A Weak Dynamic Programming Principle for Combined Optimal control / Stopping problem with $\mathcal{E}^f$-conditional Expectation

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Classical stochastic control problems

Formulation of the problem

Markovian stochastic control problems on a given horizon of time T

$$u(0,x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}[\int_0^T f(\alpha_s, X_s^\alpha) ds + g(X_T^\alpha)]$$

where

- $\mathcal{A}$ is a set of admissible control processes α_s
- (X_s^α) is a controlled process
- $g(X_T^\alpha)$: terminal reward ; $f(\alpha_s, X_s^\alpha)$: instantaneous reward process.

Formally, for all (t, y) the associated value function is defined by

$$u(t,y) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}[\int_t^T f(\alpha_s, X_s^\alpha) ds + g(X_T^\alpha) \mid X_t^\alpha = y].$$

The dynamic programming principle can formally be stated as

$$u(0,x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\Big[\int_0^t f(\alpha_s, X_s^\alpha)ds + u(t, X_t^\alpha)\Big], \quad \text{for all } t \text{ in } [0, T].$$

Classically established under assumptions which ensure that the value function u satisfies some regularity/ measurability properties.

Classical stochastic control problems

Weak Dynamic Programming Principle

- **Deterministic** control : Case of a *discontinuous* value function : Barles'86, Barles-Perthame'93

- **Stochastic** control : Case of a *discontinuous* value function, not even measurable : Bouchard-Touzi'11,
  (with a lower semi continuity assumption of the terminal reward g, to obtain the super-optimality principle).

  Bouchard-Nutz'12 (state constraints)

  Bayraktar-Yao'13 (zero-sum stochastic games).

General stochastic control problems

Goal : Generalize these results obtained for the classical case :

$$\sup_{\alpha\in\mathcal{A}} \mathbb{E}[\int_0^T f(\alpha_s, X_s^\alpha)ds + g(X_T^\alpha)] \tag{1}$$

to the case when linear expectation $\mathbb{E}$ is replaced by nonlinear expectation :

$$\sup_{\alpha\in\mathcal{A}} \mathcal{E}_{0,T}^{f^\alpha}[g(X_T^\alpha)], \tag{2}$$

where $\mathcal{E}^{f^\alpha}[\eta]$ is the nonlinear conditional expectation associated with a BSDE with jumps with controlled driver $f(\alpha_t, X_t^\alpha, y, z, k)$, and terminal condition η, and g Borelian only.

Problem (1) is a particular case of (2) when the driver f does not depend on the solution of the BSDE, that is when $f(\alpha_t, X_t^\alpha, y, z, k) \equiv f(\alpha_t, X_t^\alpha)$.

Dynamic risk-measure ρ defined for each position $\xi \in L^2(\mathcal{F}_T)$ by

$$\rho_{t,T}(\xi) := -\mathcal{E}^f_{t,T}[\xi], \qquad 0 \leq t \leq T,$$

where $\mathcal{E}^f$ is the conditional expectation associated with a BSDE with jumps with driver f.

Properties of the risk measure (Consistency, Continuity, Zero-one law, Translation invariance, monotonicity) depend on the driver of the BSDE. (Quenez-S. SPA '13).

Example of driver : $f(t,z,k) := -C_1|z| - C_2 \int_{\mathbf{E}} |k(e)|\Psi(e)\nu(de)$,
where C_1, $C_2 \geq 0$ can be interpreted as risk-aversion coefficients.
If $C_2 \leq 1$, the risk measure is monotone w.r.t. terminal wealth.

Consider the seller of a European option

- payoff $G(S_T)$, with G irregular (e.g. $G(x) = \mathbf{1}_{[a,b]}(x)$ or $\mathbf{1}_{[a,b[}(x)$, ...)
- S : price process of the underlying asset.
- Wealth process : $X^{\alpha,t,x}$, controlled by α, a portfolio-strategy.
- Shortfall risk of his terminal position :
  $\rho_{t,T}[-(X_T^{\alpha,t,x} - G(S_T))^-] = -\mathcal{E}^f_{t,T}[-(X_T^{\alpha,t,x} - G(S_T))^-]$.

Problem : At time t, for initial wealth x, minimize over all predictable $\alpha \in \mathcal{A}_t^t$ this shortfall risk.
Value function :

$$v(t,x) := -\sup_{\alpha \in \mathcal{A}_t^t} \mathcal{E}^f_{t,T}[-(X_T^{\alpha,t,x} - G(S_T))^-].$$

Mixed generalized optimal control/stopping problems

$$\sup_{\alpha\in\mathcal{A}} \sup_{\tau\in\mathcal{T}} \mathcal{E}^{f^\alpha}_{0,\tau}[\bar{h}(\tau, X^\alpha_\tau)],$$

- X^α_τ : controlled jump diffusion
- $\mathcal{T}$: set of stopping times in $[0, T]$
- $\bar{h}$ is an *irregular* reward function.

The value function is not continuous, not even measurable.

Outline

1. **Mixed control-optimal stopping problem with $\mathcal{E}^f$-expectation**
   Reduction to an optimal control problem for reflected BSDEs
2. **Weak dynamic programming principle**
3. **Nonlinear Hamilton-Jacobi-Bellman variational inequality**
   Characterisation of the value function as a *weak* viscosity solution.
   (need of comparison theorems between BSDEs and reflected BSDEs with weak hypothesis on the drivers)

Probability setup

Product space $\Omega := \Omega_W \otimes \Omega_N$

$\Omega_W := \mathcal{C}([0, T])$: Wiener space, that is the set of continuous functions ω^1 from $[0, T]$ into $\mathbb{R}$ such that $\omega^1(0) = 0$.
$\Omega_N := \mathbb{D}([0, T])$: Skorohod space of RCLL functions ω^2 from $[0, T]$ into $\mathbb{R}$, such that $\omega^2(0) = 0$.

$B = (B^1, B^2)$: canonical process defined for t and $\omega = (\omega^1, \omega^2)$ by $B^i_t(\omega) = B^i_t(\omega^i) := \omega^i_t$, $i = 1, 2$.

We denote the first coordinate process B^1 by W.

P^W : probability measure on $(\Omega_W, \mathcal{B}(\Omega_W))$ s.t. W is a Brownian motion.

Set $\mathbf{E} := \mathbb{R}^n \backslash \{0\}$, $n \geq 1$. We define the jump random measure N as :
for $t > 0$ and $\mathbf{B} \in \mathcal{B}(\mathbf{E})$, $N(., [0,t] \times \mathbf{B}) := \sum_{0<s\leq t} \mathbf{1}_{\{\Delta B_s^2 \in \mathbf{B}\}}$.
$(\mathbf{E}, \mathcal{B}(\mathbf{E}))$ is equipped with a σ-finite positive measure ν s.t.
$\int_{\mathbf{E}} (1 \wedge |e|) \nu(de) < \infty$.

P^N : probability measure on $(\Omega_N, \mathcal{B}(\Omega_N))$ s.t. N is a Poisson r.m. with compensator $\nu(de)dt$ and s.t. $B_t^2 = \sum_{0<s\leq t} \Delta B_s^2$.
Set $\tilde{N}(dr, de) = N(dr, de) - \nu(de)dt$.

$\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ completed filtration associated with canonical process B.

We define the product probability measure $P := P^W \otimes P^N$.

Nonlinear conditional expectation

Let $\mathcal{A}$ be the set of controls, defined as predictable processes valued in a compact subset A of $\mathbb{R}^p$.
For $\alpha \in \mathcal{A}$, $t \in [0, T]$ and x in $\mathbb{R}$, we introduce the nonlinear conditional expectation $\mathcal{E}^{\alpha,t,x}$ defined for all stopping time S and $\eta \in L^2(\mathcal{F}_S)$ as

$$\mathcal{E}^{\alpha,t,x}_{r,S}[\eta] := \mathcal{X}^{\alpha,t,x}_r, \ t \leq r \leq S$$

where $(\mathcal{X}^{\alpha,t,x}_r)_{t \leq r \leq S}$ is the solution of the BSDE :

$$\begin{cases} -d\mathcal{X}^{\alpha,t,x}_r = f(\alpha_r, r, X^{\alpha,t,x}_r, \mathcal{X}^{\alpha,t,x}_r, Z^{\alpha,t,x}_r, K^{\alpha,t,x}_r(\cdot))dr - Z^{\alpha,t,x}_r dW_r \\ \qquad\qquad - \int_{\mathbf{E}} K^{\alpha,t,x}_r(e)\tilde{N}(dr, de) \\ \mathcal{X}^{\alpha,t,x}_S = \eta, \end{cases}$$

and $(X^{\alpha,t,x}_s)_{t \leq s \leq T}$ is the state process :

$$X^{\alpha,t,x}_s = x + \int_t^s b(X^{\alpha,t,x}_r, \alpha_r)dr + \int_t^s \sigma(X^{\alpha,t,x}_r, \alpha_r)dW_r$$
$$+ \int_t^s \int_{\mathbf{E}} \beta(X^{\alpha,t,x}_{r^-}, \alpha_r, e)\tilde{N}(dr, de),$$

Assumptions on the driver f of the BSDE :
f is measurable, satisfies $|f(\alpha, t, x, 0, 0, 0)| \leq C(1 + |x|^p)$, is Lipschitz continuous wrt to α, x, y, z, k, and

$$f(\alpha, t, x, y, z, k_1) - f(\alpha, t, x, y, z, k_2) \geq \int_{\mathbf{E}} \gamma(x, y, z, k_1, k_2)(e)(k_1 - k_2)(e)\nu(de)$$

with $|\gamma(\cdot, e)| \leq \Psi(e)$, and $\gamma(\cdot, e) \geq -1$, where $\Psi \in L^2_\nu$.
L^2_ν : set of Borelian functions l such that $\|l\|^2_\nu := \int_{\mathbf{E}} l^2(e)\nu(de) < \infty$.
(this condition ensures the Comparison principle for BSDEs with jumps)

Assumptions on the coefficients of the SDE :
$b,\ \sigma$ are Lipschitz continuous with respect to x and α,
$|\beta(x, \alpha, e)| \leq C\,\Psi(e)$; where $\Psi \in L^2_\nu$
$|\beta(x, \alpha, e) - \beta(x', \alpha', e)| \leq C(|x - x'| + |\alpha - \alpha'|)\,\Psi(e)$

The solution $(\mathcal{X}^{\alpha,t,x}, Z^{\alpha,t,x}, K^{\alpha,t,x})$ of the BSDE belongs to

- $\mathcal{S}^2$: set of real-valued RCLL adapted processes (φ_s) with $\mathbb{E}[\sup_{0\leq s\leq T}\varphi_s^2] < \infty$
- $\mathbb{H}^2$: set of predictable processes (Z_t) such that $\mathbb{E}\int_0^T Z_s^2 ds < \infty$
- $\mathbb{H}^2_\nu$: set of predictable processes $(k_t(\cdot))$ such that $\mathbb{E}\int_0^T \|k_s\|^2_{L^2_\nu} ds < \infty$

and $(X_s^{\alpha,t,x})_{t\leq s\leq T}$ belongs to $\mathcal{S}^2$.

General mixed optimal Control/Stopping problem

Suppose the initial time is equal to 0. For each initial condition $x \in \mathbb{R}$, we consider the mixed optimal stopping/stochastic control problem :

$$u(0,x) := \sup_{\tau \in \mathcal{T}} \sup_{\alpha \in \mathcal{A}} \mathcal{E}^{\alpha,0,x}_{0,\tau}[\bar{h}(\tau, X^{\alpha,0,x}_{\tau})]$$

where

$$\bar{h}(t,x) := h(t,x)\mathbf{1}_{t<T} + g(x)\mathbf{1}_{t=T},$$

with

- $g : \mathbb{R} \to \mathbb{R}$ is Borelian.
- $h : [0, T] \times \mathbb{R} \to \mathbb{R}$ is uniformly continuous with respect to (t,x).
- $|h(t,x)| + |g(x)| \leq C(1 + |x|^p)$

Note that $\bar{h}$ is Borelian but not necessarily continuous in (t,x).

We now make the problem dynamic.

- For $t \in [0, T]$ and $\omega \in \Omega$, define the t-translated path $\omega^t = (\omega^t_s)_{s \geq t} := (\omega_s - \omega_t)_{s \geq t}$.
- $\mathbb{F}^t = (\mathcal{F}^t_s)_{t \leq s \leq T}$ completed filtration associated with the translated Brownian motion $W^t := (W_s - W_t)_{s \geq t}$ and the translated Poisson random measure $N^t := N(]t, s], .)_{s \geq t}$.
- $\mathcal{T}^t_t$: set of $\mathbb{F}^t$-stopping times with respect to $\mathbb{F}^t$ with values in $[t, T]$.
- $\mathcal{P}^t$: the predictable σ-algebra on $\Omega \times [t, T]$ equipped with the filtration $\mathbb{F}^t$.
- $\mathcal{A}^t_t$: set of controls $\alpha : \Omega \times [t, T] \mapsto \mathbf{A}$, which are predictable.

Value function

$$u(t,x) := \sup_{\alpha\in\mathcal{A}_t^t}\sup_{\tau\in\mathcal{T}_t^t}\mathcal{E}_{t,\tau}^{\alpha,t,x}[\bar{h}(\tau, X_\tau^{\alpha,t,x})].$$

Since α and τ depend only on ω^t, the SDE satisfied by $X^{\alpha,t,x}$ and the BSDE satisfied by $\mathcal{E}_{t,\tau}^{\alpha,t,x}[\bar{h}(\tau, X_\tau^{\alpha,t,x})]$ can be solved with respect to the translated Brownian motion W^t and the translated Poisson random measure N^t.

Hence the function u is well defined as a deterministic function of t and x.

$$u(t,x) := \sup_{\alpha \in \mathcal{A}_t^t} \sup_{\tau \in \mathcal{T}_t^t} \mathcal{E}_{t,\tau}^{\alpha,t,x}[\bar{h}(\tau, X_\tau^{\alpha,t,x})].$$

For each $\alpha \in \mathcal{A}_t^t$, we introduce the function u^α defined as

$$u^\alpha(t,x) := \sup_{\tau \in \mathcal{T}_t^t} \mathcal{E}_{t,\tau}^{\alpha,t,x}[\bar{h}(\tau, X_\tau^{\alpha,t,x})].$$

Key result :**Value function of optimal stopping pbs with $\mathcal{E}^f$-expectation = solution of nonlinear reflected BSDE** (Quenez-A.S. SPA'14)

So $u^\alpha(t,x) = Y_t^{\alpha,t,x}$, where $(Y^{\alpha,t,x}, Z^{\alpha,t,x}, K^{\alpha,t,x})$ is the solution of the reflected BSDE with driver $f^{\alpha,t,x} := f(\alpha_\cdot, \cdot, X_\cdot^{\alpha,t,x}, y, z, k)$, obstacle process $\bar{h}(s, X_s^{\alpha,t,x})_{t \leq s \leq T}$, and terminal condition $g(X_T^{\alpha,t,x})$, that is

Reflected BSDE

$$Y_s^{\alpha,t,x} = g(X_T^{\alpha,t,x}) + \int_s^T f(\alpha_r, r, X_r^{\alpha,t,x}, Y_r^{\alpha,t,x}, Z_r^{\alpha,t,x}, K_r^{\alpha,t,x}(\cdot))dr$$

$$+ A_T^{\alpha,t,x} - A_s^{\alpha,t,x} - \int_s^T Z_r^{\alpha,t,x} dW_r - \int_s^T \int_{\mathbb{R}^*} K^{\alpha,t,x}(r,e)\tilde{N}(dr,de)$$

$$Y_s^{\alpha,t,x} \geq h(s, X_s^{\alpha,t,x}), 0 \leq s < T \text{ a.s. },$$

$A^{\alpha,t,x}$ is a RCLL nondecreasing predictable process with $A_t^{\alpha,t,x} = 0$ and s.t.

$$\int_0^T (Y_s^{\alpha,t,x} - \bar{h}(s, X_s^{\alpha,t,x}))dA_s^c = 0; \ \Delta A_s^d = -\Delta A_s^{\alpha,t,x} \mathbf{1}_{\{Y_{s^-}^{\alpha,t,x} = \bar{h}(s^-, X_{s^-}^{\alpha,t,x})\}}$$

Here $A^{\alpha,t,x,c}$ denotes the continuous part of A and $A^{\alpha,t,x,d}$ its discontinuous part.

This reflected BSDE can be solved on the restricted space $\Omega \times [t,T]$ wrt the t-translated BM and the t-translated Poisson random measure.

Remark : When $\limsup_{t\to T, y\to x} h(t,y) \leq g(x)$, then the obstacle $\bar{h}(\cdot, X_\cdot^{\alpha,t,x})$ is left upper-semi continuous along stopping times, which implies the continuity of the process $A^{\alpha,t,x}$.

Our initial mixed optimal stopping/control problem can thus be expressed as an **optimal control problem for reflected BSDEs** :

$$\mathbf{u}(\mathbf{t},\mathbf{x}) = \sup_{\tau \in \mathcal{T}^t_{[t,T]}} \sup_{\alpha \in \mathcal{A}^t_t} \mathcal{E}^{\alpha,t,x}_{t,\tau}[\overline{h}(\tau, X^{\alpha,t,x}_\tau)] = \sup_{\alpha \in \mathcal{A}^t_t} u^\alpha(t,x) = \sup_{\alpha \in \mathcal{A}^{\mathbf{t}}_{\mathbf{t}}} \mathbf{Y}^{\alpha,\mathbf{t},\mathbf{x}}_{\mathbf{t}}$$

where $Y^{\alpha,t,x}_t$ is the solution of the RBSDE associated to driver $f^{\alpha,t,x}$, obstacle $\overline{h}(u, X^{\alpha,t,x}_u)_{t \leq u \leq T}$, and terminal condition $g(X^{\alpha,t,x}_T)$.

Remarks :

- *Also in the case of linear expectations, this approach provides alternative proofs of the dynamic programming principle*
- *Some mixed problems with nonlinear expectations are studied in Bayraktar and Yao'12 and Quenez-S. SPA '14. There, the obstacle process does not depend on the control, which yields the characterization of the value function as the solution of a reflected BSDE. This is not the case here, and hence the dynamic programming principle can not be derived directly from the flow property of reflected BSDEs.*

2. Weak Dynamic programming principle

Since for fixed s, the value function $x \mapsto u(s,x)$ is not necessarily Borelian, we cannot a priori establish a classical dynamic programming principle. We will provide a *weak* DDP involving the lower- and upper-semicontinuous envelope of u defined by

$$u_*(t,x) := \liminf_{(t',x')\to(t,x)} u(t',x')$$

$$u^*(t,x) := \limsup_{(t',x')\to(t,x)} u(t',x')$$

Define now

$$\bar{u}_*(t,x) := u_*(t,x)\mathbf{1}_{t<T} + g(x)\mathbf{1}_{t=T}$$

$$\bar{u}^*(t,x) := u^*(t,x)\mathbf{1}_{t<T} + g(x)\mathbf{1}_{t=T}$$

We have : $\bar{u}_* \leq u \leq \bar{u}^*$.

Theorem :[A *weak* dynamic programming principle]

- The function $\bar{u}^*$ satisfies the *sub–optimality principle of dynamic programming* : $\forall t \in [0, T]$ and $\theta \in \mathcal{T}_t^t$,

$$u(t,x) \leq \sup_{\alpha \in \mathcal{A}_t^t} \sup_{\tau \in \mathcal{T}_t^t} \mathcal{E}_{t,\theta\wedge\tau}^{\alpha,t,x}\left[h(\tau, X_\tau^{\alpha,t,x})\mathbf{1}_{\tau<\theta} + \bar{u}^*(\theta, X_\theta^{\alpha,t,x})\mathbf{1}_{\tau\geq\theta}\right]$$

- The function $\bar{u}_*$ satisfies the *super–optimality principle of dynamic programming* : $\forall t \in [0, T]$ and $\theta \in \mathcal{T}_t^t$,

$$u(t,x) \geq \sup_{\alpha \in \mathcal{A}_t^t} \sup_{\tau \in \mathcal{T}_t^t} \mathcal{E}_{t,\theta\wedge\tau}^{\alpha,t,x}\left[h(\tau, X_\tau^{\alpha,t,x})\mathbf{1}_{\tau<\theta} + \bar{u}_*(\theta, X_\theta^{\alpha,t,x})\mathbf{1}_{\tau\geq\theta}\right].$$

Proof of Weak Dynamic Programming Principle

- **Super-optimality principle** (ii)
  We approximate θ by a sequence of stopping times $(\theta^n)_{n\in\mathbb{N}}$.
  - **existence of weak ε- controls** (requires some measurable selection theorem) ,
  - **a "splitting" result**, (which basically states that, given an intermediary time $t \leq T$, and a fixed path up to t, the BSDE can be solved wrt to the t-translated Brownian motion and Poisson random measure).

  - a Fatou lemma for reflected BSDEs where the limit involves both terminal condition and terminal time.
  - comparison theorems for reflected BSDEs with jumps,
  - estimates on reflected BSDEs
  - flow property of reflected BSDEs

- **Sub-optimality principle** (i) : (the easiest)
  Similar arguments (but without need of existence of weak ε- controls).

Remarks :

The weak dynamic programming principle still holds with θ *replaced by* θ^α *in the inequalities, given a family of stopping times indexed by controls* $\{\theta^\alpha, \alpha \in \mathcal{A}_t^t\}$.

No regularity condition is required on the terminal reward map g *to ensure the DDP, even for the super optimality one. This is not the case in the previous literature even in the case of a classical expectation, where the reward* g *is supposed to be lower-semicontinuous.*

Let $s \in [0, T]$. For each ω, let ${}^s\omega := (\omega_{r\wedge s})_{0\leq r\leq T}$ and $\omega^s := (\omega_r - \omega_s)_{s\leq r\leq T}$.

We shall identify the path ω with $({}^s\omega, \omega^s)$, which means that a path can be splitted into two parts : the path before time s and the s-translated path after time s.

Let α be a given control in $\mathcal{A}$. We show the following :

- at time s, for fixed past path $\tilde{\omega} := {}^s\omega$, the process $\alpha(\tilde{\omega}, .)$ which only depends on the future path ω^s is an s-admissible control, that is $\alpha(\tilde{\omega}, .) \in \mathcal{A}_s^s$;
- furthermore, $Y^{\alpha,0,x}(\tilde{\omega}, .)$ from time s coincides with the solution of the reflected BSDE driven by W^s and $\tilde{N}^s$, controlled by $\alpha(\tilde{\omega}, .)$ and associated with initial time s and initial state condition $X_s^{\alpha,0,x}(\tilde{\omega})$.

Proof of Weak Dynamic Programming Principle

Existence of weak ε-optimal controls

- We first show a measurability property of the functions $u^{\alpha}(t,x)$ with respect to control α and initial condition x.
- For $s \geq t$, we introduce the set $\mathcal{A}_s^t$ of restrictions to $[s, T]$ of the controls in $\mathcal{A}_t^t$. Let $\eta \in L^2(\mathcal{F}_s^t)$. For each $\omega \in \Omega$, by definition of u we have :

$$u(s,\eta(\omega)) = \sup_{\alpha \in \mathcal{A}_s^s} u^{\alpha}(s,\eta(\omega)). \tag{3}$$

Theorem : (Existence of *weak* ε-optimal controls) : Let $t \in [0, T]$, $s \in [t, T]$ and $\eta \in L^2(\mathcal{F}_s^t)$. Let $\varepsilon > 0$. There exists $\alpha^\varepsilon \in \mathcal{A}_s^t$ such that, for almost every $\omega \in \Omega$, $\alpha^\varepsilon({}^s\omega, T^s)$ is weakly ε-optimal for Problem (3), that is

$$u_*(s,\eta(\omega)) \leq u^{\alpha^\varepsilon({}^s\omega, T^s)}(s,\eta(\omega)) + \varepsilon. \tag{4}$$

Assumption 1. Assume that $dP \otimes dt$-a.s for each $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,

$$f(t, y, z, k_1) - f(t, y, z, k_2) \geq \langle \gamma_t^{y,z,k_1,k_2}, k_1 - k_2 \rangle_\nu,$$

with γ measurable, bounded, and satisfying

$$\gamma_t^{y,z,k_1,k_2}(e) \geq -1 \quad \text{and} \quad |\gamma_t^{y,z,k_1,k_2}(e)| \leq \psi(e), \quad \text{where} \quad \psi \in L^2_\nu.$$

This assumption ensures the comparison theorem for BSDEs with jumps.

Theorem : Let $T > 0$. Let (η_t) be an RCLL process in $\mathcal{S}^2$. Let f be a Lipschitz driver satisfying Assumption 1. Let $(\theta^n)_{n\in\mathbb{N}}$ be a non increasing sequence of stopping times in $\mathcal{T}$, converging a.s. to $\theta \in \mathcal{T}$ as n tends to ∞. Let $(\xi^n)_{n\in\mathbb{N}}$ be a sequence of random variables s.t. $\mathbb{E}[\sup_n (\xi^n)^2] < +\infty$, and for each n, ξ^n is $\mathcal{F}_{\theta^n}$-measurable.
Let $Y_{.,\theta^n}(\xi^n)$; $Y_{.,\theta}(\liminf_{n\to+\infty} \xi^n)$ and $Y_{.,\theta}(\limsup_{n\to+\infty} \xi^n)$ be the solutions of the reflected BSDEs associated with driver f, obstacle $(\eta_s)_{s<\theta^n}$ (resp. $(\eta_s)_{s<\theta}$) , terminal time θ^n (resp. θ), terminal condition ξ^n (resp. $\liminf_{n\to+\infty} \xi^n$ and $\limsup_{n\to+\infty} \xi^n$).
Suppose that

$$\liminf_{n\to+\infty} \xi^n \geq \eta_\theta \quad (\text{resp.} \quad \limsup_{n\to+\infty} \xi^n \geq \eta_\theta) \quad \text{a.s., then}$$

$$Y_{0,\theta}(\liminf_{n\to+\infty} \xi^n) \leq \liminf_{n\to+\infty} Y_{0,\theta^n}(\xi^n) \ (\text{resp. } Y_{0,\theta}(\limsup_{n\to+\infty} \xi^n) \geq \limsup_{n\to+\infty} Y_{0,\theta^n}(\xi^n)).$$

Nonlinear Hamilton-Jacobi-Bellman variational inequalities

Theorem : The value function u of the optimal control/optimal stopping problem is a **weak viscosity solution** of the HJB variational inequality, i.e. u^* is a viscosity subsolution and u_* is a viscosity supersolution.

$$
\begin{aligned}
\min \Big(u(t,x) - h(t,x), \inf_{\alpha \in \mathbf{A}} (-\frac{\partial u}{\partial t}(t,x) - L^\alpha u(t,x) \\
\quad - f(\alpha, t, x, u(t,x), (\sigma \frac{\partial u}{\partial x})(t,x), B^\alpha u(t,x))) \Big) = 0, (t,x) \in [0, T) \\
u(T,x) = g(x), \quad x \in \mathbb{R}
\end{aligned}
$$

where $L^\alpha := A^\alpha + K^\alpha$, and for $\phi \in C^2(\mathbb{R})$,

- $A^\alpha \phi(x) := \frac{1}{2}\sigma^2(x,\alpha)\frac{\partial^2 \phi}{\partial x^2}(x) + b(x,\alpha)\frac{\partial \phi}{\partial x}(x)$
- $K^\alpha \phi(x) := \int_{\mathbf{E}} \left(\phi(x + \beta(x,\alpha,e)) - \phi(x) - \frac{\partial \phi}{\partial x}(x)\beta(x,\alpha,e) \right) \nu(de)$
- $B^\alpha \phi(x) := \phi(x + \beta(x,\alpha,\cdot)) - \phi(x)$.

• A usc function u is a *viscosity subsolution* of the HJBVI if for all $(t_0, x_0) \in [0, T[\times\mathbb{R}$ and $\phi \in C^{1,2}([0, T] \times \mathbb{R})$ such that $\phi(t_0, x_0) = u(t_0, x_0)$ and $\phi - u$ attains its minimum at (t_0, x_0), we have

$$\min\{u(t_0, x_0) - h(t_0, x_0),$$
$$\inf_{\alpha\in A}\left(-\frac{\partial\phi}{\partial t}(t_0, x_0) - L^\alpha\phi(t_0, x_0)\right.$$
$$\left.-f(\alpha, t_0, x_0, u(t_0, x_0), (\sigma\frac{\partial\phi}{\partial x})(t_0, x_0), B^\alpha\phi(t_0, x_0))\right)\} \leq 0.$$

• A lsc function u is a *viscosity supersolution* of the HJBVI if for all $(t_0, x_0) \in [0, T[\times\mathbb{R}$ and $\phi \in C^{1,2}([0, T] \times \mathbb{R})$ such that $\phi(t_0, x_0) = u(t_0, x_0)$ and $\phi - u$ attains its maximum at (t_0, x_0), we have

$$\min(u(t_0, x_0) - h(t_0, x_0),$$
$$\inf_{\alpha\in A}(-\frac{\partial\phi}{\partial t}(t_0, x_0) - L^\alpha\phi(t_0, x_0) - f(\alpha, t_0, x_0, u(t_0, x_0), (\sigma\frac{\partial\phi}{\partial x})(t_0, x_0), B^\alpha\phi(t_0, x_0))) \geq$$

Note that if a map u is both a viscosity subsolution and a viscosity supersolution and $u(T,x) = g(x)$, then it is continuous and it is a viscosity solution in the classical sense.
Here, since the value function u is not regular, it is not in general a viscosity solution in the classical sense.

Proof :

- **subsolution** : use a new comparison thm between a BSDE and a reflected BSDE + sub-optimality weak Dynamic Programming principle
- **supersolution** : comparison thm for BSDEs + super-optimality weak Dynamic Programming principle.

In the case of classical expectations, the proof is classical.
Here, in the case of f-expectations, the arguments are different. In particular, we need to establish a comparison thm between a BSDE and a reflected BSDE with a weak hypothesis on the corresponding drivers (the inequality between the drivers is only required along the solution of the BSDE).

Concluding remarks

We have proven a *weak* dynamic programming principle for a mixed stochastic control/optimal controlproblem with f-expectation.

This has required some specific techniques of stochastic analysis and BSDEs to handle measurability and other issues due to the nonlinearity of the expectation and the lack of regularity of the terminal reward.

Our result on the existence of weak ε-controls allows us to get rid off regularity assumption on the reward g.

Note that our approach allows us to treat ambiguity both on the drift and the volatility

Extension to generalized Dynkin games with uncertainty and irregular rewards.

R. Dumitrescu - A. Sulem- M.C Quenez (2015) : Dynamic Programming Principle for Combined Optimal Stopping / Stochastic Control with $\mathcal{E}^f$-Expectations.

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