Parametrix method for skew diffusions

Arturo Kohatsu-Higa[1]

Ritsumeikan University
joint work with
Dai Taguchi (Ritsumeikan University)
Jie Zhong (University of Central Florida)
Vlad's Birthday Worksop

6, October, 2015

[1] Supported by KAKENHI 24340022

Outline

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Some models with discontinuous coefficients

Continuous time threshold AR models Brockwell, Tweedie, Stramer, Tong, Chan etc.

$$X_t^{(p)} + \sum_{k=0}^{p-1} a_{k,i} X_t^{(k)} + b_i = \sigma_1 Z_t; \text{ if } r_{i-1} < X_t < r_i$$

Here $X^{(i)}$ denotes the i-th derivative and $r_{i-1} < r_i$.
The SDE extension which is obtained using a limiting procedure has coefficients which are discontinuous at points.
This is also associated with the so-called change point models:

$$X_t = x + \int_0^t \sigma_i(X_s) dZ_t^i$$

Here σ may have discontinuities of the type

$$\sigma_i(x) = \sum_{j=1}^{n} \sigma_i^j(x) 1\{r_{j-1} < x < r_j\}.$$

Goal:Estimate parameters, prove that they are efficient, test design

The symmetrization of the Carr-Lee (By Akahori-Imamura)

$dX_t = \sigma(X_t)dW_t + b(X_t)dt \quad \tau_k = \inf\{t; X_t = k\}$

Theorem *Let X be a solution to a 1-dim SDE and $\tilde{X}$ be the solution to*

$$d\tilde{X}_t = \tilde{\sigma}(\tilde{X}_t)\, dW_t + \tilde{b}(\tilde{X}_t)\, dt,$$

where

$$\tilde{\sigma}(x) = \sigma(x)1_{\{x>k\}} \pm \sigma(2k - x)1_{\{x\leq k\}},$$

and

$$\tilde{b}(x) = b(x)1_{\{x>k\}} - b(2k - x)1_{\{x\leq k\}}.$$

We assume $X_0 = \tilde{X}_0 > k$. Then we have

$$\begin{aligned} &E[f(X_t - K)1_{\{X_t>k\}}1_{\{\tau_k>t\}}] \\ &\quad = E[f(\tilde{X}_t - k)1_{\{\tilde{X}_t>k\}}] - E[f(k - \tilde{X}_t)1_{\{\tilde{X}_t<k\}}] \end{aligned} \tag{1}$$

for any bounded Borel function f and $t > 0$.

Goal: Volatility estimation, Simulation, Static hedging

A generic stochastic volatility model is given as follows:

$$\begin{aligned} dX_t &= \sigma_{11}(X_t, V_t)dW_t + b_1(X_t, V_t)\,dt \\ dV_t &= \sigma_{21}(V_t)dW_t + \sigma_{22}(V_t)dB_t + b_2(V_t)\,dt, \end{aligned} \tag{2}$$

where $W \perp B$, $b(x,v) = (b_1(x,v), b_2(v)) \in C(\mathbb{R}^2)$ and

$$\sigma(x,v) = \begin{pmatrix} \sigma_{11}(x,v) & 0 \\ \sigma_{21}(v) & \sigma_{22}(v) \end{pmatrix} \in C(\mathbb{R}^2)$$

In most cases, $\sigma_{11}(x,v) = x\nu(v)$ for some ν and $b_1(x,v) = rx$.

Theorem *Let $X_0 > K > 0$ and τ_K is the first hitting time of X to K*

$$\tilde{\sigma}_{11}(x,v) = \begin{cases} \sigma_{11}(x,v) & x \geq K \\ -\sigma_{11}(2K - x, v) & x < K \end{cases}.$$

Similarly for $\tilde{b}_1$ and let $\tilde{X}$ be the unique (weak) solution to

$$d\tilde{X}_t = \tilde{\sigma}_{11}(\tilde{X}_t, V_t)dW_t + \tilde{b}_1(\tilde{X}_t, V_t)\,dt,$$

Then, it holds for any bounded Borel function f and $t > 0$ that

$$E[f(X_t - K)1_{\{X_t > K\}}1_{\{\tau_K > t\}}] = E[f(\tilde{X}_t - K)1_{\{\tilde{X}_t > K\}}] - E[f(K - \tilde{X}_t)1_{\{\tilde{X}_t < K\}}],$$

Relation b/w SkD and SDE with dis-continuous coefficients

Define for $\alpha \in (0,1)$

$$s_\alpha(x) := (1-\alpha)x\mathbf{1}(x \geq 0) + \alpha x \mathbf{1}(x < 0).$$

Then (note that if $\alpha \notin (0,1)$ then s_α is not bijective)

$$X_t(x) = x + \int_0^t b(X_s(x))ds + \int_0^t \sigma(X_s(x))dW_s + (2\alpha - 1)L_t^0(X)$$

$$s_\alpha \downarrow \uparrow s_\alpha^{-1}$$

$$Z_t = z + \int_0^t \mu(Z_s)ds + \int_0^t \rho(Z_s)dW_s,$$

where

$$\mu(z) := (1-\alpha)b\left(\frac{z}{1-\alpha}\right)\mathbf{1}(z>0) + \alpha b\left(\frac{z}{\alpha}\right)\mathbf{1}(z<0) + \frac{b(0)}{2}\mathbf{1}(z=0),$$
$$\rho(z) := (1-\alpha)\sigma\left(\frac{z}{1-\alpha}\right)\mathbf{1}(z>0) + \alpha \sigma\left(\frac{z}{\alpha}\right)\mathbf{1}(z<0) + \frac{\sigma(0)}{2}\mathbf{1}(z=0).$$

Weak solution: Krylov , Le Gall , Nakao .

Physical interpretation

In physical/biological models, diffusions have the corresponding interpretation by Kolmogorov equation In the case that $\rho(x) = (1-\alpha)1(x>0) + \alpha 1(x<0) + \frac{1}{2}1(x=0)$, this is strongly related with the transmission problem

$$\begin{aligned}
\partial_t u(t,x) &= \mu(x)\partial_x u(t,x) + \frac{\rho^2(x)}{2}\partial_x^2 u(t,x) \\
\alpha\partial_x u(t,0+) &= (1-\alpha)\partial_x u(t,0-) \\
u(0,x) &= f(x).
\end{aligned}$$

This problem is related with the skew Brownian motion and the boundary condition is interpreted as a transmission condition at the boundary. It is well known that there is a unique strong solution for skew Brownian motion ($\mu = 0, \rho = 1$) iff $\alpha \in [0,1]$.

$$X_t = x + W_t + L_t^0(X).$$

Some partial conclusions/questions

- Simple models are necessary to understand the behavior of the model
- Some flexibility in the coefficients is necessary in order to fit to data
- When are regular models close to irregular models?
- Are approximations always correct?
- Many different types of transmission conditions are possible
- Extensive literature and applications
- One needs to understand the behavior of the density even at discontinuity points.

A. Lejay, E. Mordecki, S. Torres, Is Brownian skew?, Preprint.

Skew diffusion

- A skew diffusion is the unique solution of the following one-dimensional stochastic differential equation with symmetric local time:

$$X_t(x) = x + \int_0^t b(X_s(x))ds + \int_0^t \sigma(X_s(x))dW_s + (2\alpha - 1)L_t^0(X). \quad (3)$$

- $W = (W_t)_{0 \leq t \leq T}$ is a one-dimensional standard Brownian motion.
- $L^0(X) = (L_t^0(X))_{0 \leq t \leq T}$ is a symmetric local time of X at the origin.
- $t \in [0, T]$, $\alpha \in (0, 1)$.
- If $\alpha = 1/2$, then a solution to the equation (3) is a diffusion process.
- If $\alpha = 1$ or $\alpha = 0$, then a solution to the equation (3) is Reflected stochastic differential equation.
- We want to prove that

(i) Existence of the density of a skew diffusion.

(ii) The behavior of the density of a skew diffusion up to the boundary.

(iii) Probabilistic representation for $\mathbb{E}[f(X_T(x))]$ for some function f

(iv) Other applications and extensions

Theoretical result

Let $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. Our first main result on this talk is the following.

Theorem *Assume that*

(i) *There exist positive constants* $\overline{a}$ *and* $\underline{a}$, *such that for any* $x \in \mathbb{R}$,

$$\underline{a} \le a(x) := \sigma^2(x) \le \overline{a}.$$

(ii) $b \in \mathcal{M}_b$, $a \in C_b^{\eta}$ *with* $\eta \in (0, 1]$, *i.e.,* $\exists K > 0$ *such that*

$$\sup_{x \in \mathbb{R}} |b(x)| + \sup_{x,y \in \mathbb{R}, x \neq y} \frac{|a(x) - a(y)|}{|x - y|^{\eta}} \le K.$$

Then $\forall (t, x) \in (0, T] \times \mathbb{R}_0$, *there exists the density function of* $X_t(x)$, $p_t(x, \cdot)$, *so that i.e.,* $\exists C > 0$ *and* $c > 0$ *s.t.* $\forall (t, x, y) \in (0, T] \times \mathbb{R}_0 \times \mathbb{R}$,

$$p_t(x, y) \le \frac{Ce^{-\frac{(y-x)^2}{2ct}}}{\sqrt{2\pi c t}} \text{ and } |\partial_x p_t(x, y)| \le \frac{C}{t^{1/2}} \frac{e^{-\frac{(y-x)^2}{2ct}}}{\sqrt{2\pi c t}}.$$

This density is continuous in x, *its side derivatives exist at* $x = 0$, *its second derivatives is continuous. If* $a \in C_b^{1+\eta}$, $b \in C_b^{\eta}$ *then the first derivative condition at* $y = 0$ *is also satisfied.*

Some transformations with solution construction

$$s_\alpha(x) := (1-\alpha)x1(x \geq 0) + \alpha x 1(x < 0),$$

$$r_\alpha(x) := s_\alpha^{-1}(x) = \frac{x}{(1-\alpha)}1(x \geq 0) + \frac{x}{\alpha}1(x < 0)$$

$$f_\alpha(x) := \frac{D_- s_\alpha(x) + D_+ s_\alpha(x)}{2} = (1-\alpha)1(x > 0) + \alpha 1(x < 0) + \frac{1}{2}1(x = 0).$$

Note that $f_\alpha \circ r_\alpha(x) = f_\alpha \circ s_\alpha(x) = f_\alpha(x)$.

Proposition *Suppose that $\alpha \in (0,1)$, and that b and σ are measurable functions. Define $Z_t = s_\alpha(X_t)$. Then $X = (X_t)_{0\leq t\leq T}$ is solution of the X-SDE if and only if $Z = (Z_t)_{0\leq t\leq T}$ solves the following continuous flow SDE*

$$Z_t = z + \int_0^t \mu(Z_s)ds + \int_0^t \rho(Z_s)dW_s, \; z := s_\alpha(x),$$

where $\rho(z) := f_\alpha(z)\sigma(r_\alpha(z))$ and

$$\mu(z) := f_\alpha(z)b(r_\alpha(z)) = (1-\alpha)b\left(\frac{z}{1-\alpha}\right)1(z > 0) + \alpha b\left(\frac{z}{\alpha}\right)1(z < 0) + \frac{b(0)}{2}1(z$$

Parametrix method for Skdiffusion

In this section, we introduce a parametrix method for diffusion process ($\alpha = 1/2$):

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,\ X_0 = x.$$

The parametrix method is a “Taylor-like expansion” for the density of diffusion process and is used to construct a fundamental solution for parabolic type PDEs (Levi or Friedman [**?**])
Consider a “frozen process”

$$dX_t^y = b(y)dt + \sigma(y)dW_t, (\text{ or } d\bar{X}_t^y = \sigma(y)dW_t),\ \bar{X}_0^y = x.$$

Then $\overline{p}(x,y) := \frac{e^{\frac{|y-x-b(y)t|^2}{2a(y)t}}}{\sqrt{2\pi a(y)t}}$ or $\overline{p}(x,y) := \frac{e^{\frac{|y-x|^2}{2a(y)t}}}{\sqrt{2\pi a(y)t}}$ satisfies

$$\partial_s \overline{p}_{t-s}(x,y) = -L^y p^y_{t-s}(x,y),\ \lim_{t\downarrow s} \int_{\mathbb{R}} f(x)\overline{p}_{t-s}(x,y)dx = f(y).$$

$$Lf(z) = b(z)\partial_x f(z) + \frac{1}{2}\sigma^2(z)\partial_x^2 f(z)$$
$$L^y f(z) = b(y)\partial_x f(z) + \frac{1}{2}\sigma^2(y)\partial_x^2 f(x)$$

Closeness is measured by $Lf(z) - L^y f(z)$

Hence we have the following “distance” type argument

$$
\begin{aligned}
p_t(x,y) - \bar{p}_t(x,y) &= \int_0^t ds \partial_s \int_{\mathbb{R}} dz p_s(x,z)\bar{p}_{t-s}(z,y) \\
&= \int_0^t ds \int_{\mathbb{R}} dz \left(\partial_s p_s(x,z)\bar{p}_{t-s}(z,y) + p_s(x,z)\partial_s \bar{p}_{t-s}(z,y)\right) \\
&= \int_0^t ds \int_{\mathbb{R}} dz \left(L^* p_s(x,z)\bar{p}_{t-s}(z,y) - p_s(x,z)L^y \bar{p}_{t-s}(z,y)\right) \\
&= \int_0^t ds \int_{\mathbb{R}} dz p_s(x,z)\underbrace{(L - L^y)\bar{p}_{t-s}(z,y)}_{=:A_{t-s}(z,y)} \\
&= \int_0^t ds \int_{\mathbb{R}} dz p_s(x,z)A_{t-s}(z,y) \\
&=: p \circledast A(t,x,y).
\end{aligned}
$$

Given good integrable estimates on $(L - L^y)\bar{p}_{t-s}(z,y)$, this implies that

$$
p_t(x,y) = \bar{p}_t(x,y) + p \circledast A(t,x,y).
$$

$\bar{p}_t(x,y)$ is called the “parametrix” and this procedure is called the “parametrix method”.

By iterating the above procedure, we have the following “formal expansion”

$$\begin{aligned}
p_t(x,y) &= \overline{p}_t(x,y) + p \circledast A(t,x,y) \\
&= \overline{p}_t(x,y) + \overline{p} \circledast A(t,x,y) + p \circledast A^{\circledast 2}(t,x,y) \\
&= \cdots \\
\text{“}=\text{”} &\sum_{n=0}^{\infty} \overline{p} \circledast A^{\circledast n}(t,x,y),
\end{aligned}$$

where $f^{\circledast 1} := f$, $f^{\circledast n} := f \circledast f^{\circledast(n-1)}$ and $f \circledast g^{\circledast 0} := f$.
Under the condition

(i) σ is a positive, bounded and uniformly elliptic function.

(ii) b is bounded and a is η-Hölder continuous with $\eta \in (0,1]$,

the above expansion holds and $p_t(x,y)$ satisfies a Gaussian upper bound because we have the following estimate:

$$|\overline{p} \circledast A^{\circledast n}(t,x,y)| \le \frac{C^n}{\Gamma(1+n\eta/2)} \frac{e^{-\frac{|y-x|^2}{2ct}}}{\sqrt{2\pi ct}} =: \frac{C^n}{\Gamma(1+n\eta/2)} g_{ct}(y-x).$$

The notion of closeness

$$p_t(x,y) - \overline{p}_t(x,y) = \int_0^t ds \int_{\mathbb{R}} dz p_s(x,z) \underbrace{(L - L^y)\overline{p}_{t-s}(z,y)}_{=:A_{t-s}(z,y)}$$
$$= \int_0^t ds \int_{\mathbb{R}} dz p_s(x,z) A_{t-s}(z,y)$$
$$=: p \circledast A(t,x,y).$$

The distance is measured by $(L - L^y)\overline{p}_{t-s}(z,y)$ together with the integrability of the above time integral. In fact,

$$A_{t-s}(z,y) = (L - L^y)\overline{p}_{t-s}(z,y)$$
$$\sim \left\{ (b(z) - b(y)) \frac{z-y}{t-s} + \frac{1}{2}(\sigma^2(z) - \sigma^2(y)) \frac{(z-y)^2}{(t-s)^2} \right\}$$
$$\times \overline{p}_{t-s}(z,y)$$

How to make it work in our case where $b = \delta_0$? Using the same degeneracy in the approximation term!

Recall that for the parametrix method for diffusion process, a "frozen process" X^y is defined by

$$X_t^y = x + b(y)t + \sigma(y)W_t, \text{ (or } X_t^y = x + \sigma(y)W_t).$$

For a skew diffusion process:

$$X_t(x) = x + \int_0^t b(X_s(x))ds + \int_0^t \sigma(X_s(x))dW_s + (2\alpha - 1)L_t^0(X),$$

a "frozen process" X^y which is the unique strong solution to the equation

$$X_t^y = x + \sigma(y)W_t + (2\alpha - 1)L_t^0(X^y),$$

which is a slightly generalized version of "skew Brownian motion".

$$Lf(z) = (b(z) + (2\alpha - 1)\delta_0(z))\partial_x f(z) + \frac{1}{2}\sigma^2(z)\partial_x^2 f(z)$$
$$L^y f(z) = (b(y) + (2\alpha - 1)\delta_0(z))\partial_x f(z) + \frac{1}{2}\sigma^2(y)\partial_x^2 f(x)$$

Closeness is measured by $Lf(z) - L^y f(z)$

The solution of the equation

$$Y_t = x + W_t + (2\alpha - 1)L_t^0(Y),$$

is called the "skew Brownian motion" (Harrison and Shepp [2]).
The density function of Y_t, $p_{Y_t}(x,\cdot)$, can be given explicitly by using the Gaussian density :
if $x \geq 0$

$$p_{Y_t}(x,y) - \overbrace{g_t(y-x)}^{p_t^D(x,y)} = (2\alpha - 1)\overbrace{\{g_t(y+x)\,1(y \geq 0) - g_t(y-x)\,1(y<0)\}}^{p_t^\alpha(x,y)},$$

and if $x < 0$

$$p_{Y_t}(x,y) - g_t(y-x) = (2\alpha - 1)\{-g_t(y+x)\,1(y<0) + g_t(y-x)\,1(y \geq 0)\}.$$

Note that $p_{Y_t}(x,y)$ satisfies the following condition:

$$\alpha\partial_x p_{Y_t}(0+,y) = (1-\alpha)\partial_x p_{Y_t}(0-,y)$$

and if $\alpha \neq 1/2$, $p_{Y_t}(x,\cdot)$ is discontinuous at 0 because

$$p_{Y_t}(x,0+) = 2\alpha g_t(x) \text{ and } p_{Y_t}(x,0-) = 2(1-\alpha)g_t(x).$$

▸ Distance Notice all the differentiability properties.

In the same way, the density $p_t^y(x,\cdot)$of $X_t^y = x + \sigma(y)W_t + (2\alpha - 1)L_t^0(X^y)$ is given explicitly. We denote $\overline{p}_t(x,y) := p_t^y(x,y)$. Then we can prove that

$$p_t(x,y) := \sum_{n=0}^{\infty} \overline{p} \circledast A^{\circledast n}(t,x,y)$$

is the density function of a skew diffusion $X_t(x)$ and other properties hold. Moreover, $p_t(x,y)$ has the same property of $p_{Y_t}(x,y)$:

$$\alpha \partial_x p_t(0+,y) = (1-\alpha)\partial_x p_t(0-,y).$$

and if $\alpha \neq 1/2$, $p_t(x,\cdot)$ is discontinuous at 0 and satisfies

$$p_t(x,y) \leq C \frac{\exp\left(-\frac{(y-x)^2}{2ct}\right)}{\sqrt{2\pi ct}}.$$

$p_t(\cdot,y)$ is a Lipschitz function with discontinuous derivative at $x = 0$. Notice that proving existence of $\partial_t p_t(x,y)$ is possible even at $x = 0$.

Some problems in detail

- The parametrix works for the skew Brownian motion because it is close enough. It probably does not work if one uses only Brownian motion.
- The idea of the proof requires three steps: In the first step, we use the parametrix method for semigroup of $X_t(x)$, we prove that the expansion holds for any $f \in C_c^\infty$ almost every $x \in \mathbb{R}_0$. So $p_T(x, \cdot)$ is the density function of $X_t(x)$ for almost every $x \in \mathbb{R}_0$.
  This weakness in the argument is due to the duality that is used in order to apply the backward parametrix method.
- In the second step, we prove the existence of the density $p_{Z_t}(z, \cdot)$ of $Z_t(z)$, $z \in \mathbb{R}_0$. Then using the continuity of the flow defined by Z and the continuity of the function $p_{Z_t}(z, \cdot)$ for every $z \in \mathbb{R}_0$, we obtain that $p_{Z_t}(z, \cdot)$ is the density for every $z \in \mathbb{R}_0$.
- Transform back from Z to X.
- Why does it work? Use the domain for the expansion: f differentiable in R_0 and satisfying $\alpha f'(0+) = (1-\alpha) f'(0-)$. L has the local time term as well as $\bar{L}$!

Probabilistic representation

Diffusion case

(1) $\{N_t; t \geq 0\}$ Poisson (1) (2)Jump times

(3) Euler-Maruyama (EM) scheme using this time partition.

$$X^{*,\pi}_{\tau_{i+1}} = X^{*,\pi}_{\tau_i} + \sigma(X^{*,\pi}_{\tau_i})(W(\tau_{i+1}) - W(\tau_i)) - b_1(X^{*,\pi}_{\tau_i})(\tau_{i+1} - \tau_i).$$

Here $X^{*,\pi}_0 = X_0$ follows the density f (assumption). Then we have

$$\mathbb{E}[f(X_T)] = e^T \mathbb{E}\left[\bar{p}_{T-\tau_{N_T}}(X^{*,\pi}_T, x) \prod_{j=0}^{N_T-1} \theta^D_{\tau_{j+1}-\tau_j}(X^{*,\pi}_{\tau_{j+1}}, X^{*,\pi}_{\tau_j})\right]$$

$$\theta^D_t(x,y) = \frac{1}{2}(a(y)-a(x))H^2_{ta(y)}(y-x-b_1(y)t)-(b(y)-b_1(x))H^1_{a(y)t}(y-x-b_1(y)t)$$

$$H^1_a(z) = -a^{-1}z \qquad H^2_a(y) = (a^{-1}z)(a^{-1}z) - a^{-1}.$$

Note the degeneration in $t^{-(1-\frac{\alpha}{2})}$ of the Hermite polynomials and $T - \tau_{N_T}$ above.

Idea of the proof Recall that $p_t(x,y) = \sum_{n=0}^{\infty} \bar{p} \circledast A^{\circledast n}(t,x,y)$. By the definition of convolution $\circledast$, we have

$$\bar{p} \circledast A^{\circledast n}(t,x,y) = \int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{\mathbb{R}^n} dy_1 \cdots dy_n \prod_{i=0}^{n-1} A_{t_i - t_{i+1}}(y_{i+1}, y_i) \bar{p}_{t_n}(y_{n+1}, y_n).$$

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Thanks to Vlad!