Second order Backward SDEs and the Principal-Agent Problem

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En l'honneur de **Vlad**, ... déjà soixante ans

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PROBLEM FORMULATION

Output process

• Effort=Control process $\nu = (\alpha, \beta)$ with values in $A \times B$

• Output=the controlled state process in $\mathbb{R}^d$: any weak solution $\mathbb{P}$ of the SDE

$$dX = \sigma_t(X, \beta_t)\big[\lambda_t(X, \alpha_t)dt + dW_t\big)$$

where W is a Brownian motion with values in $\mathbb{R}^n$

• Observation of X does not give access to the drift $\sigma\lambda$

• Observation of X gives access to $\sigma\sigma^\top$ but not to σ

The Agent problem

Agent solves the following control problem :

$$V_0^A(\xi) := \sup_{\mathbb{P}} \mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^T k_s(X,\nu_s^{\mathbb{P}})ds}\xi(X) - \int_0^T e^{-\int_0^t k_s(X,\nu_s^{\mathbb{P}})} c_t(X,\nu_t^{\mathbb{P}})dt\right]$$

where the contract $\xi(X)$ is $\mathcal{F}_T-$measurable, and represents the compensation for the management of X

$\longrightarrow$ No interest on X, except for the compensation ξ indexed on X

Path-dependency of ξ is crucial $\Longrightarrow$ Non-Markov stochastic control

The Principal problem

Moral hazard : Principal chooses the optimal compensation scheme $\xi(X)$ based on the observation of X only, i.e. Principle does not observe the Agent effort

Principal solves the optimization problem

$$V_0^P := \sup_{\xi \in \Xi_R} \mathbb{E}^{\mathbb{P}^\star(\xi)} \Big[U(\ell(X_T) - \xi) \Big]$$

- $\mathbb{P}^*(\xi)$: solution of Agent problem given the contract ξ
- Ξ_R : collection of all ξ, such that $V_0^A(\xi) \geq R$ (reservation utility)

Existing literature

• Non-zero sum Stackelberg game, highly nonlinear problem

• Holstrom & Milgrom '87 (Econometrica), ..., Sannikov '08, un-controlled diffusion

• Cvitanić & Zhang '13 : calculus of variations $\Longrightarrow$ Pontryagin Maximum Principle leading to a system of Forward-Backward SDEs...

Our objective : Simple solution by standard dynamic programming

Outline

1. Review of stochastic control of Markov diffusions

2. Stochastic control of non-Markov diffusions
   - Smooth processes
   - Case of un-controlled diffusion
   - General controlled diffusion case

3. Solving the Principal-Agent Problem
   - Recalling Problem formulation
   - A sub-optimal Principal problem
   - Reducing the Principal problem to standard control

Stochastic control of Markov diffusions

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$, filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$
W : Brownian motion with values in $\mathbb{R}^n$

- Control process : $\nu = \{\nu_t, t \geq 0\}$ $\mathbb{F}-$progressively measurable process with values in $U \subset \mathbb{R}^k$
- Controlled state process X^ν, valued in $\mathbb{R}^d$, defined by the SDE

$$dX_t^\nu = b(t, X_t^\nu, \nu_t)dt + \sigma(t, X_t^\nu, \nu_t)dW_t$$

$\mathcal{U}$: admissible controls, i.e. X^ν well-defined, appropriate regularity

- Control problem :

$$V(t,x) := \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[g(X_T^{t,x,\nu})\right]$$

Hamiltonian and the HJB equation

Hamiltonian :

$$H(t,x,z,\gamma) \quad := \quad \sup_{u\in U}\Big\{b(t,x,u)\cdot z + \frac{1}{2}\sigma\sigma^\top(t,x,u):\gamma\Big\}$$

for all $(t,x)\in[0,T]\times\mathbb{R}^d$ and $(z,\gamma)\in\mathbb{R}^d\times\mathcal{S}_{\mathbb{R}}(d)$. Then,

The value function V solves the Dynamic Programming (Hamilton-Jacobi-Bellman) Equation :

$$\partial_t V + H(t,x,DV,D^2V) = 0, \quad t<T,\ x\in\mathbb{R}^d$$
$$V(T,.) = g \qquad \text{on } \mathbb{R}^d$$

In which sense HJB equation holds ?

• Classical sense : $V \in C^{1,2}\big([0,T),\mathbb{R}^d\big)$... Not expected, many counter-examples

• Sobolev solutions : $V \in W^{1,2}\big([0,T),\mathbb{R}^d\big)$:, see Krylov 1980, very developed in the semilinear case...

• Vicosity solutions : not in this talk

- V locally bounded, Crandall & Lions '81, Lions '83...
- No access to optimal control, in general
- Uniqueness implied by comparison result, difficult ! finite-dim underlying space

Itô's formula

All previous notions of solutions rely on differential calcul :

$$d\varphi(t, X_t^\nu) = \partial_t\varphi(t, X_t^\nu)dt + D\varphi(t, X_t^\nu) \cdot dX_t^\nu + \frac{1}{2}D^2\varphi(t, X_t^\nu)d\langle X^\nu\rangle_t$$

where $\langle X^\nu\rangle$ is the quadratic variation process :

$$d\langle X^\nu\rangle_t = \sigma\sigma^\top(t, X_t^\nu, \nu_t)dt$$

This is all we need from regularity...

Running cost, discounting

- More general control problem :

$$V(t,x) := \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[e^{-\int_t^T k(s,X_s^\nu,\nu_s)ds} g(X_T^\nu) + \int_t^T e^{-\int_t^r k(s,X_s^\nu,\nu_s)ds} f(r,X_r^\nu,\nu_r)dr\right]$$

$\Longrightarrow$ Hamiltonian :

$$H(t,x,y,z,\gamma) := \sup_{u\in U}\Big\{ b(t,x,u)\cdot z + \frac{1}{2}\sigma\sigma^\top(t,x,u):\gamma + f(t,x,u)y - k(t,x,u)\Big\}$$

The value function V solves the Dynamic Programming (Hamilton-Jacobi-Bellman) Equation :

$$\partial_t V + H(t,x,V,DV,D^2V) = 0, \quad t<T,\ x\in\mathbb{R}^d$$
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Paths space and non-anticipative process

- $\Omega = \{\omega \in C^0([0,T],\mathbb{R}^d), \omega_0 = 0\}$, $\Lambda := [0,T] \times \Omega$
- X canonical process, i.e. $X_t(\omega) = \omega(t)$
- $\mathbb{F} = \{\mathcal{F}_t\}$ the corresponding filtration, i.e. $\mathcal{F}_t = \sigma(X_s, s \le t)$
- $d\big[(t,\omega),(t',\omega')\big] = |t-t'| + \|\omega_{.\wedge t} - \omega'_{.\wedge t'}\|_\infty$
- $u : [0,T] \times \Omega \longrightarrow \mathbb{R}$ non-anticipative if $u(t,\omega) = u\big(t,(\omega_s)_{s\le t}\big)$

In particular, $u \in C^0(\Lambda) \Longrightarrow u$ non-anticipative

Probability measures on the paths space

- $\mathbb{P}_0$: Wiener measure on Ω, so that X is a $\mathbb{P}_0-$Brownian motion

- $\mathbb{P} = \mathbb{P}^{\alpha,\beta}$ such that

$$X_t = \int_0^s \alpha_s^{\mathbb{P}} ds + \int_0^s \beta_s^{\mathbb{P}} dW_t^{\mathbb{P}}, \;\; \mathbb{P} - \text{a.s.}$$

for some adapted processes $\alpha^{\mathbb{P}}$, $\beta^{\mathbb{P}}$

Smooth processes

$\phi : [0, T] \times \Omega \longrightarrow \mathbb{R} \in C^{1,2}(\Lambda)$ if

- in $C^0(\Lambda)$ (in particular, non-anticipative)
- $\exists$ processes $\theta, Z, \Gamma \in C^0(\Lambda)$ valued in $\mathbb{R}$, $\mathbb{R}^d$, $\mathcal{S}_d(\mathbb{R})$, s.t.

$$d\phi_t \;=\; \theta_t dt + Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X\rangle_t, \quad \mathbb{P}-\text{a.s. for all } \mathbb{P} = \mathbb{P}^{\alpha,\beta}$$

Then, denote :

$$\partial_t \phi_t := \theta_t, \quad \partial_\omega \phi_t := Z_t, \quad \partial^2_{\omega\omega}\phi_t := \Gamma_t$$

Or drop $C^0(\Lambda)$ requirements, replace by integrability on Y and Z

$\Longrightarrow$ Sobolev regularity...

Back to stochastic control... Path-dependent case

- Control process $\nu = \{\nu_t, t \geq 0\}$ $\mathbb{F}-$prog meas valued in $U \subset \mathbb{R}^k$
- Controlled state process X^ν, valued in $\mathbb{R}^d$, defined by the SDE

$$dX_t^\nu = b(t, X_\cdot^\nu, \nu_t)dt + \sigma(t, X_\cdot^\nu, \nu_t)dW_t$$

$\mathcal{U}$: admissible controls, i.e. X^ν well-defined, appropriate regularity

- Control problem :

$$V(t,\omega) := \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[\xi(X_\cdot^{t,\omega,\nu})\right]$$

where $\xi(x) = \xi\big(x_{\wedge T}\big)$, $\mathcal{F}_T-$measurable

Path-dependent Hamiltonian

Hamiltonian :

$$H(t,\omega,z,\gamma) \quad := \quad \sup_{u\in U}\Big\{b_t(\omega,u)\cdot z+\frac{1}{2}\sigma_t\sigma_t^\top(\omega,u):\gamma\Big\}$$

for all $(t,\omega)\in[0,T]\times\Omega$ and $(z,\gamma)\in\mathbb{R}^d\times\mathcal{S}_\mathbb{R}(d)$
Consider the Path-dependent HJB equation

$$\partial_t v+H_t\big(\omega,\partial_\omega v,\partial^2_{\omega\omega}v\big)=0,\quad t<T,\ \omega\in\Omega$$
$$v_T=\xi \qquad \text{on } \Omega$$

Verification argument

Let v smooth and solves the path-dependent HJB, then $\forall\, \nu \in \mathcal{U}$:

$$
\begin{aligned}
v_0 &= \mathbb{E}\left[v_T(X^\nu)\right] + \mathbb{E}\left[v_0 - v_T(X^\nu)\right] \\
&= \mathbb{E}\left[\xi(X^\nu)\right] - \mathbb{E}\left[\int_0^T dv_t(X^\nu)\right] \\
&= \mathbb{E}\left[\xi(X^\nu)\right] - \mathbb{E}\left[\int_0^T \partial_t v_t dt + \partial_\omega v_t \cdot dX_t^\nu + \frac{1}{2}\partial^2_{\omega\omega} v_t : \sigma_t^\nu {\sigma_t^\nu}^\top dt\right]
\end{aligned}
$$

$$
= \mathbb{E}\left[\xi(X^\nu)\right] - \mathbb{E}\left[\int_0^T \underbrace{\left(-H_t(\partial_\omega v_t, \partial^2_{\omega\omega} v_t) + \partial_\omega v_t \cdot b_t^\nu + \frac{1}{2}\partial^2_{\omega\omega} v_t : \sigma_t^\nu {\sigma_t^\nu}^\top\right)}_{\leq\, 0} dt\right]
$$

Hence $v_0 \geq V$, and equality satisfied by $\hat{\nu}$ maximier of H

Semilinear path-dependent HJB equation

Suppose $\sigma_t(\omega,u) \equiv \sigma_t(\omega)$, or even $\sigma_t(\omega,u) \equiv I_d$, $(d=n)$, for simplicity. Then the Hamiltonian reduces to

$$H_t(\omega,z,\gamma) = F_t(\omega,z) + \frac{1}{2}\mathrm{Tr}[\gamma], \quad \text{where} \quad F_t(\omega,z) := \sup_u b_t(\omega,u)\cdot z$$

We want to find a solution $v_t(\omega)$ of HJB, then

$$\begin{aligned} dv_t &= \Big(\partial_t v_t + \frac{1}{2}\mathrm{Tr}\big[\partial^2_{\omega\omega} v_t\big]\Big)dt + \partial_\omega v_t \cdot dX^\nu_t \\ &= -F_t(\partial_\omega v_t)dt + \partial_\omega v_t \cdot dX^\nu_t \end{aligned}$$

Semilinear HJB equation and backward SDE

- Denote $\mathbb{P}^\nu := \mathbb{P}_0 \circ (X^\nu)^{-1}$, $Z := \partial_\omega v \implies$ solve for (v, Z) :

$$dv_t = -F_t(Z_t)dt + Z_t \cdot dX, \quad \mathbb{P}^\nu - \text{a.s. for all } \nu$$

- Notice that $\mathbb{P}^\nu \sim \mathbb{P}_0$ in the present context

$$dv_t = -F_t(Z_t)dt + Z_t \cdot dX, \quad \text{and} \quad v_T = \xi, \quad \mathbb{P}_0 - \text{a.s.}$$

$\implies$ Backward SDE (Pardoux & Peng '91), for Lipschitz F :

For $\xi \in \mathbb{L}^2$, $\exists \mathbb{F}-$adapted solution (v, Z) with $\|v\|_{\mathbb{L}^2} + \|Z\|_{\mathbb{L}^2} < \infty$

$\implies$ Sobolev solution of the path-dependent semilinear PDE

The case of controlled diffusion : difficulties

• Similar to the Markov case, very difficult to access to the Hessian component... Need a relaxation of the C^2-regularity

• The $\mathbb{P}^\nu$'s (measures induces by the controlled state) are defined on different supports for different values of ν, so can not reduce the analysis to one single measure

$\Longrightarrow$ Quasi-sure stochastic analysis : stochastic analysis under a non-dominated family of singular measure

From fully nonlinear HJB equation to semilinear

- $H_t(\omega, z, \gamma)$ non-decreasing and convex in γ, Then

$$H_t(\omega, z, \gamma) = \sup_{a \geq 0} \left\{ \frac{1}{2} a : \gamma - H_t^*(\omega, z, a) \right\}$$

Path-dependent HJB equation is

$$\partial_t v + \sup_{a \geq 0} \left\{ \frac{1}{2} a : \partial^2_{\omega\omega} v - H_t^*(\partial_\omega v, a) \right\} = 0, \quad v_T = \xi$$

$\Longrightarrow$ stochastic representation

$$v_t(\omega) = \sup_a Y_t^a(\omega)$$

where, denoting $\mathbb{P}^a := \mathbb{P}_0 \circ \left(\int_0^\cdot a_s^{1/2} dX_s \right)$,

$$Y_t^a = \xi - \int_t^T H_s^*(Z_s^a, a_s) ds + \int_t^T Z_s^a dX_s, \quad \mathbb{P}^a - \text{a.s.}$$

Wellposedness of second order BSDEs

There exists a unique triple (Y, Z, K) $\mathbb{F}-$adapted with appropriate integrability, such that

- $Y_t = \xi - \int_t^T H_s^*(Z_s, a_s)ds - \int_t^T Z_s dX_s + \int_t^T dK_s$, $\mathbb{P}^a-$a.s. for all control process a
- K nondecreasing, $K_0 = 0$, and $\inf_a \mathbb{E}^{\mathbb{P}^a}\big[K_T\big] = 0$

Soner, NT & Zhang '10
Chao, Possamaï & Tan '15

Regularity reduces to the non-decreasing process K

Suppose $K_t = \int_0^t \dot{K}_s ds,\ t \in [0, T]$, and define the process Γ by

$$\dot{K}_t = H_t(Z_t, \Gamma_t) - \frac{1}{2} a_t : \Gamma_t + H_t^*(Z_t, a_t)$$

Substituting in the 2BSDE, we get for all a :

$$Y_t = \xi + \int_t^T \Big[H_s(Z_s, \Gamma_s) - \frac{1}{2} a_s : \Gamma_s \Big] ds - \int_t^T Z_s dX_s, \ \mathbb{P}^a - \text{a.s.}$$

$\Longrightarrow$ $Y_t(\omega)$ solves the path-dependent HJB equation :

$$\partial_t Y + H_t(\partial_\omega Y, \partial^2_{\omega\omega} Y) = 0, \quad Y_T = \xi$$

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The Principal-Agent problem

- Agent solves the control problem :

$$V_0^A(\xi) := \sup_{\mathbb{P}} \mathbb{E}^{\mathbb{P}}\Big[e^{-\int_0^T k_s(X,\nu_s^{\mathbb{P}})ds}\xi(X) - \int_0^T e^{-\int_0^t k_s(X,\nu_s^{\mathbb{P}})}c_t(X,\nu_t^{\mathbb{P}})dt\Big]$$

where the Output process is a weak solution $\mathbb{P}$ of the SDE

$$dX = \sigma_t(X,\beta_t)\big[\lambda_t(X,\alpha_t)dt + dW_t\big)$$

- Principal solves the optimization problem

$$V_0^P := \sup_{\xi\in\Xi_R} \mathbb{E}^{\mathbb{P}^\star(\xi)}\Big[U(\ell(X_T)-\xi)\Big]$$

where Ξ_R : collection of all ξ, such that $V_0^A(\xi) \geq R$

A class of revealing contracts

- Path-dependent Hamiltonian for the Agent problem :

$$H_t(\omega, y, z, \gamma) \quad := \quad \sup_{a,b}\Big\{\sigma_t(\omega,a)\lambda_t(\omega,b)\cdot z + \tfrac{1}{2}\sigma_t\sigma_t^\top(\omega,a) : \gamma \\ -k_t(\omega,a,b)y - c_t(\omega,a,b)\Big\}$$

- For $Y_0 \in \mathbb{R}$ and Z, Γ $\mathbb{F}^X-$prog meas, define

$$Y_t^{Z,\Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2}\int_0^t \Gamma_s : d\langle X\rangle_s - \int_0^t H_s(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s)ds$$

Proposition $V_0^A(Y_T^{Z,\Gamma}) = Y_0$ and any maximizer of the Hamiltonian $(a^\star, b^\star)(Y, Z, \Gamma)$ induces a solution $\mathbb{P}^\star$ of the Agent problem

Sub-optimal stochastic control problem

Under $\widehat{\mathbb{P}}^{Z,\Gamma}$, we have

$$
\begin{aligned}
dX_t &= \sigma_t^\star(X, Y_t, Z_t, \Gamma_t)\big[\lambda_t^\star(X, Y_t, Z_t, \Gamma_t)dt + dW_t\big] \\
dY_t^{Z,\Gamma} &= Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt
\end{aligned}
$$

where $\sigma_t^\star(\omega, y, z, \gamma) := \sigma_t\big(\omega, b^\star(\omega, y, z, \gamma)\big)$, $\lambda_t^\star(\omega, y, z, \gamma) := \cdots$

$$
V_0^P \geq \sup_{Y_0 \geq R} \underline{V}_0(X_0, Y_0); \quad \underline{V}_0(X_0, Y_0) := \sup_{Z,\Gamma} \mathbb{E}^{\widehat{\mathbb{P}}^{Z,\Gamma}}\Big[U(\ell(X_T) - Y_T^{Z,\Gamma})\Big]
$$

$\underline{V}$ characterized by standard HJB equation

Case of un-controlled diffusion

- $B = \{\beta^0\}$, then $Y^{Z,\Gamma} = Y^{Z,0}$
- To prove that $V_0^P = \underline{V}_0$, it suffices to show that an arbitrary $\xi \in \mathcal{F}_T^X$ has the representation $\xi = Y_T^{Z,0}$, i.e.

$$\xi = Y_0 + \int_0^T Z_t \cdot dX_t - \int_0^T H_t(Y_t, Z_t, 0)dt, \quad \mathbb{P}^{\beta^0} - \text{a.s.}$$

Backward SDE wellposedness guarantees this is true ! Hence

$$V_0^P = \sup_{Y_0 \geq R} \underline{V}_0(X_0, Y_0); \quad \underline{V}_0(X_0, Y_0) := \sup_Z \mathbb{E}^{\widehat{\mathbb{P}}^{Z,0}}\Big[U(\ell(X_T) - Y_T^{Z,0})\Big]$$

The general case

• In the fully nonlinear case, the representation $\xi = Y_T^{Z,\Gamma}$ not true for general ξ... We only have the 2BSDE representation :

$$Y_t = \xi - \int_t^T H_s^*(Y_s, Z_s, a_s)ds - \int_t^T Z_s dX_s + \int_t^T dK_s, \quad \mathbb{P} - \text{a.s. for all } \mathbb{P}$$

with K nondecreasing, $K_0 = 0$, and $\inf_{\mathbb{P}} \mathbb{E}^{\mathbb{P}}\big[K_T\big] = 0$

• It is sufficient to find an approximation ξ^ε of ξ such that $\xi^\varepsilon = Y_T^{Z^\varepsilon,\Gamma^\varepsilon}$... and pass to the limit in the Principal problem...

• $K_t^\varepsilon := \frac{1}{\varepsilon}\int_{0\vee(t-\varepsilon)}^t dK_s$ and $\xi^\varepsilon := Y_T^\varepsilon$ (replacing K by K^ε) $\Longrightarrow$ $\xi^\varepsilon = Y^{Z,\Gamma^\varepsilon}$ and $\mathbb{P}^\star$ optimal for K is also optimal for K^ε

$$\Longrightarrow \quad V_0^P \quad = \quad \sup_{Y_0 \geq R} \underline{V}_0(X_0; Y_0)$$

BON ANNIVERSAIRE VLAD !