Algorithms & Complexity
Lecture 5: Dynamic Programming

October 19, 2020

CentraleSupélec / ESSEC Business School

Dimo Brockhoff

Inria Saclay – Ile-de-France

Corona Update

Taux d'incidence - semaine glissante

ACTIONS

Chiffres-clés 2020-09-11-2020-09-17

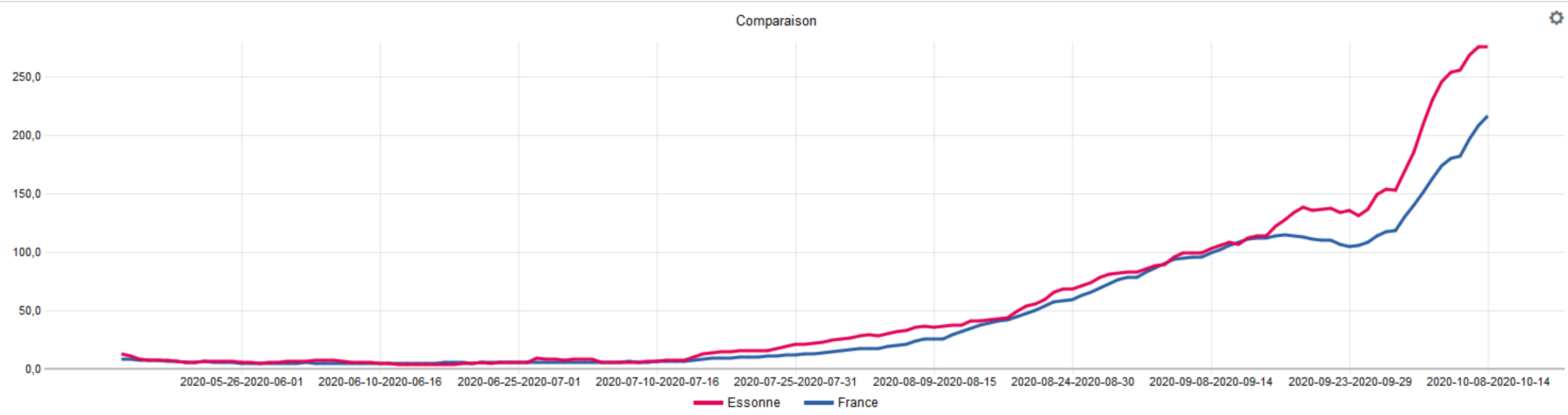
France : **107,9 pour 100 000 hab.**

Essonne : **106,1 pour 100 000 hab.**

Statistique	France
minimum	14,6 (Creuse - 23)
maximum	293,2 (Guadeloupe - 971)
médiane	66,4
observations valides	104 sur 104

Graphiques et comparaisons

Évolution temporelle comparée



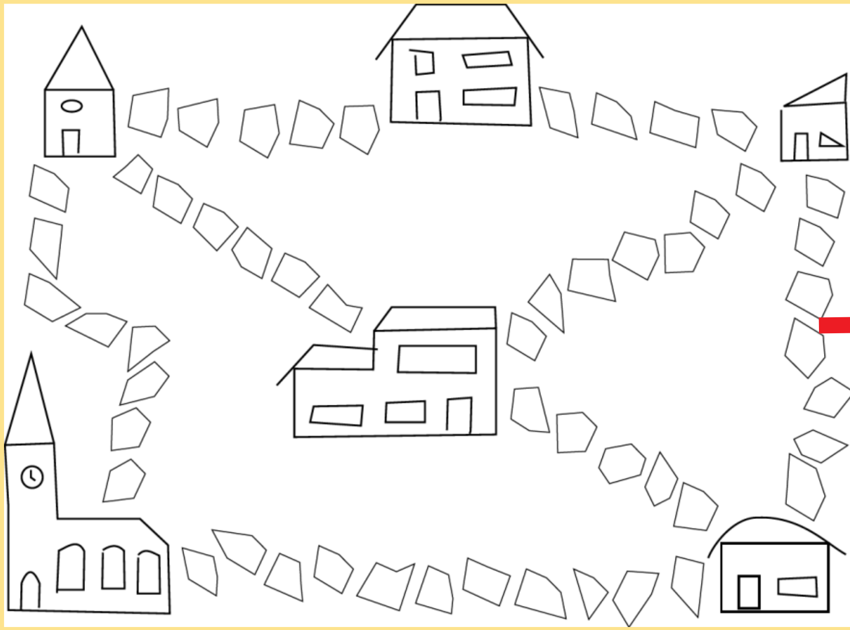
https://geodes.santepubliquefrance.fr/#bbox=38985,6323608,423056,255910&c=indicator&i=sp_ti_tp_7j.tx_pe_gliss&s=2020-09-11-2020-09-17&selcodgeo=91&t=a01&view=map2

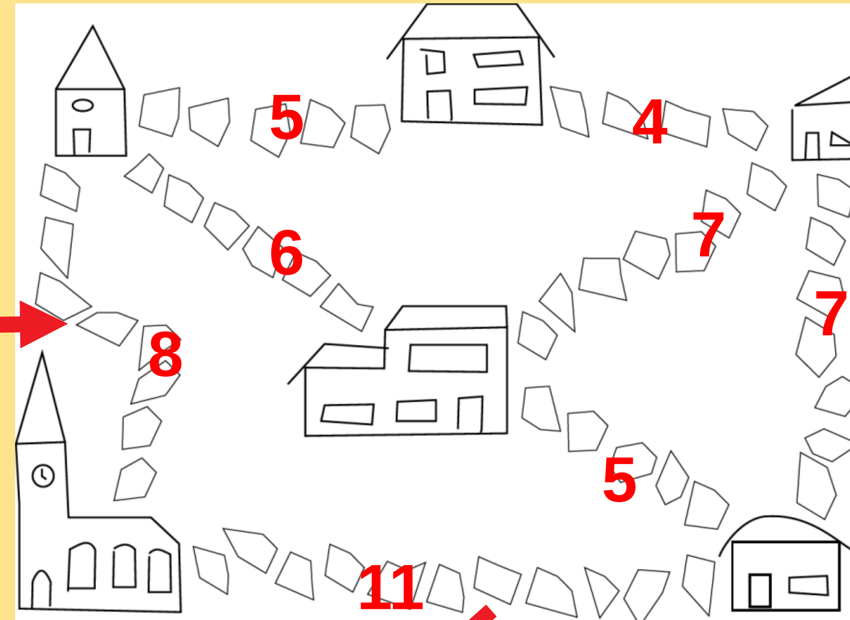
Course Overview

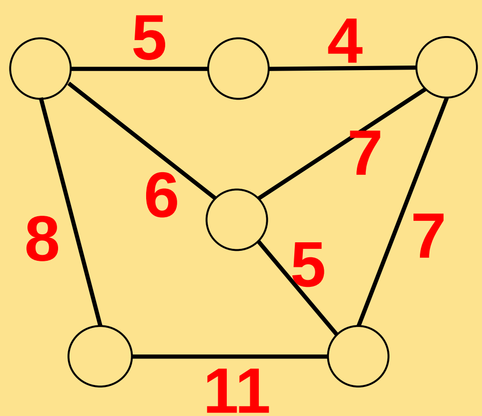
Thu		Topic
Mon, 21.09.2020	PM	Introduction, Combinatorics, O-notation, data structures
Mon, 28.09.2020	AM	Data structures II
Mon, 5.10.2020	AM	Sorting algorithms, recursive algorithms
Mon, 12.10.2020	PM	Greedy algorithms
➨ Mon, 19.10.2020	PM	Dynamic programming
Mon, 2.11.2020	PM	Randomized Algorithms and Blackbox Optimization
Mon, 16.11.2020	AM	Complexity theory I
Mon, 23.11.2020	AM	Complexity theory II
Mon, 14.12.2019	PM	Exam

Discussion Home Exercise

Exercise 1: Little Slopy Village

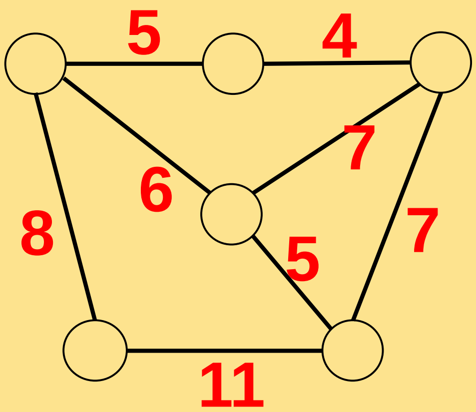






Discussion Home Exercise

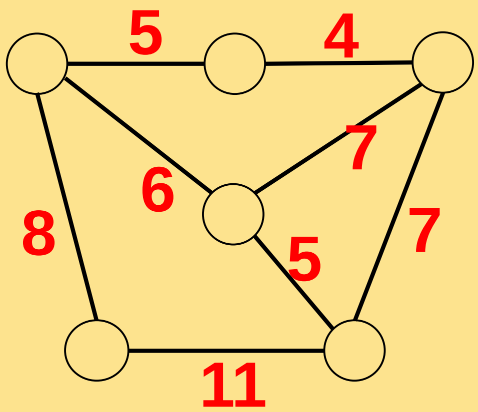
Exercise 1: Little Slopy Village

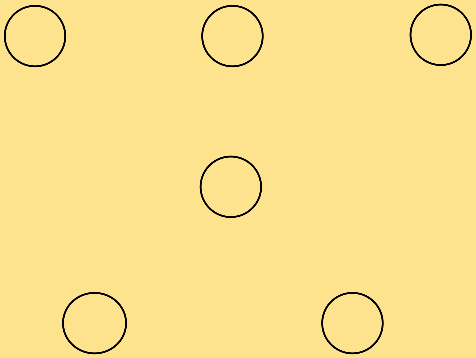


Algorithm of Kruskal:

take always the shortest edge that does not produce a cycle

Exercise 1: Little Slopy Village

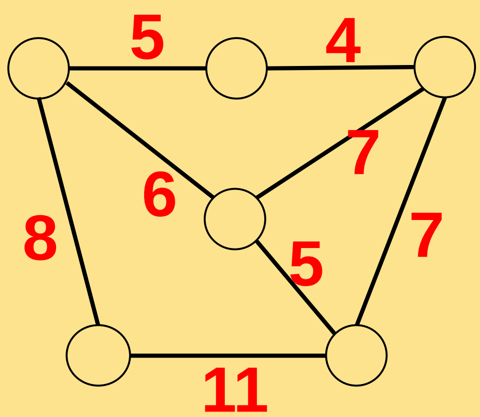


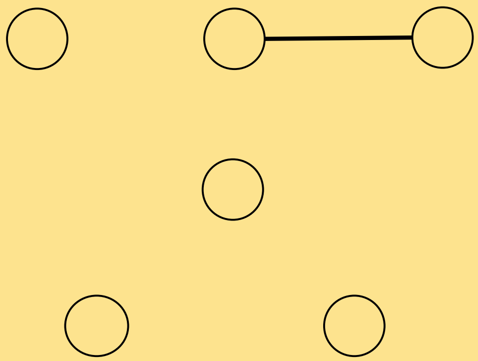


Algorithm of Kruskal:

take always the shortest edge that does not produce a cycle

Exercise 1: Little Slopy Village

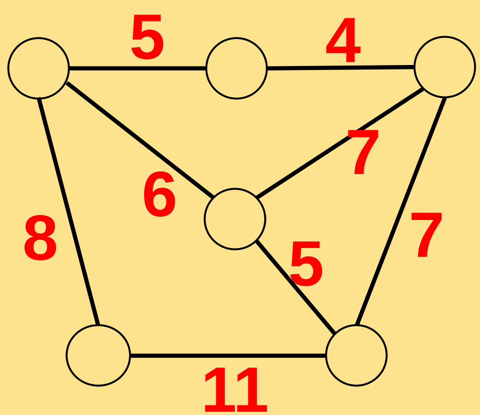


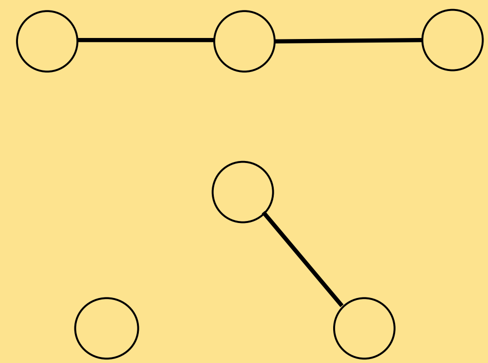


Algorithm of Kruskal:

take always the shortest edge that does not produce a cycle

Exercise 1: Little Slopy Village

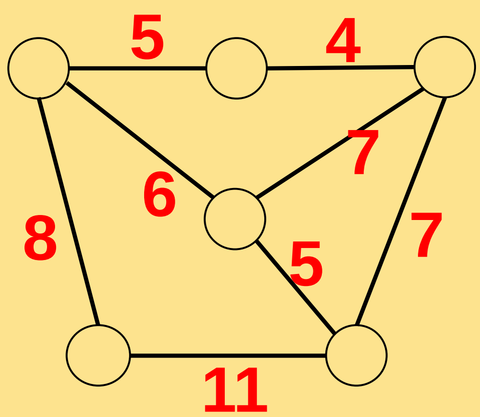


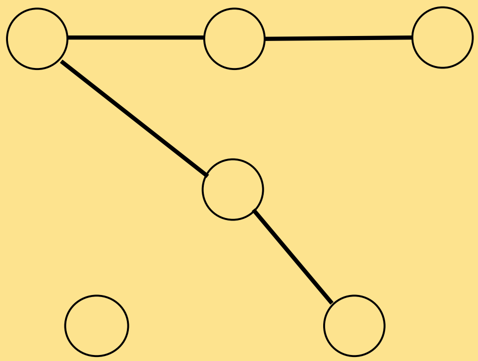


Algorithm of Kruskal:

take always the shortest edge that does not produce a cycle

Exercise 1: Little Slopy Village



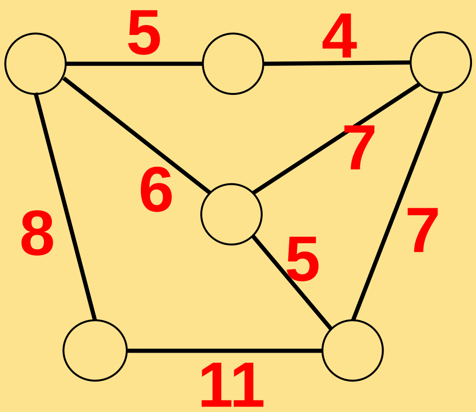


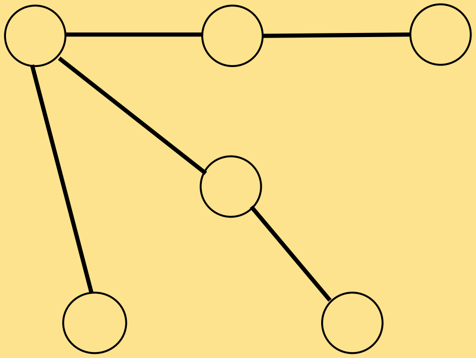
Algorithm of Kruskal:

take always the shortest edge that does not produce a cycle

Discussion Home Exercise

Exercise 1: Little Slopy Village





Algorithm of Kruskal:
take always the shortest edge that does not produce a cycle

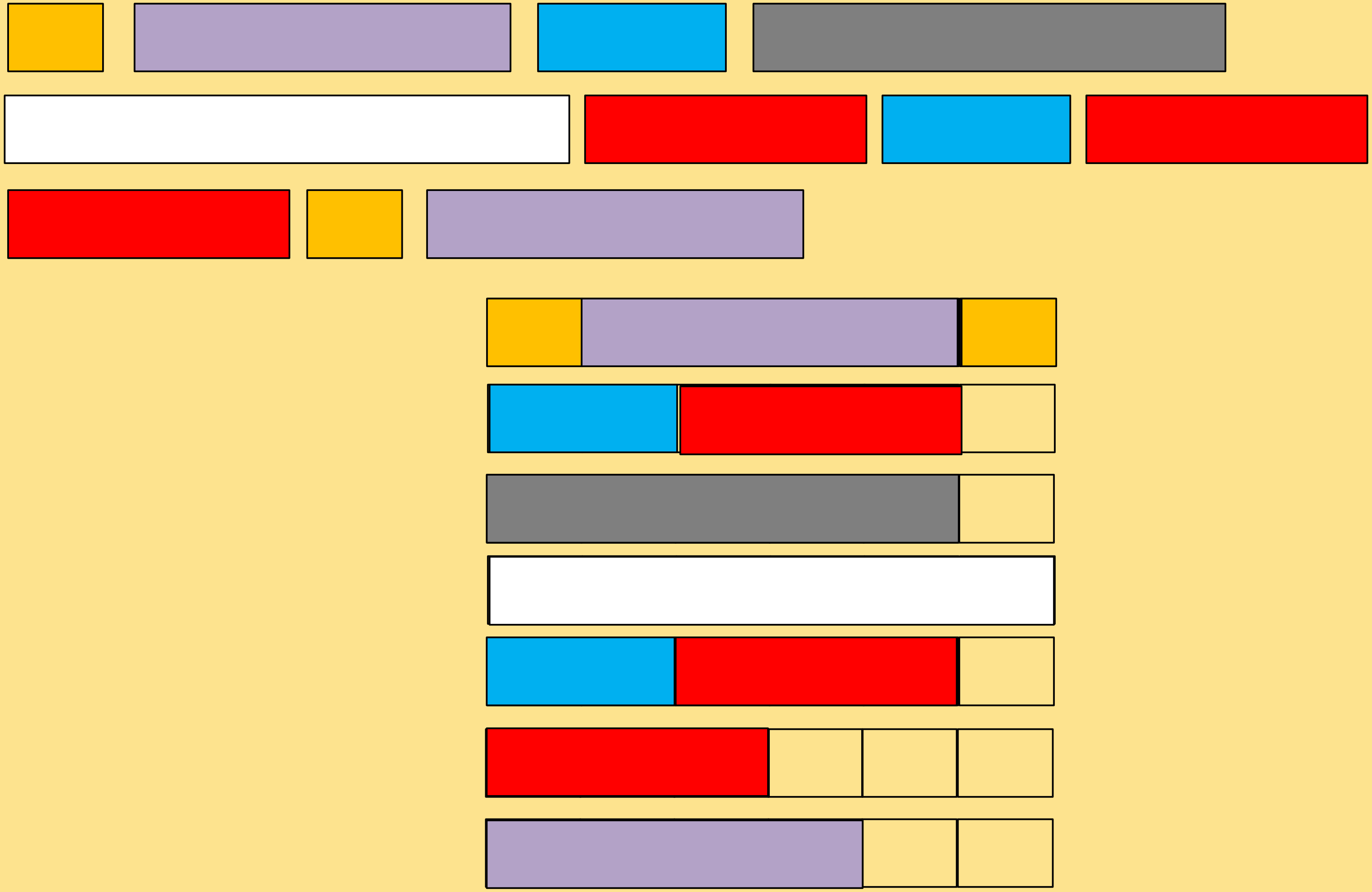
Here: total weight of 4+5+5+6+8 = 28

Note: solution not always unique, but always optimal

Discussion Home Exercise

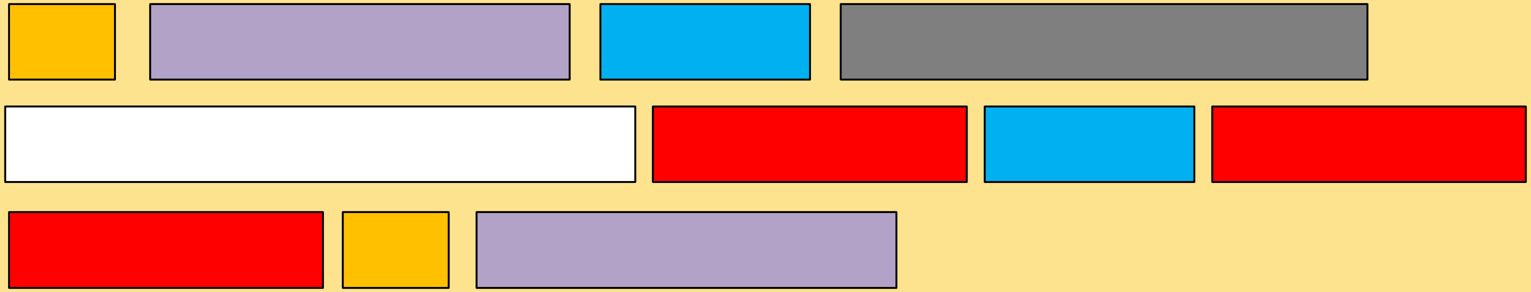
Exercise 2: Bin Packing

- items of size 1, 4, 2, 5, 6, 3, 2, 3, 3, 1, 4
- bin size: 6, first fit strategy

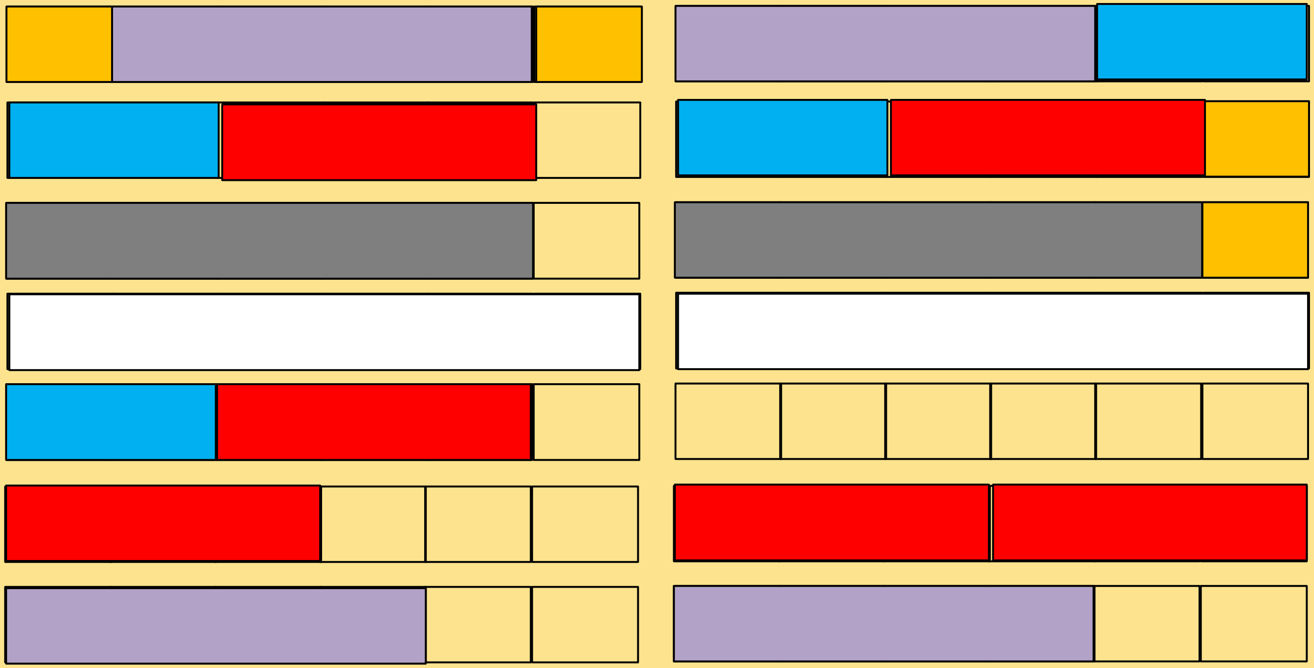


Exercise 2: Bin Packing

- items of size 1, 4, 2, 5, 6, 3, 2, 3, 3, 1, 4
- bin size: 6, first fit strategy



Optimal?



Discussion Home Exercise

Exercise 3: Assisting in a Robbery

- n items with weights $w_1, \ldots, w_n$ and values $v_1, \ldots, v_n$, max. load W
- calls for the knapsack problem

a) Potential Greedy Algorithm:

- take items according to their value-per-weight ratio v_i/w_i until total weight W is reached

b) Implementation: see .ipynb file

Discussion Home Exercise

Exercise 3: Assisting in a Robbery

- n items with weights $w_1, \ldots, w_n$ and values $v_1, \ldots, v_n$, max. load W
- calls for the knapsack problem

c) Is the greedy algorithm optimal?

no (see also complexity theory lectures in the end)

proof by counter example:

W=4	**1st item**	**2nd item**	**3rd item**	**4th item**
Value	4	3	2	1
Weight	1	1	1	1
Value/Weight	4	3	2	1

Greedy algorithm chooses item 1, although 2+3 is better

Dynamic Programming

Dynamic Programming

Wikipedia:

"[...] **dynamic programming** is a method for solving a complex problem by breaking it down into a collection of simpler subproblems."

But that's not all:

- dynamic programming also makes sure that the subproblems are not solved too often but only once by keeping the solutions of simpler subproblems in memory ("trading space vs. time")
- it is an exact method, i.e. in comparison to the greedy approach, it always solves a problem to optimality

Two Properties Needed

Optimal Substructure

A solution can be constructed efficiently from optimal solutions of sub-problems

Overlapping Subproblems

Wikipedia: “[...] a problem is said to have **overlapping subproblems** if the problem can be broken down into subproblems which are reused several times or [if] a recursive algorithm for the problem solves the same subproblem over and over rather than always generating new subproblems.”

Note: in case of optimal substructure but independent subproblems, often greedy algorithms are a good choice; in this case, dynamic programming is often called “divide and conquer” instead

Main Idea Behind Dynamic Programming

Main idea: solve larger subproblems by breaking them down to smaller, easier subproblems in a recursive manner

Typical Algorithm Design:

❶ decompose the problem into subproblems and think about how to solve a larger problem with the solutions of its subproblems

❷ specify how you compute the value of a larger problem recursively with the help of the optimal values of its subproblems ("Bellman equation")

❸ bottom-up solving of the subproblems (i.e. computing their optimal value), starting from the smallest by using a table structure to store the optimal values and the Bellman equality (top-down approach also possible, but less common)

❹ eventually construct the final solution (can be omitted if only the value of an optimal solution is sought)

Bellman Equation (aka “Principle of Optimality”)

- introduced by R. Bellman as “Principle of Optimality” in 1957
- the basic equation underlying dynamic programming
- necessary condition for optimality

citing Wikipedia:

> “Richard Bellman showed that a dynamic optimization problem in discrete time can be stated in a recursive, step-by-step form known as backward induction by writing down the relationship between the value function in one period and the value function in the next period. The relationship between these two value functions is called the "Bellman equation".”

- The value function here is the objective function.
- The Bellman equation exactly formalizes how to compute the optimal function value for a larger subproblem from the optimal function value of smaller subproblems.

we will see examples later today...

Historical Note

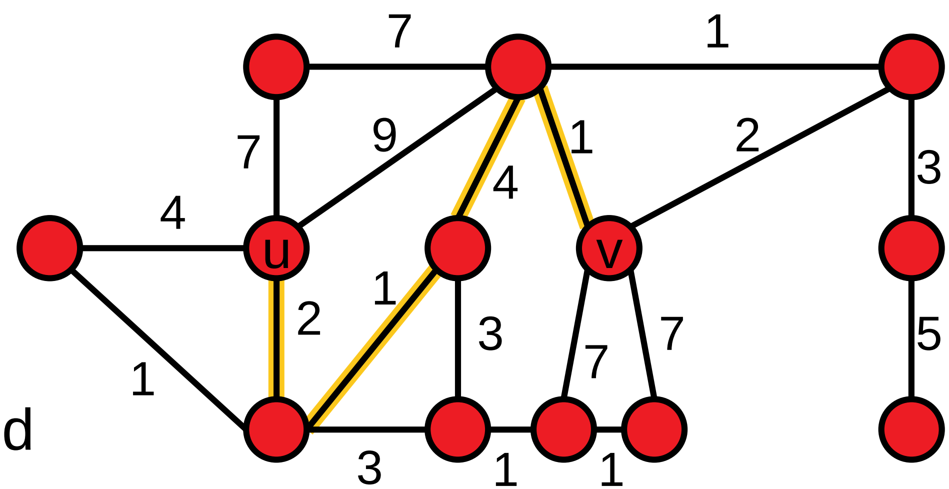
Why is it called "dynamic" and why "programming"?

- R. Bellman worked at the time, when he "invented" the idea, at the RAND Corporation who were strongly connected with the Air Force
- In order to avoid conflicts with the head of the Air Force at this time, R. Bellman decided against using terms like "mathematical" and he liked the word *dynamic* because it "has an absolutely precise meaning" and cannot be used "in a pejorative sense"
- in addition, it had the right meaning: "I wanted to get across the idea that this was dynamic, this was multistage, this was time-varying."
- Citing Wikipedia: "The word *programming* referred to the use of the method to find an optimal *program*, in the sense of a military schedule for training or logistics."

A First Example: Shortest Path Problem

Shortest Path problem:

Given a graph G=(V,E) with edge weights w_i for each edge e_i.
Find the shortest path from a vertex v to a vertex u, i.e., the path $(v, e_1=\{v, v_1\}, v_1, ..., v_k, e_k=\{v_k,u\}, u)$ such that $w_1 + ... + w_k$ is minimized.



Note:

We can often assume that the edge weights are stored in a distance matrix D of dimension |V|x|V| where an entry $D_{i,j}$ gives the weight between nodes i and j and "non-edges" are assigned a value of ∞

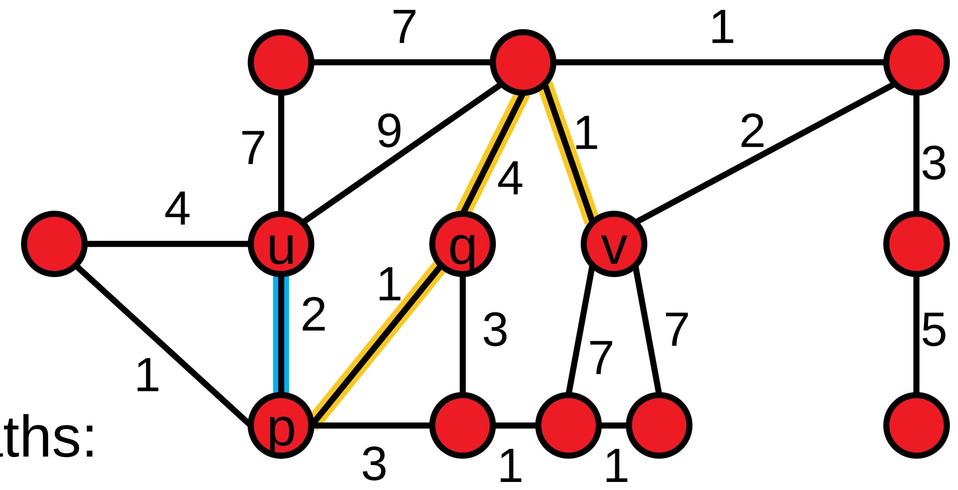
Opt. Substructure and Overlapping Subproblems

Optimal Substructure

The optimal path from u to v, if it contains another vertex p can be constructed by simply joining the optimal path from u to p with the optimal path from p to v.

Overlapping Subproblems

Optimal shortest sub-paths can be reused when computing longer paths: e.g. the optimal path from u to p is contained in the optimal path from u to q and in the optimal path from u to v.



The Algorithm of E. Dijkstra (1956)

Basic Idea:

- distinguish between visited and unvisited nodes
- in each step visit only one new node
- How?
  - choose the one with smallest distance to the current set of nodes
  - update all shortest path lengths of the new point's neighbors
  - keep track of second-to-last node on those shortest paths

The Algorithm of E. Dijkstra (1956)

ShortestPathDijkstra(G, D, source, target):

Initialization:

- dist(source) = 0 and for all v ∈ V: dist(v)= $D_{source,v}$
- for all v ∈ V:
  if $D_{source,v}$ finite: prev(v) = source # predecessors on opt. path
  else: prev(v) = None
- U = V \ {source} # U: unexplored vertices

Unless U empty do:

- newNode = $\text{argmin}_{u \in U}$ {dist(u)}
- remove newNode from U
- for each neighbor v of newNode do:
  - alternativeDist = dist(newNode) + $D_{newNode,v}$
  - if alternativeDist < dist(v):
    - dist(v) = alternativeDist
    - prev(v) = newNode

The Algorithm of R. Floyd (1962)

Idea:

- if we knew that the shortest path between source and target goes through node v, we would be able to construct the optimal path from the shorter paths "source→v" and "v→target"
- subproblem P(k): compute all shortest paths where the intermediate nodes can be chosen from $v_1, \dots, v_k$

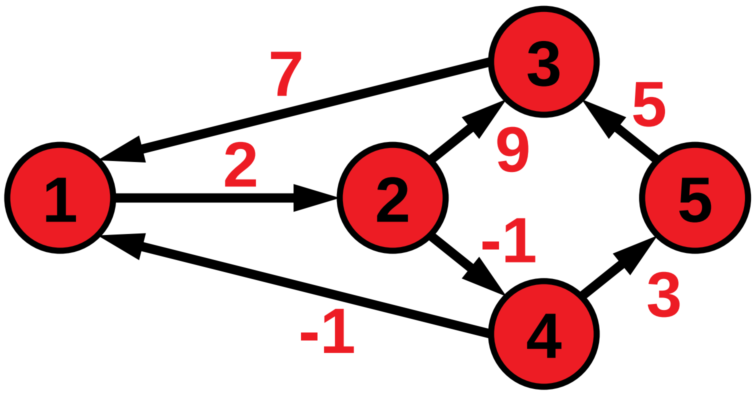
ShortestPathFloyd(G, D, source, target) [= AllPairsShortestPath(G)]

- Init: for all 1 ≤ i,j ≤ |V|: dist(i,j) = $D_{i,j}$
- For k = 1 to |*V*| # solve subproblems P(k)
  - for all pairs of nodes (i.e. 1 ≤ i,j ≤ |V|):
    - dist(i,j) = min { dist(i,j), dist(i,k) + dist(k,j) }

Note: This algorithm has the advantage that it can handle negative weights as long as no cycle with negative total weight exists

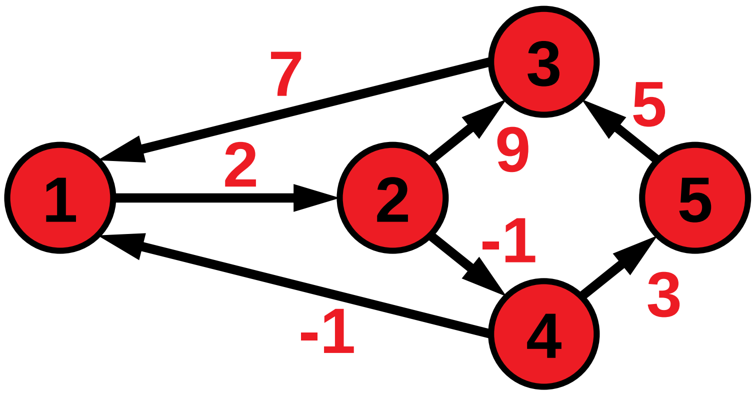
Note 2: distance $D_{i,i}$ could also be set to zero

Example



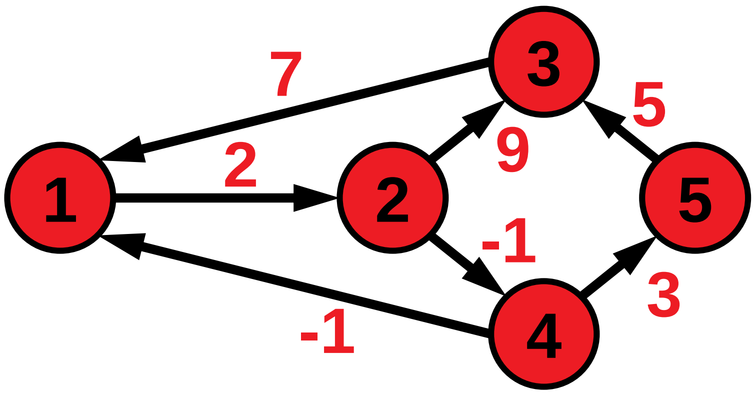
k=0	1	2	3	4	5
1					
2					
3					
4					
5					

Example



k=0	1	2	3	4	5
1	∞	2	∞	∞	∞
2	∞	∞	9	-1	∞
3	7	∞	∞	∞	∞
4	-1	∞	∞	∞	3
5	∞	∞	5	∞	∞

Example

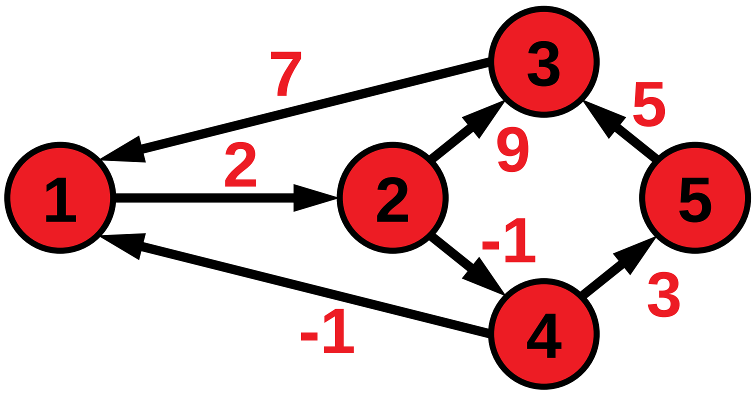


allow 1 as intermediate node

k=0	1	2	3	4	5
1	∞	2	∞	∞	∞
2	∞	∞	9	-1	∞
3	7	∞	∞	∞	∞
4	-1	∞	∞	∞	3
5	∞	∞	5	∞	∞

k=1	1	2	3	4	5
1					
2					
3					
4					
5					

Example

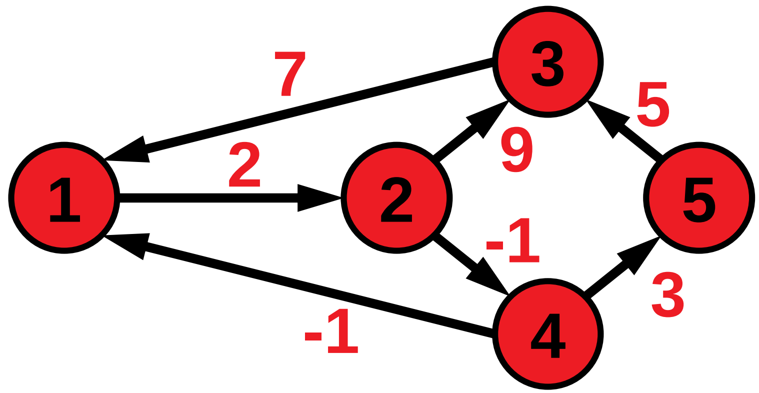


allow 1 as intermediate node

k=0	1	2	3	4	5
1	∞	2	∞	∞	∞
2	∞	∞	9	-1	∞
3	7	∞	∞	∞	∞
4	-1	∞	∞	∞	3
5	∞	∞	5	∞	∞

k=1	1	2	3	4	5
1					
2					
3					
4					
5					

Example

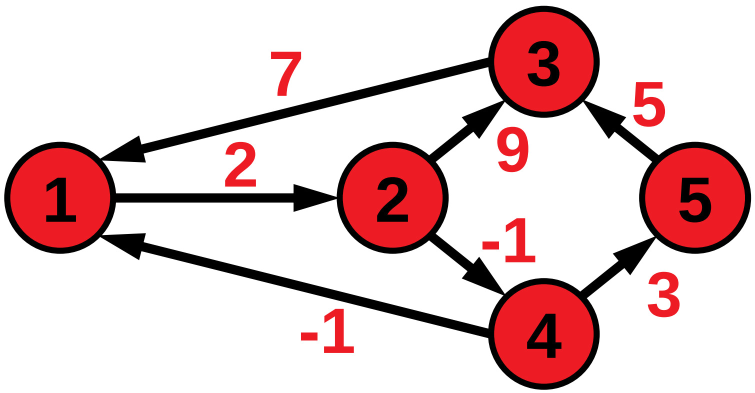


allow 1 as intermediate node

k=0	1	2	3	4	5
1	∞	2	∞	∞	∞
2	∞	∞	9	-1	∞
3	7	∞	∞	∞	∞
4	-1	∞	∞	∞	3
5	∞	∞	5	∞	∞

k=1	1	2	3	4	5
1					
2					
3					
4					
5					

Example

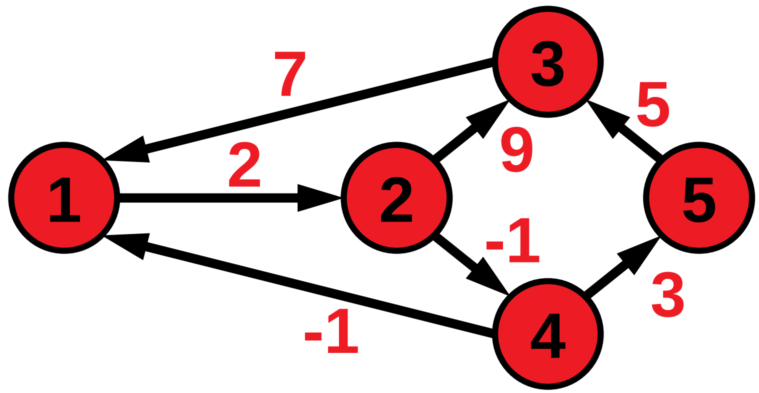


allow 1 as intermediate node

k=0	1	2	3	4	5
1	∞	2	∞	∞	∞
2	∞	∞	9	-1	∞
3	7	∞	∞	∞	∞
4	-1	∞	∞	∞	3
5	∞	∞	5	∞	∞

k=1	1	2	3	4	5
1					
2					
3		9			
4		1			
5					

Example

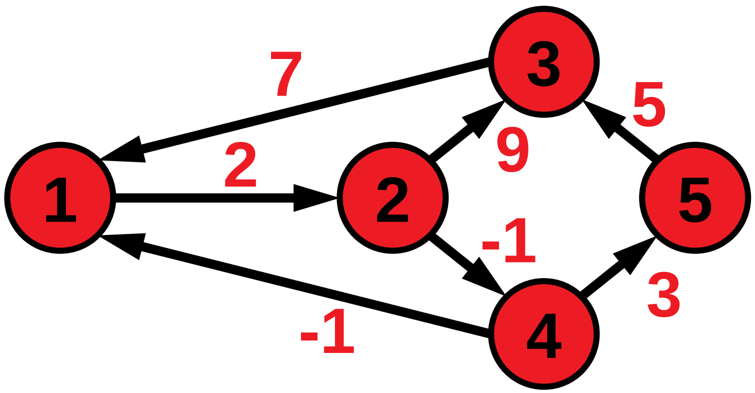


allow 1 as intermediate node

k=0	1	2	3	4	5
1	∞	2	∞	∞	∞
2	∞	∞	9	-1	∞
3	7	∞	∞	∞	∞
4	-1	∞	∞	∞	3
5	∞	∞	5	∞	∞

k=1	1	2	3	4	5
1	∞	2	∞	∞	∞
2	∞	∞	9	-1	∞
3	7	9	∞	∞	∞
4	-1	1	∞	∞	3
5	∞	∞	5	∞	∞

Example

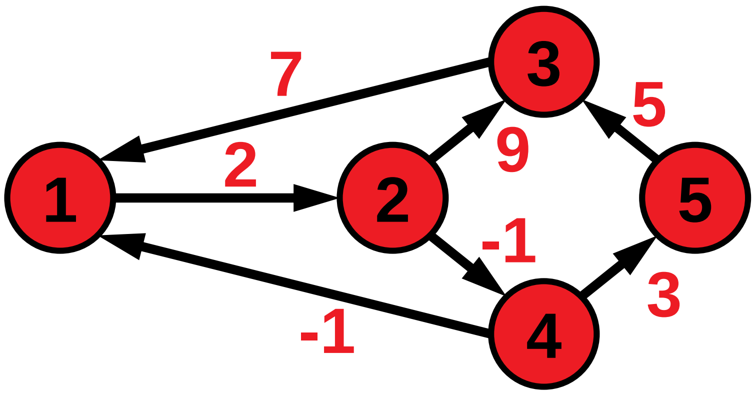


allow 1 & 2 as intermediate nodes

k=1	1	2	3	4	5
1	∞	2	∞	∞	∞
2	∞	∞	9	-1	∞
3	7	9	∞	∞	∞
4	-1	1	∞	∞	3
5	∞	∞	5	∞	∞

k=2	1	2	3	4	5
1	∞	2	∞	∞	∞
2	∞	∞	9	-1	∞
3	7	9	∞	∞	∞
4	-1	1	∞	∞	3
5	∞	∞	5	∞	∞

Example

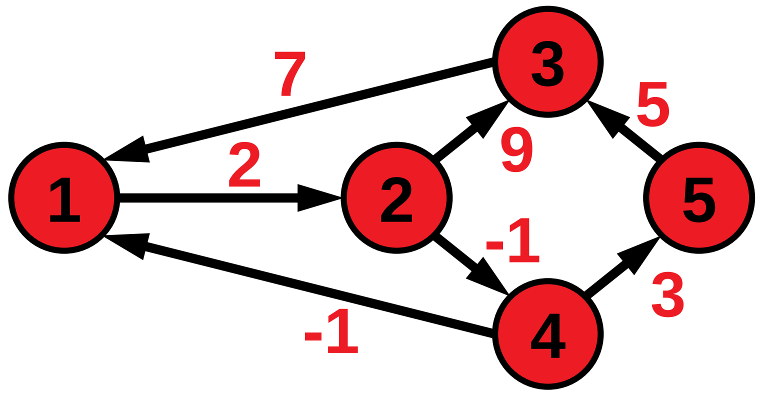


allow 1 & 2 as intermediate nodes

k=1	1	2	3	4	5
1	∞	2	∞	∞	∞
2	∞	∞	9	-1	∞
3	7	9	∞	∞	∞
4	-1	1	∞	∞	3
5	∞	∞	5	∞	∞

k=2	1	2	3	4	5
1	∞	2	∞	∞	∞
2	∞	∞	9	-1	∞
3	7	9	∞	∞	∞
4	-1	1	∞	∞	3
5	∞	∞	5	∞	∞

Example

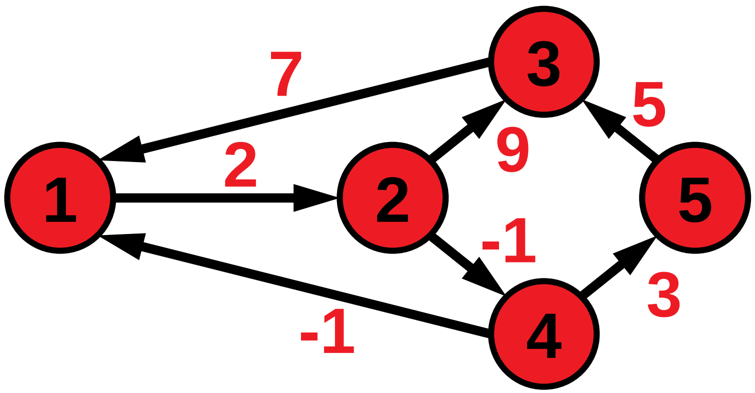


allow 1 & 2 as intermediate nodes

k=1	1	2	3	4	5
1	∞	2	∞	∞	∞
2	∞	∞	9	-1	∞
3	7	9	∞	∞	∞
4	-1	1	∞	∞	3
5	∞	∞	5	∞	∞

k=2	1	2	3	4	5
1	∞	2	11	1	∞
2	∞	∞	9	-1	∞
3	7	9	18	8	∞
4	-1	1	10	0	3
5	∞	∞	5	∞	∞

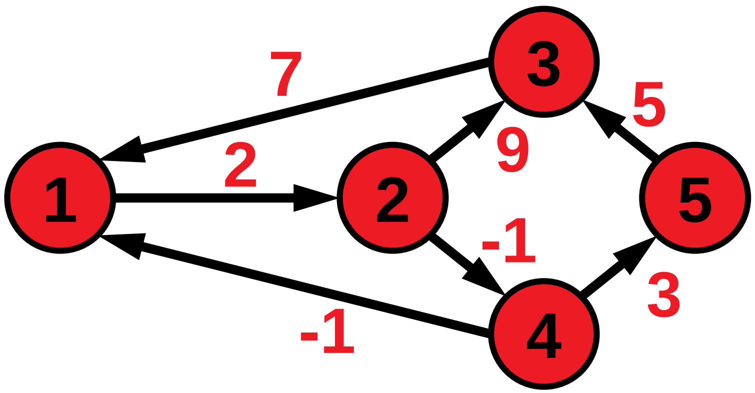
Example



allow {1,2,3} as intermediate nodes

k=2	1	2	3	4	5
1	∞	2	11	1	∞
2	∞	∞	9	-1	∞
3	7	9	18	8	∞
4	-1	1	10	0	3
5	∞	∞	5	∞	∞

k=3	1	2	3	4	5
1	∞	2	11	1	∞
2	∞	∞	9	-1	∞
3	7	9	18	8	∞
4	-1	1	10	0	3
5	∞	∞	5	∞	∞

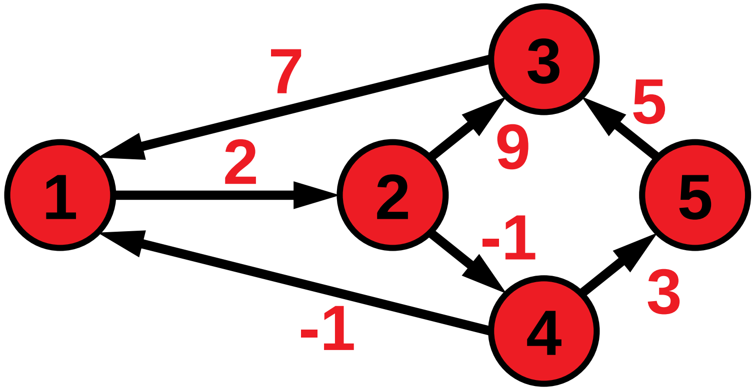


allow {1,2,3} as intermediate nodes

k=2	1	2	3	4	5
1	∞	2	11	1	∞
2	∞	∞	9	-1	∞
3	7	9	18	8	∞
4	-1	1	10	0	3
5	∞	∞	5	∞	∞

k=3	1	2	3	4	5
1			11		∞
2			9		∞
3	7	9	18	8	∞
4			10		3
5			5		∞

Example

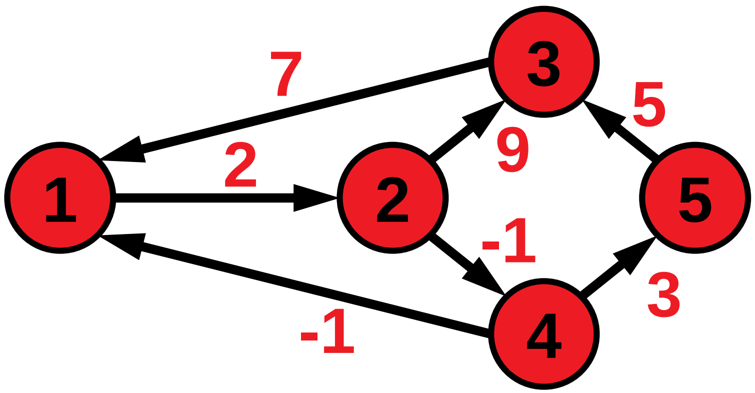


allow {1,2,3} as intermediate nodes

k=2	1	2	3	4	5
1	∞	2	11	1	∞
2	∞	∞	9	-1	∞
3	7	9	18	8	∞
4	-1	1	10	0	3
5	∞	∞	5	∞	∞

k=3	1	2	3	4	5
1	18	2	11	1	∞
2	16	18	9	-1	∞
3	7	9	18	8	∞
4	-1	1	10	0	3
5	12	14	5	13	∞

Example

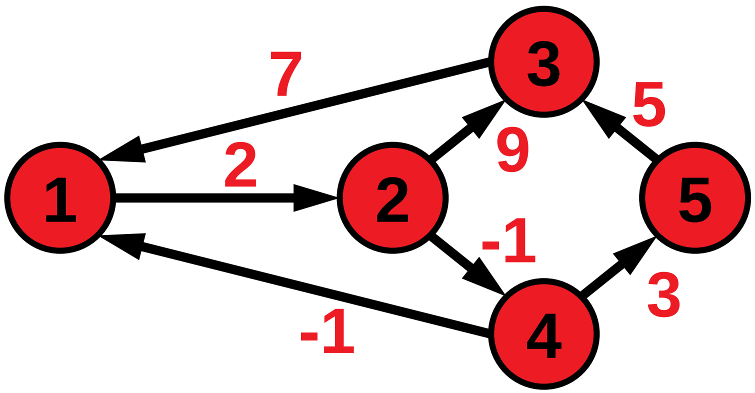


allow {1,2,3,4} as intermediate nodes

k=3	1	2	3	4	5
1	18	2	11	1	∞
2	16	18	9	-1	∞
3	7	9	18	8	∞
4	-1	1	10	0	3
5	12	14	5	13	∞

k=4	1	2	3	4	5
1	18	2	11	1	∞
2	16	18	9	-1	∞
3	7	9	18	8	∞
4	-1	1	10	0	3
5	12	14	5	13	∞

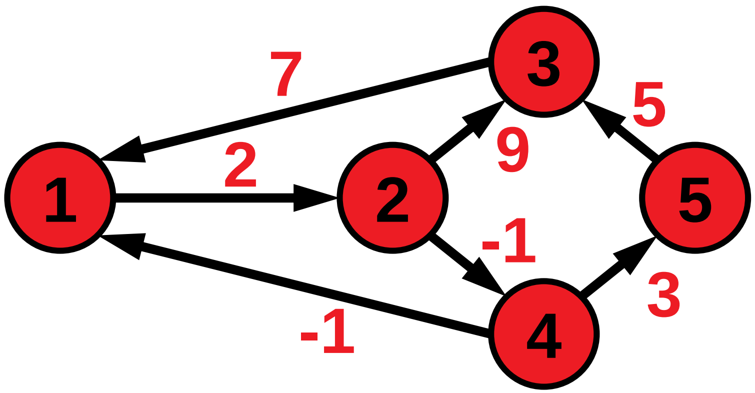
Example



allow {1,2,3,4} as intermediate nodes

k=3	1	2	3	4	5
1	18	2	11	1	∞
2	16	18	9	-1	∞
3	7	9	18	8	∞
4	-1	1	10	0	3
5	12	14	5	13	∞

k=4	1	2	3	4	5
1				1	
2				-1	
3				8	
4	-1	1	10	0	3
5				13	

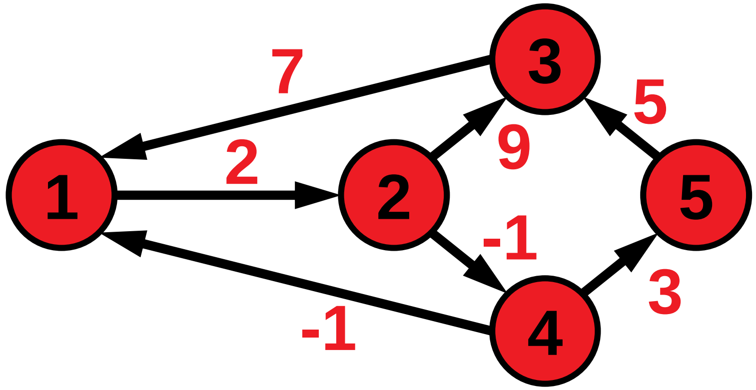


allow {1,2,3,4} as intermediate nodes

k=3	1	2	3	4	5
1	18	2	11	1	∞
2	16	18	9	-1	∞
3	7	9	18	8	∞
4	-1	1	10	0	3
5	12	14	5	13	∞

k=4	1	2	3	4	5
1	0	2	11	1	4
2	-2	0	9	-1	2
3	7	9	18	8	11
4	-1	1	10	0	3
5	12	14	5	13	16

Example

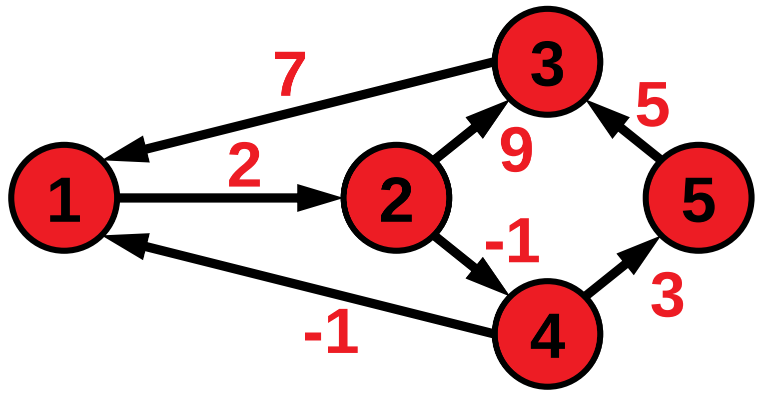


allow all nodes as intermediate nodes

k=4	1	2	3	4	5
1	0	2	11	1	4
2	-2	0	9	-1	2
3	7	9	18	8	11
4	-1	1	10	0	3
5	12	14	5	13	16

k=5	1	2	3	4	5
1	0	2	11	1	4
2	-2	0	9	-1	2
3	7	9	18	8	11
4	-1	1	10	0	3
5	12	14	5	13	16

Example



allow all nodes as intermediate nodes

k=4	1	2	3	4	5
1	0	2	11	1	4
2	-2	0	9	-1	2
3	7	9	18	8	11
4	-1	1	10	0	3
5	12	14	5	13	16

k=5	1	2	3	4	5
1	0	2	9	1	4
2	-2	0	7	-1	2
3	7	9	16	8	11
4	-1	1	8	0	3
5	12	14	5	13	16

Runtime Considerations and Correctness

$O(|V|^3)$ easy to show

- $O(|V|^2)$ many distances need to be updated $O(|V|)$ times

Correctness

- given by the Bellman equation

  dist(i,j) = min { dist(i,j), dist(i,k) + dist(k,j) }

- only correct if cycles do not have negative total weight (can be checked in final distance matrix if diagonal elements are negative)

But How Can We Actually Construct the Paths?

- Construct matrix of predecessors P alongside distance matrix
- $P_{i,j}(k)$ = predecessor of node j on path from i to j (at algo. step k)
- no extra costs (asymptotically)

$$P_{i,j}(0) = \begin{cases} 0 & \text{if } i = j \text{ or } d_{i,j} = \infty \\ i & \text{in all other cases} \end{cases}$$

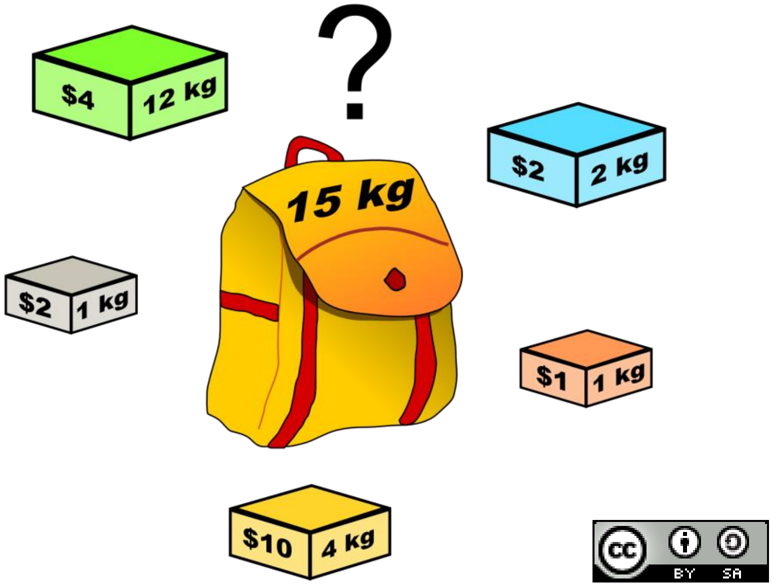
$$P_{i,j}(k) = \begin{cases} P_{i,j}(k-1) & \text{if } \mathrm{dist}(i,j) \leq \mathrm{dist}(i,k) + \mathrm{dist}(k,j) \\ P_{k,j}(k-1) & \text{if } \mathrm{dist}(i,j) > \mathrm{dist}(i,k) + \mathrm{dist}(k,j) \end{cases}$$

A Second Example: The 0-1 Knapsack Problem

0-1 Knapsack Problem (KP)

$$\max. \sum_{j=1}^{n} p_j x_j \text{ with } x_j \in \{0,1\}$$

$$\text{s.t.} \sum_{j=1}^{n} w_j x_j \leq W$$



Goal: a dynamic programming algorithm for KP

Questions:

a) what could be subproblems?
b) how to solve subproblems with the help of smaller ones?
c) how to solve the smallest subproblems exactly?

Opt. Substructure and Overlapping Subproblems

Possible subproblem:

$P(i)$: optimal profit when we allow to pack *only* i *items* into a knapsack

But how to construct solutions to the larger problems?

What about this possible subproblem?

$P(i,j)$: optimal profit when we allow to pack *only* i *items* into a knapsack of size j

Look like it's not possible to construct solutions to the larger problems from smaller ones either!

Opt. Substructure and Overlapping Subproblems

Consider now the following subproblem:

$P(i,j)$: optimal profit when allowed to pack *only the **first** i items* into a knapsack of size j

Opt. Substructure and Overlapping Subproblems

Consider now the following subproblem:

$P(i,j)$: optimal profit when allowed to pack *only the **first** i items* into a knapsack of size j

Optimal Substructure

The optimal choice of whether taking item i or not can be made easily for a knapsack of weight j if we know the optimal choice for items $1 \dots i-1$:

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1, j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Overlapping Subproblems

a recursive implementation of the Bellman equation is simple, but the $P(i,j)$ might need to be computed more than once!

Dynamic Programming Approach to the KP

To circumvent computing the subproblems more than once, we can store their results (in a matrix for example)...

knapsack weight →

items ↓

P(i,j)	0	1	2	3	...			W-1	W
0									
1					P(i,j)				
2									
...									
n-1									
n									

best achievable profit with items 1...i and a knapsack of size j

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits (5,4), (7,10), (2,3), (4,5), and (3,3). Weight restriction is W=11.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0												
1												
2												
3												
4												
5												

initialization:

$P(i,j) = 0$ if $i = 0$ or $j = 0$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits (5,4), (7,10), (2,3), (4,5), and (3,3). Weight restriction is W=11.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0											
2	0											
3	0											
4	0											
5	0											

initialization:

$P(i,j) = 0$ if $i = 0$ or $j = 0$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5,4)$, $(7,10)$, $(2,3)$, $(4,5)$, and $(3,3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0											
2	0											
3	0											
4	0											
5	0											

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1,j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5,4)$, $(7,10)$, $(2,3)$, $(4,5)$, and $(3,3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0										
2	0											
3	0											
4	0											
5	0											

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1,j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5,4)$, $(7,10)$, $(2,3)$, $(4,5)$, and $(3,3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0									
2	0											
3	0											
4	0											
5	0											

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1,j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5{,}4)$, $(7{,}10)$, $(2{,}3)$, $(4{,}5)$, and $(3{,}3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0							
2	0											
3	0											
4	0											
5	0											

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1,j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5,4)$, $(7,10)$, $(2,3)$, $(4,5)$, and $(3,3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	4						
2	0											
3	0											
4	0											
5	0											

$+p_1(=4)$

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1,j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5,4)$, $(7,10)$, $(2,3)$, $(4,5)$, and $(3,3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	4	4					
2	0											
3	0											
4	0											
5	0											

$+p_1 (= 4)$

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1, j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5,4)$, $(7,10)$, $(2,3)$, $(4,5)$, and $(3,3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	4	4	4	4	4	4	4
2	0											
3	0											
4	0											
5	0											

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1,j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5,4)$, $(7,10)$, $(2,3)$, $(4,5)$, and $(3,3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	4	4	4	4	4	4	4
2	0	0	0	0	0	4	4					
3	0											
4	0											
5	0											

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1,j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5,4)$, $(7,10)$, $(2,3)$, $(4,5)$, and $(3,3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	4	4	4	4	4	4	4
2	0	0	0	0	0	4	4	10				
3	0											
4	0											
5	0											

$+p_2(=10)$

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1,j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5,4)$, $(7,10)$, $(2,3)$, $(4,5)$, and $(3,3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	4	4	4	4	4	4	4
2	0	0	0	0	0	4	4	10	10	10	10	10
3	0											
4	0											
5	0											

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1,j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5,4)$, $(7,10)$, $(2,3)$, $(4,5)$, and $(3,3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	4	4	4	4	4	4	4
2	0	0	0	0	0	4	4	10	10	10	10	10
3	0	0	3	3	3							
4	0											
5	0											

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1,j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5{,}4)$, $(7{,}10)$, $(2{,}3)$, $(4{,}5)$, and $(3{,}3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	4	4	4	4	4	4	4
2	0	0	0	0	0	4	4	10	10	10	10	10
3	0	0	3	3	3	4						
4	0											
5	0											

$+p_3(= 3)$

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1,j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5,4)$, $(7,10)$, $(2,3)$, $(4,5)$, and $(3,3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	4	4	4	4	4	4	4
2	0	0	0	0	0	4	4	10	10	10	10	10
3	0	0	3	3	3	4	4					
4	0											
5	0											

$+p_3 (= 3)$

for $i = 1$ to n:

  for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1, j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5,4)$, $(7,10)$, $(2,3)$, $(4,5)$, and $(3,3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	4	4	4	4	4	4	4
2	0	0	0	0	0	4	4	10	10	10	10	10
3	0	0	3	3	3	4	4	10	etc.			
4	0											
5	0											

$+p_3 (= 3)$

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1, j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5,4)$, $(7,10)$, $(2,3)$, $(4,5)$, and $(3,3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	4	4	4	4	4	4	4
2	0	0	0	0	0	4	4	10	10	10	10	10
3	0	0	3	3	3	4	4	10	10	13	13	13
4	0	0	3	3	5	5	8	10	10	13	13	15
5	0	0	3	3	5	6	8	10	10	13	13	15

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1,j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Example instance with 5 items with weights and profits $(5,4)$, $(7,10)$, $(2,3)$, $(4,5)$, and $(3,3)$. Weight restriction is $W = 11$.

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	4	4	4	4	4	4	4
2	0	0	0	0	0	4	4	10	10	10	10	10
3	0	0	3	3	3	4	4	10	10	13	13	13
4	0	0	3	3	5	5	8	10	10	13	13	15
5	0	0	3	3	5	6	8	10	10	13	13	**15**

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1, j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Dynamic Programming Approach to the KP

Question: How to obtain the actual packing?

Answer: we just need to remember where the max came from!

knapsack weight →

items ↓

P(i,j)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	4	4	4	4	4	4	4
2	0	0	0	0	0	4	4	10	10	10	10	10
3	0	0	3	3	3	4	4	10	10	13	13	13
4	0	0	3	3	5	5	8	10	10	13	13	15
5	0	0	3	3	5	6	8	10	10	13	13	15

$x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$, $x_5 = 0$

for $i = 1$ to n:

for $j = 1$ to W:

$$P(i,j) = \begin{cases} P(i-1,j) & \text{if } w_i > j \\ \max\{P(i-1,j), p_i + P(i-1,j-w_i)\} & \text{if } w_i \leq j \end{cases}$$

Runtime Considerations

- If we try all possible combinations, we can solve the KP in time $O(2^n)$
- With the dynamic programming approach, we can do it in $O(nW)$
- For small enough weights (of the knapsack), this is quicker
- We might come back to this in the lectures on computational complexity…

Conclusions

I hope it became clear...

...what the algorithm design idea of dynamic programming is
...for which problem types it is supposed to be suitable
...and how to apply the idea to the knapsack problem