

# Simulation of SDE with discontinuous drift

Antoine Lejay (Inria Nancy Grand-Est)

From a joint work with  
Arturo Kohatsu-Higa (Ritsumeikan University)  
& Kazuhiro Yasuda (Hosei University)

Conference in honor of Professor Vlad Bally

## Our problem

$$dX_t = \sigma(t, X_t) dB_t + b(t, X_t) dt$$

- $\sigma$  uniformly elliptic, bounded, “regular enough”
- $b$  bounded but discontinuous

How to construct an approximation of  $X_T$  with a control on the error?

The Euler-Maruyama scheme is

- Simple to set-up, whatever the dimension
- Efficient in general

## Euler-Maruyama (EM) scheme

- Time horizon  $T > 0$
- Number of steps  $n$
- $t_i = iT/n$  and  $\varphi(t) = t_i$  with  $t_i \leq t < t_{i+1}$
- $\xi_i \sim \mathcal{N}(0, 1)$ , iid.

EM scheme: Compute recursively  $\hat{X}_0 = 0$  and

$$\hat{X}_{i+1} = \hat{X}_i + \sigma(t_i, \hat{X}_i) \sqrt{\frac{T}{n}} \xi_i + b(t_i, \hat{X}_i) \frac{T}{n}$$

Indeed, we consider the continuous EM scheme

$$\bar{X}_t = x + \int_0^t \sigma(\varphi(s), \bar{X}_{\varphi(s)}) dB_s + \int_0^t b(\varphi(s), \bar{X}_{\varphi(s)}) ds$$

so that  $\bar{X}$  and  $X$  are on the same probability space and

$$(\hat{X}_i)_{i=0, \dots, n} \stackrel{\text{dist.}}{=} (\bar{X}_{t_i})_{i=0, \dots, n}$$

## Rate of convergence of the EM scheme

The number of steps  $n$  is the parameter to adjust.

★ **strong convergence** at rate  $\delta$  if

$$\mathbb{E}[|X_T - \bar{X}_T|^2]^{1/2} \leq \frac{C}{n^\delta},$$

★ **weak convergence** at rate  $\delta$  if

$$|\mathbb{E}[f(X_T)] - \mathbb{E}[f(\bar{X}_T)]| \leq \frac{C(f)}{n^\delta}$$

for  $f \in \mathfrak{F}$ , a class of **test functions**.

- Strong conv  $\implies$  weak conv with  $\mathfrak{F}$  = Lipschitz functions
- Weak rate is more difficult to establish than strong rate.

But

- Gives better rate than strong rates
- Corresponds to what is actually computed (think to option prices)

## Traditional approach for the weak rate of convergence

$\mathcal{L}$  is the infinitesimal generator of  $X$

$\implies u(0, x) = \mathbb{E}_x[f(X_T)]$  with

$$\mathcal{L} = \frac{1}{2} a_{i,j}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + b_i(t, x) \frac{\partial}{\partial x_i}, \quad a = \sigma \sigma^T$$

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} + \mathcal{L}u(t, x) = 0 \\ u(T, x) = f(x) \end{cases}$$

Write

$$\begin{aligned} \mathbb{E}_x[f(\bar{X}_T)] - \mathbb{E}_x[f(X_T)] &= \mathbb{E}[f(\bar{X}_T)] - u(0, \bar{X}_0) \\ &= \sum_{i=0}^{n-1} \mathbb{E}[u(t_{i+1}, \bar{X}_{t_{i+1}}) - u(t_i, \bar{X}_{t_i})] \end{aligned}$$

## Traditional approach for the weak rate of convergence

$$\begin{aligned} & \mathbb{E}[f(\bar{X}_T)] - \mathbb{E}[f(X_T)] \\ &= \sum_{i=0}^{n-1} \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \frac{1}{2} (a_{i,j}(s, \bar{X}_s) - a_{i,j}(t_i, \bar{X}_{t_i})) \partial_{ij}^2 u(r, \bar{X}_r) dr \right] \\ & \quad + \sum_{i=0}^{n-1} \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \frac{1}{2} (b_i(s, \bar{X}_s) - b_i(t_i, \bar{X}_{t_i})) \partial_i u(r, \bar{X}_r) dr \right] \end{aligned}$$

Perform Taylor development up to order 4 to approximate the integrals. □

**Theorem** (Talay, ...) If  $f$ ,  $a$  and  $b$  are  $\mathcal{C}^4$ , the weak rate of convergence is of order 1.

**Rem.** The strong rate of convergence is of order 1/2.

## Extension to Hölder continuous coefficients

Fix  $\alpha \in (0, 3) \setminus \{1, 2\}$ .

$H^{\alpha/2, \alpha}$  space of Hölder continuous functions  $f$  on  $[0, T] \times \mathbb{R}^d$  which are

- $\partial_t^r \partial_x^s f$  is  $(\alpha - \lfloor \alpha \rfloor)$ -Hölder continuous in space for  $2r + s = \lfloor \alpha \rfloor$
- $\partial_t^r \partial_x^s f$  is bounded for  $2r + s \leq \alpha$
- $\partial_t^r \partial_x^s f$  is  $(\alpha - 2r - s)/2$ -Hölder continuous in time for  $0 < \alpha - 2r - s < 2$

**Theorem.** Let  $a, b \in H^{\alpha/2, \alpha}$ ,  $f \in H^{2+\alpha}$ , and  $u$  solution to  $\partial_t u(t, x) + \mathcal{L}u(t, x) = 0$  with  $u(T, x) = f(x)$ .  
Then  $u \in H^{1+\alpha/2, 2+\alpha}$ .

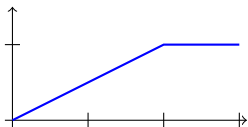
## Weak rate of convergence with Hölder coefficients

**Theorem** (Mikulevicius & Platen)

If  $a, b \in H^{\alpha/2, \alpha}$  and  $f \in H^{2+\alpha}$ , then

$$|\mathbb{E}[f(X_T)] - \mathbb{E}[f(\bar{X}_T)]| \leq \frac{K}{n^{E(\alpha)}}$$

$$E(\alpha) = \begin{cases} \frac{\alpha}{2} & \alpha \in (0, 1) \cup (1, 2) \\ 1 & \alpha \in (2, 3) \end{cases}$$





## Our approach

- The previous approaches require  $a$  and  $b$  to be regular enough, and to share the same regularity.
- They do not apply when  $b$  is discontinuous
- We do not consider applying EM scheme to  $X$  but to an approximation of  $X$  with a **regularized drift**  $b^\varepsilon \in \mathfrak{M}$

$$dX_t^\varepsilon = \sigma(t, X_t^\varepsilon) dB_t + b_\varepsilon(t, X_t^\varepsilon) dt \quad (1)$$

- We consider statements of type

$$|\mathbb{E}[f(X_T)] - \mathbb{E}[f(\bar{X}_T^\varepsilon)]| \leq \frac{C}{n^\delta}, \quad \forall f \in \mathfrak{F}$$

- When  $\sigma$  is constant
  - we consider  $|\mathbb{E}[f(X_T)] - \mathbb{E}[f(\bar{X}_T^\varepsilon)]|$  using  $X^\varepsilon$  and  $\bar{X}^\varepsilon$ .
  - we use the fact that  $\text{Law}(\bar{X}^\varepsilon) \equiv \text{Law}(\bar{X})$  (false when  $\sigma \neq \text{constant}$ )

## Works on simulation of SDE with discontinuous drift

- ✎ PhD thesis **L. Lenôtre** (2015),  $a$  and  $b$  both discount.
  - ✎ **P. Przybyłowicz** (2013), optimal rate of convergence and adaptive algo ( $d = 1$ , localized discontinuities)
  - ✎ PhD thesis **S. Niklitschek-Soto** (2013), local study around one discontinuity
  - ✎ PhD thesis **S. Arnold** (2006), Zvonkin transform ( $d = 1$ )
  - ✎ **N. Halidias & P. Kloeden** (2006): Heaviside drift
  - ✎ PhD thesis **L. Yan** (2002), convergence but no rate
  - ✎ **K.S. Chan & O. Stramer** (1998), discontinuity on polygons, no rate
  - ✎ **R. Janssen** (1984), discontinuity on a surface, no rate
- Rem.** Dealing with  $a$  discontinuous leads to another class of problems, partially covered by some of the above refs.

## The difficulty

The difficulty for dealing with weak rate of convergence comes from

- The regularity of  $u$  is determined by the regularity of  $a$ ,  $b$  and  $f \in \mathfrak{F}$ .
- However, even if  $b$  is discontinuous,  $u$  belongs to some Sobolev space.
- Using regularized drift in  $\mathfrak{M}$  allows one to use “classical results” yet with exploding constants.
- The rate of convergence depends on  $b_\varepsilon - b$  in a given norm, the regularity of  $a$  and  $\mathfrak{F}$ , the class of test functions.

How to “separate” the effects of  $\mathfrak{M}$  and  $\mathfrak{F}$ ?

## Perturbation formula

$$\mathcal{L} = \frac{1}{2} a_{ij} \partial_{ij}^2 + b_i \partial_i \text{ with semi-group } (P_t)_{t>0}$$

$$\mathcal{M} = \frac{1}{2} a_{ij} \partial_{ij}^2 + c_i \partial_i \text{ with semi-group } (Q_t)_{t>0}$$

Perturbation formula

$$\begin{aligned} Q_t &= P_t + \int_0^t Q_s (\mathcal{M} - \mathcal{L}) P_{t-s} ds \\ &= P_t + \int_0^t Q_s (b - c) \nabla P_{t-s} ds \end{aligned}$$

Proof.

$$Q_t - P_t = \int_0^t d(Q_s P_{t-s}) = \int_0^t Q_s \mathcal{M} P_{t-s} ds - \int_0^t Q_s \mathcal{L} P_{t-s} ds.$$

□

## Perturbation formula: stochastic version

- A stochastic version is

$$\mathbb{E}^c[f(X_t)] = \mathbb{E}^b[f(X_t)] + \mathbb{E}^b \left[ \int_0^t (b - c)(s, X_s) \nabla u(s, X_s) ds \right]$$

with

$$\partial_t u(t, x) + \mathcal{L}u(t, x) = 0, \quad u(T, x) = f(x)$$

- Another version is ( $Z$  Doléan exponential  $b \rightsquigarrow c$ )

$$\begin{aligned} \mathbb{E}^c[f(X_t)] - \mathbb{E}^b[f(X_t)] &= \mathbb{E} \left[ \int_0^t Z_s (b - c)^\top \nabla u(s, X_s) ds \right] \\ &\quad + \mathbb{E} \left[ (Z_T - 1) \int_0^t b^\top \nabla v(s, X_s) ds \right] \end{aligned}$$

with

$$\partial_t v + \frac{1}{2} a_{ij} \partial_{ij}^2 v(t, x) = 0 \text{ with } v(T, x) = f(x)$$

## A control using a perturbation formula

- For a process  $X$  and  $p \geq 1$ ,

$$\|g\|_{X,p} \stackrel{\text{def}}{=} \mathbb{E} \left[ \int_0^T |g(s, X_s)|^p \right]^{1/p}$$

- $X^b \stackrel{\text{def}}{=} \text{process generated by } \frac{1}{2} a_{ij} \partial_{ij}^2 + b_i \partial_i$
- $\mathcal{C}_{sl} \stackrel{\text{def}}{=} \text{set of continuous functions with "slow growth":}$   
 $\lim_{x \rightarrow 0} |f(x)| e^{-k|x|^2} = 0 \text{ for all } k > 0$
- $a$  continuous, bounded, uniformly elliptic:  
 $0 < \lambda |\xi|^2 \leq a \xi \cdot \xi \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^d$
- $b$  bounded

**Prop.** If  $f \in \mathcal{C}_{sl}$  and  $\|\nabla v\|_{X^{0,q}} < +\infty$  for  $1 < q \leq \infty$ , then

$$|\mathbb{E}[f(X_T^b)] - \mathbb{E}[f(X_T^c)]| \leq C \|b - c\|_{X^{0,p}} \|\nabla v\|_{X^{0,q}}$$

with  $p^{-1} + q^{-1} < 1$  and  $2 \leq p$ .

**Prop'.** If  $f \in \mathcal{C}_{sl}$  and  $\|\nabla u\|_{X,q} < +\infty$  for  $1 < q \leq \infty$ , then

$$|\mathbb{E}[f(X_T^b)] - \mathbb{E}[f(X_T^c)]| \leq C \|b - c\|_{X^{0,p}} \|\nabla u\|_{X,q}$$

with  $p^{-1} + q^{-1} < 1$  and  $1 \leq p$ .

**Proofs.** Combine Girsanov and repetition of Hölder inequalities. □

## Reason of this formula

- Choice of  $\mathfrak{F}$  (regularity of  $f$ )  
 $\implies \|\nabla v\|_{X^{0,q}} < +\infty$  for some  $q$ .
- The distance  $\|b - c\|_{X^{0,p}}$  depends on  $\mathfrak{F}$  through  $q$  since  $p > \frac{q}{q-1}$ .
- **Gaussian control** on the density transition of  $X^0$   
(e.g. if  $a \in H^{\alpha/2,\alpha}$ )  $\implies$

$$\mathbb{E} \left[ \int_0^T |(b - c)(s, X_s)|^p ds \right]^{1/p} \\ \leq C \left( \int_0^T \left( \int_{\mathbb{R}^d} |(b - c)(s, x)|^q dx \right)^{r/q} ds \right)^{1/r}$$

with  $\frac{d}{2q} + \frac{1}{r} < \frac{1}{p}$ , or with Krylov estimates.



## General heuristic

① Choose  $\mathfrak{F}$ , the space of terminal conditions  
 $\implies$  choice of  $q \implies$  choice of  $p$

② Choose  $\mathfrak{M}$ , the space of regularized drift, so that  
➤ For some  $\gamma > 0$ ,

$$\|b - b_\varepsilon\|_{X^{0,p}} \leq O(\varepsilon^\gamma)$$

➤ For  $b_\varepsilon \in \mathfrak{M}$  and  $f \in \mathfrak{F}$ .

$$|\mathbb{E}[f(X_T^{b_\varepsilon})] - \mathbb{E}[f(\bar{X}_T^{b_\varepsilon})]| \leq \frac{C}{\varepsilon^\beta n^\delta}$$

③ Optimize over the choice of  $\varepsilon$   
 $\implies \varepsilon = O(n^{-\delta/(\gamma+\beta)})$

$$\implies |\mathbb{E}[f(X_T^b)] - \mathbb{E}[f(\bar{X}_T^{b_\varepsilon})]| = O(n^{-\delta\gamma/(\gamma+\beta)}).$$

## General heuristic

$$\begin{array}{ccc} \mathbb{E}[f(\bar{X}^{b_\epsilon})] & \xrightarrow[n \nearrow \infty]{O(\epsilon^{-\beta} n^{-\delta})} & \mathbb{E}[f(X^{b_\epsilon})] \\ & & \begin{array}{c} \epsilon \searrow 0 \quad \downarrow O(\epsilon^\gamma) \\ \mathbb{E}[f(X^b)] \end{array} \\ O(n^{-\delta \frac{\gamma}{\gamma+\beta}}) & \dashrightarrow & \\ \text{with } \epsilon = O(n^{-\delta/(\gamma+\beta)}) & & \end{array}$$

## Examples of terminal condition

☆ If  $f \in \mathcal{C}_{\text{sl}}$  then

$$\|\nabla v\|_{X^{0,2}} \leq C \sqrt{\text{Var } f(X_T^0)}$$

☆ If  $d = 1$ ,  $f \in \mathcal{C}_{\text{sl}} \cap \mathcal{C}^1$ ,  $\nabla f$  bounded then

$$\|\nabla u\|_{X^{b,\infty}} \leq C \|\nabla f\|_{\infty}.$$

☆ Using the notion of [fractional derivative](#) (Geiss & Gobet), one may consider various values of  $q$ , even for  $f$  discontinuous.

## Rate of convergence with smooth coefficients

**Theorem.** If  $f \in \mathcal{C}^3$  with polynomial growth,  $\sigma, b \in \mathcal{C}^{1,3}$  then

$$\mathbb{E}[f(X_T)] - \mathbb{E}[f(\bar{X}_T)] \leq \frac{C}{n}$$

$C$  depends polynomially on the sup-norm of the derivatives of  $b$  (up to degree 4),

**Rem.**  $b$  bounded and  $b_\varepsilon = b \star \rho_\varepsilon$  (mollifiers)

$$\implies \|\nabla^k b_\varepsilon\|_\infty \leq K\varepsilon^{-k}.$$

We need to keep track of the **dependence in the derivatives of  $b$**  (gives the  $\varepsilon^{-\beta}$  in the rate of conv. of the EM scheme with  $b_\varepsilon$ )

The proof relies on some idea introduced in *E. Clément, A. Kohatsu-Higa, D. Lamberton (2006)*.

## Rate of convergence with smooth coefficients

Central idea of the proof.

Without drift terms, the idea is to write

$$\mathbb{E}[f(X_T)] - \mathbb{E}[f(\bar{X}_T)] = \mathbb{E}[\nabla f(\theta X_T + (1 - \theta)\bar{X}_T)E_T]$$

where  $\theta$  is a uniform in  $[0, 1]$  and

$$E_T = \int_0^t \nabla \left( \int_0^1 \sigma(s, \tau X_s + (1 - \tau)\bar{X}_s) d\tau \right) E_s dW_s \\ + \int_0^t (\sigma(s, \bar{X}_s) - \sigma(\varphi(s), \bar{X}_{\varphi(s)})) ds.$$

Then use repeatedly the duality formula of Malliavin calculus to transform

$$\mathbb{E} \left[ H \int_0^t u_s dW_s \right] = \mathbb{E} \left[ \int_0^t D_s H \cdot u_s ds \right]$$

and get the desired control (long computations).

## Rate of convergence with smooth coefficients

With a drift term, use Girsanov formula to write

$$\begin{aligned}\mathbb{E}[f(X_T)] - \mathbb{E}[f(\bar{X}_T)] \\ = \mathbb{E}[\exp(L_T)f(X_T)] - \mathbb{E}[\exp(\bar{L}_T)f(\bar{X}_T)]\end{aligned}$$

with

$$\begin{aligned}L_t &= \int_0^t b^\top \sigma^{-1}(s, X_s) dW_s - \frac{1}{2} \int_0^t b^\top a^{-1} b(s, X_s) ds \\ \bar{L}_t &= \int_0^t b^\top \sigma^{-1}(\varphi(s), \bar{X}_{\varphi(s)}) dW_s - \frac{1}{2} \int_0^t b^\top a^{-1} b(\varphi(s), X_{\varphi(s)}) ds\end{aligned}$$

and apply the same kind of computations.



## Examples

- ☆  $\sigma \in \mathfrak{M} = \mathcal{C}_b^{1,3}$ ,  $\mathfrak{F} = \mathcal{C}_p^3$   
 $\implies$  rate at most  $n^{-\gamma/(\gamma+4)}$  when  $\|b - b_\varepsilon\|_{X^{0,p}} \leq C\varepsilon^{-\gamma}$ .
- ☆  $d = 1$ ,  $b = \square$ ,  $\sigma \in \mathfrak{M} = \mathcal{C}_b^{1,3}$ ,  $\mathfrak{F} = \mathcal{C}_p^3$   
 $\implies$  rate at most  $n^{-1/5+\varepsilon}$ .
- ☆  $\sigma \in \mathfrak{M} = H^{\alpha/2,\alpha}$ ,  $\mathfrak{F} = H^{2+\alpha}$   
 $\implies$  rate at most  $n^{-E(\alpha)\gamma/(\alpha+\gamma)}$  when  $\|b - b_\varepsilon\|_{X^{0,p}} \leq C\varepsilon^{-\gamma}$ .
- ☆  $d = 1$ ,  $b = \square$ ,  $\sigma \in \mathfrak{M} = H^{\alpha/2,\alpha}$ ,  $\mathfrak{F} = H^{2+\alpha}$   
 $\implies$  rate at most  $n^{-E(\alpha)/(\alpha+1)+\varepsilon}$

## Case of constant diffusivity

$$X_t^b = x + B_t + \int_0^t b(X_s^b) ds.$$

Thanks to Girsanov theorem,  
the distributions of  $X^b$ ,  $X^{b_\epsilon}$ ,  $\bar{X}^b$  and  $\bar{X}^{b_\epsilon}$   
are absolutely continuous wrt Wiener measure.

The perturbation formula may be adapted to

$$|\mathbb{E}[f(\bar{X}_T^b)] - \mathbb{E}[f(\bar{X}_T^{b_\epsilon})]| \leq C(f) \|b - b_\epsilon\|_{L^p}, \quad p > d \vee 2.$$

Various results may be given on

$$|\mathbb{E}[f(X_T^b)] - \mathbb{E}[f(\bar{X}_T^b)]|$$

and not only on

$$|\mathbb{E}[f(X_T^b)] - \mathbb{E}[f(\bar{X}_T^{b_\epsilon})]|$$



## Constant diffusivity

$$\begin{array}{ccc}
 \mathbb{E}[f(\bar{X}^{b_\varepsilon})] & \xrightarrow[n \nearrow \infty]{O(\varepsilon^{-\beta} n^{-\delta})} & \mathbb{E}[f(X^{b_\varepsilon})] \\
 \begin{array}{c} O(\varepsilon^\gamma) \\ \downarrow \\ \varepsilon \searrow 0 \end{array} & & \begin{array}{c} \varepsilon \searrow 0 \\ \downarrow \\ O(\varepsilon^\gamma) \end{array} \\
 \mathbb{E}[f(\bar{X}^b)] & \dashrightarrow & \mathbb{E}[f(X^b)] \\
 & O(n^{-\delta \frac{\gamma}{\gamma+\beta}}) & \\
 & \text{with } \varepsilon = O(n^{-\delta/(\gamma+\beta)}) & 
 \end{array}$$

## Yet, this approach is sub-optimal

- A weak rate of order 1 could be achieved.

$$dX_t = dB_t + \begin{cases} -\theta & \text{if } X_t > 0 \\ 0 & \text{if } X_t = 0 \\ \theta & \text{if } X_t < 0 \end{cases}$$

$$|\mathbb{E}_0[f(X_T)] - \mathbb{E}_0[f(\bar{X}_T)]| \leq \frac{C}{n}$$

**Proof.** A lot of Taylor expansions and long computations.  $\square$

## Conclusions

- ⇒ Our approach relies on a perturbation formula and is then a “global” approach ( $\neq$  local analysis around the discontinuity)
- ⇒ It is flexible and allows to combine various results
- ⇒ Allows to “separate” the effects of  $\mathfrak{F}$  and  $\mathfrak{M}$
- ⇒ Mixes stochastic analysis and PDE arguments
- ⇒ But provides only sub-optimal rates
- ⇒ Still a lot of works to perform...