Singular perturbation control problems: a BSDE approach

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Conference in honour of Vlad Bally

We consider a two scale system of controlled ∞-dimensional SDEs:

$$dX_t^{u,v} = \Big(AX_t^{u,v} + b(X_t^{u,v}, Q_t^{\epsilon,u,v}, u_t)\Big)\, dt + dW^1(t), \qquad X_0^{u,v} = x^0,$$

$$\epsilon dQ_t^{\epsilon,u,v} = (BQ_t^{\epsilon,u,v} + F(X_t^{u,v}, Q_t^{\epsilon,u,v}) + G\rho(v_t))\, dt + \sqrt{\epsilon}\, G dW^2(t) \qquad Q_0^{\epsilon} = q,$$

the slow variable X takes its values in the Hilbert space H

the fast variable Q^{ϵ} takes its values in the Hilbert space K,

$\epsilon \in]0,1]$ is a small parameter.

$A : D(A) \subset H \to H$ and $B : D(B) \subset K \to K$ are unbounded linear operators generating C_0- semigroups.

$(W_t^i)_{t \geq 0}$, $i = 1, 2$, are independent cylindrical Wiener processes with values in H and K respectively, moreover $G \in \mathcal{L}(K)$

u and v are controls adapted to the filtration generated by (W^1, W^2). They take values in suitable topological spaces U and V respectively.

$$dX_t^{u,v} = \left(AX_t^{u,v} + b(X_t^{u,v}, Q_t^{\epsilon,u,v}, u_t)\right) dt + dW^1(t), \qquad X_0^{u,v} = x^0,$$

$$\epsilon dQ_t^{\epsilon,u,v} = (BQ_t^{\epsilon,u,v} + F(X_t^{u,v}, Q_t^{\epsilon,u,v}) + G\rho(v_t))\, dt + \sqrt{\epsilon}\, G dW^2(t) \qquad Q_0^\epsilon = q_0$$

- F and b Lipschitz and Gateaux differentiable w.r.t. X and Q

- b and ρ are bounded

- the semigroups generated by A and B are Hilbert Schmidt and

$$|e^{sA}|_{L_2(\Xi,H)} + |e^{sB}|_{L_2(\Xi,K)} \leq \frac{L}{(1 \wedge s)^\gamma}, \qquad 0 \leq \gamma \leq 1/2$$

- $B + F$ is dissipative with respect to Q e.g.

$$\langle (q - q'), B(q - q') + F(x, q - q') \rangle \leq -\eta |q - q'|^2$$

for a suitable $\eta > 0$ and all $x \in H$, q, $q' \in K$

$$dX_t^{u,v} = \left(AX_t^{u,v} + b(X_t^{u,v}, Q_t^{\epsilon,u,v}, u_t)\right) dt + dW^1(t), \qquad X_0^{u,v} = x^0,$$

$$\epsilon dQ_t^{\epsilon,u,v} = (BQ_t^{\epsilon,u,v} + F(X_t^{u,v}, Q_t^{\epsilon,u,v}) + G\rho(v_t))\, dt + \sqrt{\epsilon}\, G dW^2(t) \qquad Q_0^\epsilon = q_0$$

We consider the following optimal control problem

$$J^\epsilon(u,v) = \mathbb{E} \int_0^T \left(l_1(X_t^{u,v}, Q_t^{\epsilon,u,v}, u_t) + l_2(X_t^{u,v}, Q_t^{\epsilon,u,v}, u_t)\right) dt$$

and the corresponding value function $V(\epsilon) = \inf_{u,v} J^\epsilon(u,v)$

Our purpose is to study the limit of $V(\epsilon)$ as $\epsilon \to 0$.

Idea: if we freeze the slow evolution then the control problem for the quick one behaves like the optimal state of an ergodic control problem.

Thus Ergodic BSDEs must be involved here!

Ergodic BSDEs

Consider the following system in infinite horizon

$$\begin{cases} -d\check{Y}_s = \left[\Psi(U_s, \check{\Xi}_s) - \lambda\right] ds - \check{\Xi}_s dW_s, \quad s \geq 0 \\ dU_s = [LU_s + F(U_s)]ds + GdW_s \quad s \geq 0 \\ U_0 = u_0 \end{cases}$$

Assume that $L + F(\cdot)$ is dissipative and Ψ is Lipschitz w.r.t. $\check{\Xi}$ bounded w.r.t. U then the above system admits a unique solution $((U_t), (\check{Y}_t), (\check{\Xi}_t), \lambda)$ with

$$\check{Y}_s \leq C(1 + |U_s|)$$

where C can be chosen to depend only on the Lipschitz constant of Ψ and on the dissipativity constant of $L + F(\cdot)$.
Moreover λ is the value function of an ergodic control problem (both in Cesaro and in Abel sense)

see [M.Fuhrman, Y. Hu, G.T. 2009], [A. Debussche, Y. Hu 2011], [Y. Hu, P.Y. Madec, A. Richou 2013]])

BSDE reformulation of the problem

Recall that we have

$$dX_t^{u,v} = \left(AX_t^{u,v} + b(X_t^{u,v}, Q_t^{\epsilon,u,v}, u_t)\right) dt + dW^1(t),\ X_0^{u,v} = x^0,$$
$$\epsilon dQ_t^{\epsilon,u,v} = (BQ_t^{\epsilon,u,v} + F(X_t^{u,v}, Q_t^{\epsilon,u,v}) + G\rho(v_t))\, dt + \sqrt{\epsilon}\, GdW^2(t),\ Q_0^\epsilon = q,$$

thus if

$$\psi(x,q,p,\xi) = \inf_{u\in U}\{pb(x,q,u) + l_1(x,q,u)\} + \inf_{v\in V}\{l_2(x,q,v) + \xi\rho(v)\}$$

$$\begin{cases} dX_t &= AX_t + dW_t^1, \\ \epsilon dQ_t^\epsilon &= (BQ_t^\epsilon + F(X_t^\epsilon, Q_t^\epsilon)\, dt + \epsilon^{1/2}\, GdW_t^2, \\ -dY_t^\epsilon &= \psi(X_t^\epsilon, Q_t^\epsilon, Z_t^\epsilon, \Xi_t^\epsilon/\sqrt{\epsilon})\, dt - Z_t^\epsilon d\,W_t^1 - \Xi_t^\epsilon dW_t^2, \\ X_0^\epsilon &= x_0, \quad Q_0^\epsilon = q_0, \quad Y_1^\epsilon = 0. \end{cases}$$

then

$$V(\epsilon) = Y_0^\epsilon$$

Proof: Usual elimination of control by change of $\mathbb{P}$ argument.
Notice that the 'fast' controlled equation reeds

$$dQ_t^{\epsilon,u,v} = \quad .... \quad + \epsilon^{-1/2} G(\epsilon^{-1/2}\rho(v_t) + dW_t^2)$$

The parametrized ergodic BSDE

Fix $x \in H$ and $p \in H^*$ we consider the following version of the fast equation (notice that time has been stretched that is $\widehat{Q}_t = Q_{\epsilon t}$, $\widehat{W}^2_t = e^{-1/2} W^2_{\epsilon t}$)

$$d\widehat{Q}^{x,q_0}_s = B\widehat{Q}^{x,q_0}_s + F(x, \widehat{Q}^{x,q_0}_s)\, ds + d\widehat{W}^2_s; \quad Q^{x,q_0}_0 = q_0$$

Theorem 1 $\forall x \in H, p \in H^*$ *(and* $Q_0 \in K$*),* $\exists!$ *solution*

$$(Y^{x,q_0,p}, \Xi^{x,q,p}, \lambda^{x,p})$$

of the infinite horizon ergodic BSDE

$$-d\breve{Y}^{x,q_0,p}_t = [\psi(x, \widehat{Q}^{x,q_0}, p, \breve{\Xi}^{x,q_0,p}_t) - \lambda(x,p)]\, dt - \breve{\Xi}^{x,q_0,p}_t dW^2_t, \quad \forall\, t \geq 0$$

Moreover

$$|\breve{Y}^{x,q_0,p}_t| \leq c(1 + |\widehat{Q}^{x,q_0}_t|)$$

where $c > 0$ *only depends on the Lipschitz constants of* ψ *with respect to* Q *and on the dissipativity constant of* $B + F(x, \cdot\,)$*.*

The corresponding parametyrized - ergodic Control Problem

$\lambda(x,p)$ is the value function of an ergodic control problem with state equation

$$d\widehat{Q}_s = B\widehat{Q}_s^v + F(x,\widehat{Q}_s^v)\,ds + G\rho(v_s)ds + Gd\widehat{W}_s^2, \qquad \widehat{Q}_0^v = q_0$$

and cost

$$J(x,p,v) = \lim_{\delta\to 0} \mathbb{E}\,\delta \int_0^\infty e^{-\delta s}\psi^1(x,Q_s^{x,v},p) + l_2(x,Q_s^{x,v},v_s))\,ds$$

where we recall $\psi^1(x,q,p) = \inf_{u\in U}\left\{pb(x,Q_s^{x,v},u) + l_1(x,q,u)\right\}$

This implies that λ is Lipschitz in p and x.

Limit equation and main result

We can now introduce the limit forward-backward system:

$$\begin{cases} d\bar{Y}_t &= -\lambda(X_t, \bar{Z}_t)\, dt + \bar{Z}\, dW^1_t, \quad t \in [0,1), \quad \bar{Y}_1 = 0, \\ dX_t &= AX_t\, dt + dW^1_t, \quad X_0 = x_0 \end{cases}$$

Recall the f.b. system for the original, two scales control problem:

$$\begin{cases} dX_t &= AX_t + dW^1_t, \quad t \in [0,1) \\ \epsilon dQ^\epsilon_t &= (BQ^\epsilon_t + F(X^\epsilon_t, Q^\epsilon_t)\, dt + \sqrt{\epsilon}\, dW^2_t, \quad t \in [0,1) \\ -dY^\epsilon_t &= \psi(X^\epsilon_t, Q^\epsilon_t, Z^\epsilon_t, \Xi^\epsilon_t/\sqrt{\epsilon})\, dt - Z^\epsilon_t d\,W^1_t - \Xi^\epsilon_t dW^2_t, \quad t \in [0,1) \\ X^\epsilon_0 &= x_0 \quad Q^\epsilon_0 = q_0, \quad Y^\epsilon_1 = 0. \end{cases}$$

Theorem 2 (Main result)

$$\lim_{\epsilon \to 0} |Y^\epsilon_0 - \bar{Y}_0| = 0$$

[Alvarez-Bardi 2001-2007] for the finite dimensional counterpart by viscosity solutions techniques.
Also see [Kabanov-Pergamenshchikov 2003], ...

Proof: a freezing/discretization argument

The idea is to freeze the slow equation to give time to the fast equation to behave as the optimal ergodic state. We start from

$$\begin{aligned} Y_0^\epsilon - \bar{Y}_0 &= \int_0^1 (\psi(X_t, Q_t^\epsilon, Z_t^\epsilon, \Xi_t^\epsilon/\sqrt{\epsilon}) - \lambda(X_t, \bar{Z}_t))\, dt \\ &\quad + \int_0^1 (Z_t^\epsilon - \bar{Z}_t)\, dW_t^1 + \int_0^1 \Xi_t^\epsilon\, dW_t^2. \end{aligned}$$

Adding and subtracting the term: $\int_0^1 (\psi(X_t, Q_t^\epsilon, \bar{Z}_t, \Xi_t^\epsilon/\sqrt{\epsilon})dt$ that eventually will be easily treated by a change of probability we are left with

$$\int_0^1 (\psi(X_t, Q_t^\epsilon, \bar{Z}_t, \Xi_t^\epsilon/\sqrt{\epsilon}) - \lambda(X_t, \bar{Z}_t))\, dt + \int_0^1 (Z_t^\epsilon - \bar{Z}_t)\, dW_t^1 + \int_0^1 \Xi_t^\epsilon\, dW_t^2.$$

Let $t_k = k2^{-N}$, $k = 0, 1, \ldots, 2^N - 1$ and define for $t_k \leq t < t_{k+1}$:

$$X^N(t) = X(t_k), \quad Z^N(t) = 2^N \int_{t_{k-1}}^{t_k} \bar{Z}_s \, ds.$$

Fixed k we consider the system (with stretched time) for $t \geq t_k/\epsilon$:

$$-d\check{Y}_t^{N,k} = [\psi(X_{t_k}, \widehat{Q}_t^{N,k}, Z_{t_k}^{N,k}, \check{\Xi}_t^{N,k}) - \lambda(X_{t_k}, Z_{t_k}^{N,k})] \, dt - \widehat{\Xi}_t^{N,k} d\widehat{W}_t^2,$$

$$d\widehat{Q}_t^{N,k} = (B\widehat{Q}_t^{N,k} + F(X_{t_k}, \widehat{Q}_t^{N,k})) \, dt + d\widehat{W}_t^2, \quad Q_{t_k/\epsilon}^{N,k} = Q_{t_k/\epsilon}^{N,k-1},$$

Recall that the above system admits a unique solution $(\widehat{Y}^{N,k}, \widehat{\Xi}^{N,k}, \lambda(X_{t_k}^N, Z_{t_k}^N))$ such that $|\widehat{Y}_t^{N,k}| \leq c(1 + |\widehat{Q}_t^{N,k}|)$

If we set $\widehat{Q}_t^N = \widehat{Q}_t^{N,k}$, $\check{\Xi}_t^N = \check{\Xi}_t^{N,k}$ for $t \in [t_k/\epsilon, t_{k+1}/\epsilon[$ we have

$$\check{Y}^{N,k}_{t_{k+1}/\epsilon} - \check{Y}^{N,k}_{t_k/\epsilon} = \int_{t_k/\epsilon}^{t_{k+1}/\epsilon} [\psi(X^N_{\epsilon t}, \widehat{Q}^N_t, Z^N_{\epsilon t}, \check{\Xi}^N_t) - \lambda(X^N_{\epsilon t}, Z^N_{\epsilon t})]\, dt + \int_{t_k/\epsilon}^{t_{k+1}/\epsilon} \check{\Xi}^N_t \, d\widehat{W}^2_t.$$

therefore:

$$\sum_{k=1}^{2^N} \left[(\check{Y}^{N,k}_{t_k/\epsilon} - \check{Y}^{N,k}_{t_{k+1}/\epsilon}) + \int_{t_k/\epsilon}^{t_{k+1}/\epsilon} \check{\Xi}^N_t \, d\widehat{W}^2_t \right] + - \int_0^{1/\epsilon} [\psi(X^N_{\epsilon t}, \widehat{Q}^N_t, Z^N_{\epsilon t}, \check{\Xi}^N_t) - \lambda(X^N_{\epsilon t}, Z^N_{\epsilon t})]\, dt = 0$$

Recall that we had to estimate (after change of time, that is for: $\widehat{Q}^\epsilon_t := Q^\epsilon_{\epsilon t}$, $\widehat{\Xi}^\epsilon_t := \Xi^\epsilon_{\epsilon t}/\sqrt{\epsilon}$)

$$\epsilon \int_0^{1/\epsilon} (\psi(X_{\epsilon t}, \widehat{Q}^\epsilon_t, \bar{Z}_{\epsilon t}, \widehat{\Xi}^\epsilon_t) - \lambda(X_{\epsilon t}, \bar{Z}_{\epsilon t}))\, dt$$
$$+\sqrt{\epsilon} \int_0^{1/\epsilon} (Z^\epsilon_{\epsilon t} - \bar{Z}_{\epsilon t})\, d\widehat{W}^1_t + \int_0^{1/\epsilon} \widehat{\Xi}^\epsilon_t \, dW^2_t.$$

Adding the term (in blue) that we have proved to be null we get

$$\begin{aligned} Y^\epsilon_0 - \bar{Y}_0 = \ & \epsilon \int_0^{1/\epsilon} \mathcal{R}^{\epsilon,N}_t \, dt + \epsilon \sum_{k=1}^{N} (\check{Y}^{N,k}_{t_k/\epsilon} - \check{Y}^{N,k}_{t_{k+1}/\epsilon}) \\ & + \epsilon \int_0^{1/\epsilon} (\check{\Xi}^N_t - \widehat{\Xi}^\epsilon_t)\, dW^2_t + \epsilon^{\frac{1}{2}} \int_0^{1/\epsilon} (Z^\epsilon_{\epsilon t} - \bar{Z}_{\epsilon t})\, dW^1_t \\ & + \epsilon \int_0^{1/\epsilon} [\psi(X^N_{\epsilon t}, \widehat{Q}^N_t, Z^N_{\epsilon t}, \widehat{\Xi}^\epsilon_t) - \psi(X^N_{\epsilon t}, \widehat{Q}^N_t, Z^N_{\epsilon t}, \check{\Xi}^N_t)]\, dt \\ & + \epsilon \int_0^{1/\epsilon} [\psi(X_{\epsilon t}, \widehat{Q}^\epsilon_t, Z^\epsilon_{\epsilon t}, \widehat{\Xi}^\epsilon_t) - \psi(X_{\epsilon t}, \widehat{Q}^\epsilon_t, \bar{Z}_{\epsilon t}, \widehat{\Xi}^\epsilon_t)]\, dt \end{aligned}$$

where

$$|\mathcal{R}^{\epsilon,N}_t| \le L(|X^\epsilon_{\epsilon t} - X^N_{\epsilon t}| + |\widehat{Q}^\epsilon_t - \widehat{Q}^N_t| + |\bar{Z}_{\epsilon t} - Z^N_{\epsilon t}|) \qquad (1)$$

We get rid of some terms by Girsanov. Let:

$$\delta^1(t) = \frac{\psi(X_{\epsilon t}, \widehat{Q}^\epsilon_t, Z^\epsilon_{\epsilon t}, \widehat{\Xi}^\epsilon_t) - \psi(X_{\epsilon t}, \widehat{Q}^\epsilon_t, \bar{Z}_{\epsilon t}, \widehat{\Xi}^\epsilon_t)}{Z^\epsilon_{\epsilon t} - \bar{Z}_{\epsilon t}}$$

and

$$\delta^2(t) = \frac{\psi(X^N_{\epsilon t}, \widehat{Q}^N_t, Z^N_{\epsilon t}, \widehat{\Xi}^\epsilon_t) - \psi(X^N_{\epsilon t}, \widehat{Q}^N_t, Z^N_{\epsilon t}, \check{\Xi}^N_t)}{\widehat{\Xi}^\epsilon_t - \check{\Xi}^N_t}$$

We set for $s \in [0,1]$:

$$\widetilde{W}^1_s =: \int_0^s \delta^1(t/\epsilon)\, dt + W^1_s, \quad \widetilde{W}^2_s =: \epsilon^{-1/2} \int_0^s \delta^2(t/\epsilon)\, dt + W^2_s$$

We denote by $\widetilde{\mathbb{E}}^\epsilon$ the expectation with respect to the probability $\mathbb{Q}^\epsilon$ under which $(\widetilde{W}^1_s, \widetilde{W}^2_s)$ is a brownian motion (notice that both δ^1 and δ^2 are bounded uniformly in ϵ and N).
Since the left hand side is deterministic, we have

$$Y^\epsilon_0 - \bar{Y}_0 = \widetilde{\mathbb{E}}^\epsilon \int_0^1 \mathcal{R}^{\epsilon,N}_t\, dt + \epsilon \widetilde{\mathbb{E}}^\epsilon \sum_{k=1}^N (\check{Y}^{N,k}_{t_k/\epsilon} - \check{Y}^{N,k}_{t_{k+1}/\epsilon}) \tag{2}$$

We now have to estimate the expectation on the 'error' in the new probability

$$|\widetilde{\mathbb{E}}^\epsilon \int_0^1 \mathcal{R}^{\epsilon,N}_{t/\epsilon}\, dt| \leq \widetilde{\mathbb{E}}^\epsilon L \int_0^1 (|X_t - X_t^N| + |\widehat{Q}^\epsilon_{t/\epsilon} - \widehat{Q}^N_{t/\epsilon}| + |\bar{Z}_t - Z_t^N|)\, dt$$

Let us start from $\widetilde{\mathbb{E}}^\epsilon \int_0^1 |X_t - X_t^N| dt$

We notice that, with respect to $\widetilde{W}^1$, $(X_t)_{t\geq 0}$ satisfies

$$dX(t) = AX(t)dt - \delta^1(t)dt + d\widetilde{W}^1(t), \quad X_0 = x^0$$

Again by Girsanov since $\delta^1\psi^\epsilon$ is uniformly bounded

$$\widetilde{\mathbb{E}}^\epsilon \int_0^1 |X_t - X_t^N| dt \leq C_{\delta^1}\mathbb{E}\left[\int_0^1 |X_t - X_t^N|^2 dt\right]^{1/2} := C\Delta^X(N)$$

By the continuity of trajectories of $(X_t)_{t\geq 0}$ and integrability of $\sup_{t\in[0,1]}|X_t|$ we get

$$\lim_{N\to\infty} \Delta^X(N) = 0 \tag{3}$$

Concerning the term

$$\widetilde{\mathbb{E}}^{\epsilon} \int_0^1 |\bar{Z}_t - Z_t^N| dt$$

by the same argument

$$\widetilde{\mathbb{E}}^{\epsilon} \int_0^1 |\bar{Z}_t - Z_t^N| dt \leq C_{\delta 1} \mathbb{E} \left[\int_0^1 |\bar{Z}_t - Z_t^N|^2 dt \right]^{1/2} := C_{\delta 1} \Delta^Z(N)$$

and $\Delta^Z(N) \to= 0$ by construction of Z^N

Let us come to the term:

$$\widetilde{\mathbb{E}}^\epsilon \int_0^1 |Q_t^\epsilon - \widehat{Q}^N_{t/\epsilon}|\, dt = \epsilon\, \widetilde{\mathbb{E}}^\epsilon \int_0^{1/\epsilon} L|\widehat{Q}_t^\epsilon - \widehat{Q}_t^N|\, dt$$

With respect to $\widehat{\widetilde{W}}{}^2_t := \epsilon^{1/2}\widetilde{W}^2_{\epsilon t}$ the process $(\widehat{Q}_t^\epsilon)_{t\in[0,1/\epsilon]}$ solves

$$d\widehat{Q}_t^\epsilon = (B\widehat{Q}_t^\epsilon + F(X_{\epsilon t}, \widehat{Q}_t^\epsilon))\, dt + \delta^2(t)\, dt + d\widehat{\widetilde{W}}{}^2_t, \quad t \geq 0, \quad \widehat{Q}_0^\epsilon = q_0,$$

and $\widehat{Q}_t^N$ solves

$$d\widehat{Q}_t^N = (B\widehat{Q}_t^N + F(X^N_{\epsilon t}, \widehat{Q}_t^N))\, dt + \delta^2\, dt + d\widehat{\widetilde{W}}{}^2_t, \quad t \geq 0, \quad Q_0 = q_0,$$

thus $\widehat{Q}_t^\epsilon - \widehat{Q}_t^N$ is a the solution to:

$$d[\widehat{Q}_t^\epsilon - \widehat{Q}_t^N] = B(\widehat{Q}_t^\epsilon - \widehat{Q}_t^N)\, dt + F(X_{\epsilon t}, \widehat{Q}_t^\epsilon) - F(X^N_{\epsilon t}, \widehat{Q}_t^N))\, dt, \quad Q_0 = 0.$$

And since $B + F(x, \cdot)$ is dissipative we still can say that, $\mathbb{P}$-a.s.

$$\epsilon \int_0^{1/\epsilon} |\widehat{Q}_t^\epsilon - \widehat{Q}_t^N|\, dt \leq \epsilon \int_0^{1/\epsilon} |X_{\epsilon t} - X^N_{\epsilon t}|\, dt = \int_0^1 |X_t - X_t^N|\, dt.$$

thus

$$\mathbb{E}^\epsilon \int_0^1 L|Q^\epsilon_{\epsilon t} - Q^N_{\epsilon t}|\, dt \leq C\Delta^X(N)$$

Now we come to the last term.

Recalling that

$$|\check{Y}_s^{N,k}| \leq c(1 + |\widehat{Q}_s^N|) \text{ and } \widetilde{\mathbb{E}}^\epsilon \sup_{s\geq 0} |\widehat{Q}_s^N|^2 \leq \widetilde{C}$$

we get

$$|\epsilon\widetilde{\mathbb{E}}^\epsilon \sum_{k=1}^N (\widehat{Y}_{t_k/\epsilon}^{N,k} - \widehat{Y}_{t_{k+1}/\epsilon}^{N,k})| \leq \epsilon \sum_{k=1}^N \widetilde{\mathbb{E}}^\epsilon (1 + |\widehat{Q}_{t_k/\epsilon}^N| + |\widehat{Q}_{t_{k+1}/\epsilon}^N|) \leq \epsilon N(1 + 2\widetilde{C})$$

At last we sum up all results to get

$$\begin{aligned} |Y_0^\epsilon - \bar{Y}_0| &\leq \widetilde{\mathbb{E}}^\epsilon \int_0^1 |\mathcal{R}_{t/\epsilon}^{\epsilon,N}|\, dt + \epsilon |\widetilde{\mathbb{E}}^\epsilon \sum_{k=1}^N (\check{Y}_{t_k/\epsilon}^{N,k} - \check{Y}_{t_{k+1}/\epsilon}^{N,k})| \\ &\leq \Delta^X(N) + \Delta^Z(N) + \epsilon N(1 + 2\widetilde{C}) \end{aligned}$$

So our claim follow letting ϵ tend to 0 and then N to ∞.

Things to do

I) Allow degenerate noise in the slow equation.

In this case the state equation reeds

$$dX_t^{u,v} = \left(AX_t^{u,v} + b(X_t^{u,v}, Q_t^{\epsilon,u,v})\right) dt + \Gamma r(u_t) dt + \Gamma dW^1(t),$$

and after Ghirsanov transf. the solw equation still depends on Q

$$dX_t = (AX_t + b(X_t, Q_t))\, dt + \Gamma dW^1(t),$$

We have to use averaging arguments similar to [Cerrai 99] or [Bréhier 2012] takin into account that the law of the solution to the fast equation will converge towards an optimal invariant measure.

II ?): Guess some information on the convergence of the optimal controls

Grazie

e buon compleanno!