On the Richardson acceleration of finite elements schemes for parabolic SPDEs

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Framework

$W = (W^\rho)_1^\infty$ independent Wiener processes

$$du_t(x) = \big[\mathcal{L}(t)u_t(x) + f(t,x)\big]\,dt + \big[\mathcal{M}(t)^\rho u_t(x) + g^\rho(t,x)\big]\,dW_t^\rho,$$

for $(t,x) \in [0,T] \times \mathbb{R}^d$, $u_0 \in H^0 = L_2(\mathbb{R}^d)$,

$$\mathcal{L}(t)\phi = D_\alpha(a^{\alpha\beta}(t,.)D_\beta\phi), \quad \mathcal{M}^\rho(t)\phi = b^{\alpha\rho}(t,.)D_\alpha\phi,$$

for $\alpha,\beta \in \{0,1,\cdots,d\}$ and $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable bounded

- real-valued bounded processes $a^{\alpha\beta}$
- l_2-valued bounded processes $b^\alpha = (b^{\alpha\rho})_{\rho=1}^\infty$

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- real-valued bounded processes $a^{\alpha\beta}$
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f and $g = (g^\rho)_{\rho=1}^\infty$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable processes with values in $\mathbb{R}$ and l_2

First assumptions

Fix an integer $m \geq 0$ and a positive constant $K > 0$

• **(A1) Bounds on the coefficients** For $(t,x) \in [0,T] \times \mathbb{R}^d$, the coefficients $a^{\alpha\beta}(t,x)$ (resp. $b^{\alpha}(t,x) = (b^{\alpha\rho}(t,x))_{\rho=1}^{\infty}$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ measurable and their partial derivatives in x up to order $m+1$ are a.s. bounded by K in $\mathbb{R}$ (resp. l_2).

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• **(A2) Regularity of free terms and initial condition**

f (resp. $g = (g^{\rho})_{\rho=1}^{\infty}$) predictable H^{m-1} (resp. $H^m(l_2)$)-valued,

$$\mathcal{K}_m^2(T) := \int_0^T \left(|f(t)|^2_{H^{m-1}} + |g(t)|^2_{H^m(l_2)}\right) dt < \infty (a.s.)$$

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• **(A3) Stochastic parabolicity** For $\alpha, \beta \in \{1, \cdots, k\}$, $a^{\alpha\beta}(t,x) = a^{\beta\alpha}(t,x)$ and there exists a constant $\kappa > 0$ s.t.

$$\sum_{\alpha,\beta=1}^{d} \left[2a^{\alpha\beta}(t,x) - b^{\alpha,\rho}(t,x) b^{\beta,\rho}(t,x) \right] z^{\alpha} z^{\beta} \geq \kappa |z|^2$$

for all $(\omega, t, x) \in \Omega \times [0,T] \times \mathbb{R}^d$, $z \in \mathbb{R}^d$.

Well-posedeness

Theorem

Under the above assumptions the semi-linear parabolic SPDE has a unique solution $u = u(t,.)$ such that

$$E\Big(\sup_{t\in[0,T]} |u(t,.)|^2_{H^m} + \int_0^T |u(t,.)|^2_{H^{m+1}} dt\Big) \le C\big[E|u_0|^2_{H^m} + E\mathcal{K}_m^2(T)\big]$$

for a constant C depending on κ, T, m and K.

Power expansion of u^h

Aim Define

- some *particular "regular" finite elements approximations* u^h (depending on some scaling factor h and the corresponding grid points $\mathbb{G}_h$)
- random fields $u^{(0)}, u^{(1)}, \cdots, u^{(k)}$ and r_{kh} for $m > k+1+\frac{d}{2}$ s.t. $u^{(0)}(t,x) = u(t,x)$ for $t \in [0,T]$, $x \in \mathbb{G}_h$
  • $u^h(t,x) = u^{(0)}(t,x) + \sum_{j=1}^k u^{(j)}(t,x)\frac{h^j}{j!} + r_{k,h}(t,x)$ a.s. for $t \in [0,T]$ and $x \in \mathbb{G}_h$
  • there exists a constant $C := C(T,K,m,k,\kappa)$ such that for every $h > 0$

$$E\Big(\sup_{t\le T} h^d \sum_{x\in\mathbb{G}_h} |r_{k,h}(t,x)|^2\Big) \le C h^{2(k+1)}\Big(E|\phi|^2_{H^m} + E\mathcal{K}^2_m(T)\Big)$$

Richardson extrapolation

Once the above expansion
$u^h(t,x) = u(t,x) + \sum_{j=1}^{k} u^{(j)}(t,x)\frac{h^j}{j!} + r_{k,h}(t,x)$
is proved for a "regular grid" $\mathbb{G}_h$ let V be a Vandermonde matrix defined by $V(i,j) = 2^{-(i-1)(j-1)}$ for $i,j = 1,2,...,k+1$ and set

$$\bar{u}^h = \sum_{j=0}^{k} \lambda_j u^{h2^{-j}}, \text{where } (\lambda_0,..,\lambda_k) = (1,0...,0)V^{-1}$$

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Then we have the following **Richardson extrapolation**: There exists a constant N depending only on T, K, κ and k such that for every $h > 0$ and $m > k+1+\frac{d}{2}$

$$E\Big(\sup_{t\in[0,T]} h^d \sum_{x\in\mathbb{G}_h} |u(t,x) - \bar{u}^h(t,x)|^2\Big) \leq N h^{2(k+1)}\big(E|\phi|^2_{H^m} + E\mathcal{K}^2_m(T)\big),$$

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Example: For $k=1$ then $\bar{u}^h = 2u^{h/2} - u^h$ and
$\bar{u}^h - u = 2(u^{h/2} - u) - (u^h - u)$

Some known related results

• many results on diffusion about these power expansions and the corresponding Richardson-Romberg acceleration method of the **weak speed** of convergence of the **Euler** scheme with various **time meshes** (coarsest $h = T/n > 0$). (Talay&Tubaro, Bally&Talay, Malliavin &Thalmaier, Lemaire & Pagès, ...)

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• (Gyöngy & Krylov) Richardson's method for **strong speed** of semi-linear parabolic SPDEs and (space) **finite difference** schemes $\tilde{u}^h(t,.)$ based on $\mathbb{G}_h = \{h\lambda_1 + \cdots + h\lambda_n : n \geq 1, \lambda_i \in \Lambda \cup (-\Lambda)\}$ where Λ finite subset of $\mathbb{R}^d$, $h > 0$

There exist processes $\tilde{u}^{(j)}(t,.)$, $j = 0, 1, \cdots, k$ with $u^{(0)} = u$, $C > 0$ such that for $m > k + 1 + \frac{d}{2}$,

$\tilde{u}^h(t,x) = \sum_{j=0}^k \frac{h^j}{j!} \tilde{u}^{(j)}(t,x) + \tilde{R}_{h,k}(t,x)$,

$$E\Big(\sup_{t\in[0,T]} \sup_{x\in\mathbb{G}_h} |\tilde{R}_{h,k}(t,x)|^2 \Big) \leq C h^{2(k+1)}$$

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• Finite elements approximations for parabolic SPDEs (Brzezniak, Carelli, Debussche, Hausenblas, Larson, Printems, Prohl, Walsh, Yan, ...)

d=1 ; piecewise linear finite elements

Let $\psi(x) = 1 - |x]$ for $-1 \le x \le 1$ and $\psi(x) = 0$ otherwise
Fix $h > 0$, set $\mathbb{G}_h = \{x_i := ih : i \in \mathbb{Z}\}$, $\psi_i^h(x) = (1 - |x - x_i|/h)^+$
and $V_h = \{\sum_{i\in\mathbb{Z}} U_i \psi_i^h : (U_i)_{i\in\mathbb{Z}} \in l_2(\mathbb{Z})\}$
The **finite approximation** $u^h := \big(u^h(t), t \in [0,T]\big)$ of u is a V_h-valued process such that a.s.

$$(u^h(t), \psi_j^h) = (u_0, \psi_j^h) + \int_0^t \big[(-1)^{|\alpha|}\big(a^{\alpha\beta}(s) D_\beta u^h(s), D_\alpha \psi_j^h\big) + (f(s), \psi_j^h)\big]\, ds$$
$$+ \sum_\rho \int_0^t \big[(b^{\alpha\rho}(s) D_\alpha u^h(s) + g^\rho(s), \psi_j^h)\big]\, dW^\rho(s), \quad j \in \mathbb{Z}$$

Set $u^h(t,x) = \sum_{i\in\mathbb{Z}} U_i^h(t) \psi_i^h(x)$
equivalent with a system of SDEs for a $l_2(\mathbb{Z})$-valued process
$U^h = (U_i^h(t), t \in [0,T])$

d=1 ; piecewise linear finite elements - continued

The definition of $u^h(t)$ can be rewritten as a system of SDEs on $(U_i^h(t))_{i\in\mathbb{Z}} \in l_2(\mathbb{Z})$:

$$M^h_{ij} U^h_i(t) = U^h_j(0) + \int_0^t \Big(\sum_{\alpha,\beta=0,1} A^{\alpha,\beta}_{ij}(h,s) U^h_i(s) + F_j(h,s)\Big)\, ds$$

$$+ \sum_\rho \int_0^t \Big(\sum_{\alpha=0,1} B^{\alpha,\rho}_{ij}(h,s) U^h_i(s) + G^\rho_j(h,s)\Big)\, dW^\rho(s), \quad j \in \mathbb{Z},$$

where for $i,j \in \mathbb{Z}$ one sets $\mathcal{R}^h_{ij} = (\psi^h_i, \psi^h_j)$, $U^h_j(0) = (u_0, \psi^h_j)$,

$$F_j(h,s) = (f(s), \psi^h_j), \quad G^\rho_j(h,s) = (g^\rho(s), \psi^h_j),$$

$$A^{\alpha,\beta}_{ij}(h,s) = (-1)^\alpha \big(a^{\alpha,\beta}(s) D_\beta \psi^h_i, D_\alpha \psi^h_j \big)$$

$$B^{\alpha,\rho}_{ij}(h,s) = (b^{\alpha,\rho}(s) D_\alpha \psi^h_i, \psi^h_j).$$

For $\varphi : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ let $T_a\varphi(x) = \varphi(x+a)$
for $a \neq 0$, set $\delta^a = \frac{1}{a}(T_a - Id)$
For $a \in \{-h, h\}$ T_a and δ^a are operators on $v : \mathbb{G}_h \to \mathbb{R}$
Identify $U_i^h(s)$ and $U^h(s, x_i)$ for $s \in [0, T]$ and $i \in \mathbb{Z}$.
matrix $\mathcal{R}^h = hR$ where R is associated with the operator **R** on $l^2(\mathbb{Z}) \equiv l^2(\mathbb{G}_h)$

$$\mathbf{R} = Id + \frac{1}{6}(T_1 - 2Id + T_{-1}) = Id + \frac{h^2}{6}\delta^h\delta^{-h}$$

For $U \in l^2(\mathbb{Z})$, set $\|U\|^2 = \sum_i |U_i|^2$; then $\|\frac{h^2}{6}\delta^h\delta^{-h}U\| \leq \frac{2}{3}\|U\|$
The operators on $l^2(\mathbb{G}_h)$ **R** (or $\mathbb{Z} \times \mathbb{Z}$-matrices $\mathcal{R}^h$) are invertible.
Multiply by $(\mathcal{R}^h)^{-1}$; rewrite the system as a linear SPDE on the Hilbert space $l^2(\mathbb{Z})$

$$U^h(t) = (\mathcal{R}^h)^{-1}U^h(0) + \int_0^t \big[(\mathcal{R}^h)^{-1}A(h,s)^*U^h(s) + (\mathcal{R}^h)^{-1}F(h,s)\big]\,ds$$
$$+ \int_0^t \big[(\mathcal{R}^h)^{-1}B^\rho(h,s)^*U^h(s) + (\mathcal{R}^h)^{-1}G^\rho(h,s)\big]\,dW^\rho(s).$$

"discrete L^2" (resp. H^1) space $\mathcal{U}_{0,h}$ (resp. $\mathcal{U}_{1,h}$) with the norm

$$|U|_{0,h}^2 := h \sum_{i\in\mathbb{Z}} U^2(x_i), \quad |U|_{1,h}^2 := h \sum_{i\in\mathbb{Z}} \left[U^2(x_i) + |\delta^h U(x_i)|^2\right]$$

Set $\Psi_h : U \to V_h$ defined by $\Psi_h(U) = \sum_i U(x_i)\psi_i^h$; then

$$\frac{1}{3}|U|_{k,h}^2 \leq |\Psi_h(U)|_{H^k}^2 \leq |U|_{k,h}^2, \quad k = 0, 1$$

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Theorem

Let the Assumptions (A1)-(A3) be satisfied with $m = 0$. Then for every $h > 0$ there exists a unique FE approximation $(U^h(t))_i \in l^2(\mathbb{Z}) \equiv U^h(t, x_i) \in L^2(\mathbb{G}_h)$ on $[0, T]$. Furthermore,

$$E\Big(\sup_{t\in[0,T]} |U^h(t)|_{0,h}^2 + \int_0^T |U^h(t)|_{1,h}^2 dt\Big) \leq C\big(E|u_0|_{L^2}^2 + E\mathcal{K}_0^2(T)\big)$$

for a constant C depending on κ, K and T, that is

$$E\Big(\sup_{t\in[0,T]} |\Psi_h(U)(t)|_{L^2}^2 + \int_0^T |\Psi_h(U)(t)|_{H^1}^2 dt\Big) \leq C\big(E|u_0|_{L^2}^2 + E\mathcal{K}_0^2(T)\big)$$

Expansion of the finite elements approximation u^h

- **Aim:** Find processes $u^{(1)}, u^{(2)}, \cdots, u^{(k)}$ and r_{kh} for $m > k+2$ (that is for integer-valued m for $m \geq k+3$) s.t.
  - $u^h(t,x) = u(t,x) + \sum_{j=1}^{k} u^{(j)}(t,x)\frac{h^j}{j!} + r_{k,h}(t,x)$ a.s. for $t \in [0,T]$ and $x \in \mathbb{G}_h$
  - there exists a constant $C := C(T,K,m,k,\kappa)$ such that for every $h > 0$

$$E\Big(\sup_{t\leq T} h \sum_{x\in\mathbb{G}_h} |r_{k,h}(t,x)|^2\Big) \leq C h^{2(k+1)} \Big(E|u_0|^2_{H^m} + E\mathcal{K}^2_m(T)\Big)$$

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- **Sketch of proof**
  • rewrite the equation of u^h with operators $\mathcal{L}^h(t)$ and $\mathcal{M}^{\rho,h}(t)$ on functions defined on $\mathbb{R}$ and get power expansions of these operators, of the free terms and initial conditon

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Expansion of the finite elements approximation u^h

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  • Define inductively the processes $v^{(j)}$ on $\mathbb{R}$ in terms of the coefficients of these expansions
  • Prove the upper estimate of the discrete norm of error term using the previous "general" result

Rewrite the system of SDEs
$\mathbf{R} = Id + \frac{h^2}{6}\delta^h\delta^{-h}$ invertible operator on $l^2(\mathbb{G}_h)$; set

$$\begin{aligned}
\mathcal{L}^h(s)U &= \frac{1}{2}\mathbf{R}^{-1}\Big\{\delta^{-h}(a^{11}_{(h)}(s)\delta^h U) + \delta^h(a^{11}_{(-h)}(s)\delta^{-h}U)\Big\} \\
&\quad + \frac{1}{2}\mathbf{R}^{-1}\Big\{\delta^{-h}\big(\bar{a}^{10}_{+h}(s)U\big) + \delta^h\big(\bar{a}^{10}_{-h}(s)U\big) + \bar{a}^{01}_{-h}(s)\delta^{-h}U \\
&\quad + \bar{a}^{01}_{+h}(s)\delta^h U\Big\} + \frac{1}{6}\mathbf{R}^{-1}\Big\{\tilde{a}^{00}_{-h}(s)T_{-h}U + 4\hat{a}^{00}_h(s)U + \tilde{a}^{00}_{+h}(s)T_h U\Big\}, \\
\mathcal{M}^{h\rho}(s)U &= \frac{1}{2}\mathbf{R}^{-1}\Big\{\bar{b}^{1\rho}_{-h}(s)\delta^{-h}U + \bar{b}^{1\rho}_{+h}(s)\delta^h U\Big\} \\
&\quad + \frac{1}{6}\mathbf{R}^{-1}\Big\{\tilde{b}^{0\rho}_{-h}(s)T_{-h}U + 4\hat{b}^{0\rho}_h(s)U + \tilde{b}^{0\rho}_{+h}(s)T_h U\Big\}.
\end{aligned}$$

for various space averages of the coefficients "of the form"
$\Phi_{\epsilon h}(x) := C\int_0^1 \phi(t, x+\epsilon hy)\xi(y)dy$ for some function ξ.
Set $\check{\varphi}_h(x) = \eta_h * \varphi$ where $\eta_h(x) = \frac{1}{h}\eta(x/h)$ and
$\eta(x) = (1-|x|)1_{[-1,+1]}(x)$.

The related scheme

The system of equations defining $U^h(t, x_j)$ can be rewritten

$$U^h(t,x_j) = \mathbf{R}^{-1}\check{u}_0(x_j) + \int_0^t \left(\mathcal{L}^h(s)U^h(s,.)(x_j) + \mathbf{R}^{-1}\check{f}_h(s,.)(x_j)\right) ds$$
$$+ \int_0^t \left(\mathcal{M}^{h\rho}(s)U^h(s,.)(x_j) + \mathbf{R}^{-1}\check{g}_h^{\rho}(s,.)(x_j)\right) dW^{\rho}(s), \; j \in \mathbb{Z}$$

The operators $\mathcal{L}^h(s)$ and $\mathcal{M}^{h\rho}(s)$ can be extended to functions defined on $\mathbb{R}$. Related evolution equation (of functions on $\mathbb{R}$)

$$v^h(t) = \mathbf{R}^{-1}(\check{u}_0)_h + \int_0^t \left(\mathcal{L}^h(s)v^h(s) + \mathbf{R}^{-1}\check{f}_h(s)\right) ds$$
$$+ \int_0^t \left(\mathcal{M}^{h\rho}(s)v^h(s) + \mathbf{R}^{-1}\check{g}_h^{\rho}(s)\right) dW^{\rho}(s),$$

Expansions of the initial condition and free term

- **Free terms** For $n \leq m$ and $j = 0, \cdots, n$, there exist H^0-valued $\mathcal{F}_0$ r.v. $u_0^{(j)}$ and H^0 (resp. $H^0(l^2)$)-valued adapted processes $f^{(j)}(t)$ (resp. $g^{(j)\rho}(t)$) with $u_0^{(0)} = u_0$, and $\Phi^{(0)}(t) = \Phi(t)$ for $\Phi = f$ or $\Phi = g$, and if we set,

$$\Phi_{h,n}(t) = \mathbf{R}^{-1}\check{\Phi}_h(t) - \Phi(t) - \sum_{1 \leq j \leq n} h^j \Phi^{(j)}(t)/j!$$

  then $|\Phi_{h,n}(t)|_{H^r} \leq C h^{n+1} |\Phi(t)|_{H^{r+n+1}}$ for $r + n + 1 \leq m$.
- **Operators** Suppose **(A1)** and **(A3)**; then for $v \in H^{m+1}$, $\frac{d^i}{dh^i}\mathcal{L}_h(t)v$, $\frac{d^i}{dh^i}\mathcal{M}_h^\rho(t)v$ continuous from $(0, \infty)$ to H^{m-1-i}
  Their **limits as** $h \to 0$: $\mathcal{L}^{(i)}(t)v$ and $\mathcal{M}^{(i)\rho}(t)v$ exist in H^{m-1-i} for all $i = 0, ..., m-1$
  $\mathcal{L}^{(0)}(t) = \mathcal{L}(t)$ and $\mathcal{M}^{(0)\rho}(t) = \mathcal{M}^\rho(t)$ for every $t \in [0, T]$.
  Furthermore, for $i + l \leq m - 1$, $t \in [0, T]$ and $v \in H^{m+1}$.

$$|\mathcal{L}^{(i)}(t)v|_{H^l} \leq C\,|v|_{H^{l+2+i}}, \quad |\mathcal{M}^{(i)}(t)v|_{H^l} \leq C|v|_{H^{l+1+i}}$$

Consider the system of SPDEs (solved inductively) for $j = 1, \cdots, k$

$$du_t^{(j)} = \Big(\mathcal{L}(t)u_t^{(j)} + \sum_{l=1}^{j}\binom{j}{l}\mathcal{L}^{(l)}(t)u_t^{(j-l)} + f^{(j)}(t)\Big)\,dt$$
$$+\Big(\mathcal{M}^{\rho}(t)u_t^{(j)} + \sum_{l=1}^{j}\binom{j}{l}\mathcal{M}^{(l)\rho}(t)u_t^{(j-l)} + g^{(j)\rho}(t)\Big)\,dW^{\rho}(t),$$

with initial condition $u_0^{(j)} = u_0^{(j)}$

Theorem

Let Assumptions **(A1)**, **(A2)** *and* **(A3)** *hold; let $k \in [1, m]$ be an integer. The system of SPDEs for $u^{(1)}, \cdots, u^{(k)}$ has a unique solution $\big(u^{(1)}, \cdots, u^{(k)}\big)$ where each $u^{(j)}$ is a continuous H^{m-j} valued processes and for some constant C depending on κ, K, T and m we have for $j = 1, \cdots, k$*

$$E\sup_{t\le T}|u_t^{(j)}|^2_{H^{m-j}} + E\int_0^T |u_t^{(j)}|^2_{H^{m+1-j}}\,dt \le C\big(E|\phi|^2_{H^m} + E\mathcal{K}_m^2(T)\big).$$

For $n \leq m-1$, $t \in [0, T]$ and $h > 0$, let

$$\mathbb{L}_{h,n}(t) = \mathcal{L}^h(t) - \sum_{i=0}^{n} \frac{h^i}{i!} \mathcal{L}^{(i)}(t), \ \mathbb{M}^{\rho}_{h,n}(t) = \mathcal{M}^{h,\rho}(t) - \sum_{i=0}^{n} \frac{h^i}{i!} \mathcal{M}^{(i)\rho}(t)$$

Then for $\varphi \in H^{l+n+3}$ and $\psi \in H^{l+n+2}$,

$$|\mathbb{L}_{h,n}(t)\varphi|_{H^l} \leq Ch^{n+1}|\varphi|_{H^{n+l+3}}, \ |\mathbb{M}_{h,n}(t)\psi|_{H^l(l^2)} \leq Ch^{n+1}|\psi|_{H^{l+n+2}}$$

For $n \leq m-1$, $t \in [0,T]$ and $h>0$, let

$$\mathbb{L}_{h,n}(t) = \mathcal{L}^h(t) - \sum_{i=0}^{n} \frac{h^i}{i!} \mathcal{L}^{(i)}(t), \; \mathbb{M}^{\rho}_{h,n}(t) = \mathcal{M}^{h,\rho}(t) - \sum_{i=0}^{n} \frac{h^i}{i!} \mathcal{M}^{(i)\rho}(t)$$

Then for $\varphi \in H^{l+n+3}$ and $\psi \in H^{l+n+2}$,

$$|\mathbb{L}_{h,n}(t)\varphi|_{H^l} \leq C h^{n+1} |\varphi|_{H^{n+l+3}}, \; |\mathbb{M}_{h,n}(t)\psi|_{H^l(l^2)} \leq C h^{n+1} |\psi|_{H^{l+n+2}}$$

Let $r_{h,k}(t) = u^h_t - u_t - \sum_{1\leq j\leq k} \frac{h^j}{j!} u^{(j)}_t$; then (set $u^{(0)} = u$)

$$\begin{aligned} d\, r_{h,k}(t) &= \big(\mathcal{L}^h_t r_{h,k}(t) + F_{h,k}(t) + f_{hk}(t)\big)\, dt \\ &\quad + \big(\mathcal{M}^{h,\rho}_t r_{h,k}(t) + G^{\rho}_{h,k}(t) + g^{\rho}_{hk}(t)\big)\, dW^{\rho}_t, \quad r_{h,k}(0) = \phi_{h,k}, \end{aligned}$$

$$F_{h,k}(t) = \sum_{j=0}^{k} \frac{h^j}{j!} \mathbb{L}_{h,k-j}(t) u^{(j)}_t, \; G^{\rho}_{h,k}(t) = \sum_{j=0}^{k} \frac{h^j}{j!} \mathbb{M}^{\rho}_{h,k-j}(t) u^{(j)}_t.$$

Expansion of the solution

Recall that $r_{h,k}(t) = u^h_t - u_t - \sum_{1\le j\le k} \frac{h^j}{j!} u^{(j)}_t$ where u^h is the finite elements approximation and u is the solution to the semi-linear SPDE. By the Sobolev embedding theorem $(m > d/2)$ we have the existence of a modification of $r_{h,k}$ continuous on $[0, T] \times \mathbb{R}$ whose restriction to $[0, T] \times \mathbb{G}_h$ is an adapted $\mathcal{U}_h$-valued process. Then the **restriction of** $r_{h,k}(t, x)$ **to** $[0, T] \times \mathbb{G}_h$ is the **solution an abstract equation similar to that of** U^h. For $k \le m - 3$

$$E \sup_{t\le T} |r_{h,k}(t)|^2_{h,0} + E \int_0^T |r_{h,k}(t)|^2_{h,1}\, dt$$

$$\le CE \int_0^T \Big[|F_{k,h}(t)|^2_{0,h} + |f_{hk}|^2_{h,0} + \sum_\rho \big(|G^\rho_{k,h}(t)|^2_{0,h} + |g^\rho_{h,k}|^2_{h,0} \big) \Big] dt$$

$$\le CE \int_0^T |F_{k,h}(t)|^2_{H^1} + |G_{k,h}(t)|^2_{H^1} + |f_{hk}|^2_{H^1} + |g_{h,k}|^2_{H^1}\, dt$$

$$\le Ch^{2(k+1)} E \Big(|\phi|^2_{H^m} + \mathcal{K}_m(T) \Big)$$

Another convergence estimate

If we want to get **estimates uniformly on** $[0,T]\times\mathbb{G}_h$, one needs stronger "stochastic parabolicity assumptions"
(A3Bis) There exists a positive constant $\kappa>0$ such that
$a^{11}(t,x)-\frac{3}{2}\sum_\rho|b^{1\rho}(t,x)|^2\ge\kappa$ a.s. for every $t\in[0,T]$, $x\in\mathbb{R}$
Then for any function $f\in L^2(\mathbb{R})$,

$$(\mathcal{L}^h f,f)+\frac{1}{2}\sum_\rho|\mathcal{M}^{h\rho}f|^2_{L^2}\le-\frac{\kappa}{2}|\delta^h f|^2_{L^2}+C|f|^2_{L^2}$$

This yields the existence of a unique solution v^h such that

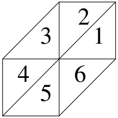
$$E\Big(\sup_{t\in[0,T]}|v^h(t)|^2_{H^m}+\int_0^T|\delta^h v^h(t)|^2_{H^m}dt\Big)\le CE\big(|\phi|^2_{H^m}+\mathcal{K}^2_m(T)\big)$$

One proves H^l estimates of $r_{h,k}(t)$; since $d=1$ and $H^1\subset\mathcal{C}$, we deduce that for $k+3\le m$ and a constant $C:=C(K,k,\kappa,T)$

$$E\Big(\sup_{t\in[0,T]}\sup_{x\in\mathbb{G}_h}|r_{h,k}(t,x)|^2\Big)\le Ch^{2(k+1)}E\big(|\phi|^2_{H^{k+3}}+\mathcal{K}^2_{k+3}(T)\big)$$

This yields a Richardson extrapolation for the sup norm.

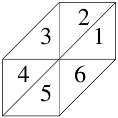
An example for $d = 2$; linear finite elements



Fix $h > 0$ and let ψ be defined on $\mathbb{R}^2$ as follows:

- on **1**$=\{x : 0 \leq x_2 \leq x_1 \leq 1\}$, $\psi(x) = 1 - x_1$,
- on **2**$=\{x : 0 \leq x_1 \leq x_2 \leq 1\}$, $\psi(x) = 1 - x_2$,
- on **3**$=\{x : -1 \leq x_1 \leq 0, 0 \leq x_2 \leq x_1 + 1\}$, $\psi(x) = 1 + x_1 - x_2$,
- on **4**$=\{x : -1 \leq x_1 \leq x_2 \leq 0\}$, $\psi(x) = 1 + x_1$,
- on **5**$=\{x : -1 \leq x_2 \leq x_1 \leq 0\}$, $\psi(x) = 1 + x_2$,
- on **6**$=\{x : 0 \leq x_1 \leq 1, x_1 - 1 \leq x_2 \leq 0\}$, $\psi(x) = 1 + x_2 - x_1$

An example for $d = 2$; linear finite elements



Fix $h > 0$ and let ψ be defined on $\mathbb{R}^2$ as follows:

- on $\mathbf{1}$=$\{x : 0 \leq x_2 \leq x_1 \leq 1\}$, $\psi(x) = 1 - x_1$,
- on $\mathbf{2}$=$\{x : 0 \leq x_1 \leq x_2 \leq 1\}$, $\psi(x) = 1 - x_2$,
- on $\mathbf{3}$=$\{x : -1 \leq x_1 \leq 0, 0 \leq x_2 \leq x_1 + 1\}$, $\psi(x) = 1 + x_1 - x_2$,
- on $\mathbf{4}$=$\{x : -1 \leq x_1 \leq x_2 \leq 0\}$, $\psi(x) = 1 + x_1$,
- on $\mathbf{5}$=$\{x : -1 \leq x_2 \leq x_1 \leq 0\}$, $\psi(x) = 1 + x_2$,
- on $\mathbf{6}$=$\{x : 0 \leq x_1 \leq 1, x_1 - 1 \leq x_2 \leq 0\}$, $\psi(x) = 1 + x_2 - x_1$

For $\mathbf{i} \in \mathbb{Z}^2$ let $\psi_{\mathbf{i}}^h(x) = \psi(\frac{1}{h}(x - h\mathbf{i}))$ (centered at $(i_1 h, i_2 h)$ and rescaled by h)

- Set on $\mathbb{R}^d$

$$\psi(x) = \Pi_{k=1}^d \big(1 - |x_k|\big) \text{ for } \ x \in [-1,1]^d \text{ and } 0 \text{ otherwise}$$

For $\mathbf{i} = (i_1, i_2, \cdots, i_d) \in \mathbb{Z}^d$ and $h > 0$ set
$\psi_{\mathbf{i}}^h(x) = \psi\big(\frac{x_1 - i_1 h}{h}, \frac{x_2 - i_2 h}{h}, \cdots, \frac{x_d - i_d h}{h}\big)$.

- In both examples, if $\Psi_h : U_h \to V_h$ is the extension operator $\Psi_h(U) = \sum_{\mathbf{i}\in\mathbb{Z}^d} U_{\mathbf{i}} \psi_{\mathbf{i}}^h$
$|\Psi_h(U)|_{L^2} \sim |U|_{0,h}$ and $|\nabla\Psi_h(U)|_{L^2} \sim |U|_{1,h}$ recall the discrete L^2 (resp. H^1) norms

$$|U|_{0,h}^2 := h^d \sum_{i\in\mathbb{Z}^d} U_{\mathbf{i}}^2, \quad |U|_{1,h}^2 := h^d \sum_{i\in\mathbb{Z}^d} \left[U_{\mathbf{i}}^2 + |\delta^h U_{\mathbf{i}}|^2\right]$$

- the infinite matrix $\mathcal{R}_{\mathbf{i},\mathbf{j}}^h = \big(\psi_{\mathbf{i}}^h, \psi_{\mathbf{j}}^h\big) = h^2\big(Id - R\big)$, where R is associated with a linear invertible operator $\mathbf{R}$ on $l^2(\mathbb{G}_h) \equiv l^2(\mathbb{Z}^2)$ such that (for $d = 2$)

$$\|\mathbf{R}U\|^2 \leq \frac{5}{6}\|U\|^2 \text{ in example 1 (linear FE)}$$

$$\|\mathbf{R}U\|^2 \leq \frac{17}{18}\|U\|^2 \text{ in example 2 (quadratic FE for } d = 2)$$

- Then the finite elements approximation $u^h(t,x) = \sum_{\mathbf{i}\in\mathbb{Z}^d} U^h_{\mathbf{i}}(t)\psi^h_{\mathbf{i}}(x)$ satisfies with for $\mathbf{i} \in \mathbb{Z}^d$

$$
\begin{aligned}
(u^h(t), \psi^h_{\mathbf{i}}) = (u_0, \psi^h_{\mathbf{i}}) + \int_0^t \Big[(-1)^{|\alpha|}\big(a^{\alpha\beta}(s) D_\beta u^h(s), D_\alpha \psi^h_{\mathbf{i}}\big) \\
+ (f(s), \psi^h_{\mathbf{i}})\Big] ds + \int_0^t \Big[\big(b^{\alpha\rho}(s) D_\alpha u^h(s) + g^\rho(s), \psi^h_{\mathbf{i}}\big)\Big] dW^\rho(s)
\end{aligned}
$$

- It can be rephrased using the inverse of the operator **R** (which can be expressed as combinations of compositions of translations $T_{\epsilon e_l}$ for $\epsilon \in \{-1, +1\}$, $l = 1, \cdots, d$)
"discrete" differential operators (using various averages of the coefficients which depend on the finite elements and the free terms), and the operators $\delta_{\epsilon e_k}$ and $T_{\epsilon e_l}$
A similar expansion of the corresponding functions extended on the grid $\mathbb{G}_h$ and the processes defined on $\mathbb{R}^d$ is proved
Hence the Richardson extrapolation is true

Going further

- One can formulate an abstract convergence result based on the expansion of the free terms and the operators; the convergence on the grid holds for the approximation (discrete norms); that of the functions is unclear in an abstract setting.
- Dealing with an infinite system is not realistic.
  Choose a radius R and introduce a "smooth" cut-off function for the coefficients outside this ball
  gives a solution $u_R(t,x)$ defined on the whole space (there is well-posedness)
  exponentially good control of the difference $u - u_R$
  The finite elements approximation u_R^h of u_R is a finite sum of the $\psi_{\mathbf{i}}^h$
  There is well-posedness and good estimates for u_R^h but the uniform stochastic parabolicity fails
  (work in progress) the expansion of $u_R^h - u_R$ holds.

Last but not least

LA MULȚI ANI VLAD !