau u^{\mathrm{T}}\Gamma(\theta)u.$$

BON ANNIVERSAIRE VLAD ! ! ! !