Exercise: Polynomial Reductions

Introduction to Optimization lecture
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Abstract

In the lecture, we have introduced the general concept of *polynomial reductions* to prove the NP-completeness of decision problems. It is the purpose of this exercise to prove two more polynomial reductions between other optimization problems. On the one hand, we will polynomially reduce the (undirected) HAMILTONIAN CIRCUIT (HC) problem to the TRAVELING SALESPERSON PROBLEM (TSP) and, on the other hand, we will polynomially reduce VERTEX COVER to CLIQUE.

1 Part I: HC $\leq_p$ TSP

Given an undirected graph $G_1 = (V_1, E_1)$, the decision problem HAMILTONIAN CIRCUIT, or HC for short, asks the question of whether G contains a Hamiltonian cycle, i.e., a closed path through G which touches all vertices of G. The TRAVELING SALESPERSON PROBLEM, or TSP for short, is defined by an undirected graph $G_2 = (V_2, E_2)$ with edge weights w_i for each edge $e_i \in E_2$ (also called lengths of the travels between the nodes) and a given threshold length $L > 0$ and asks whether there is a tour through the graph which visits each vertex once and which total length (in terms of the lengths of the traversed edges) is upper bounded by L.

Both problems seem quite related and it is the purpose of this exercise to prove a polynomial reduction from HC to TSP (HC $\leq_p$ TSP). Together with the proofs of 3-SAT $\leq_p$ DHC and DHC $\leq_p$ HC from the lecture and the theorem of Cook that 3-SAT is NP-complete, we thus prove here that also TSP is NP-complete (because it is trivial to show that TSP $\in$ NPC.

Note that if you do not have any idea of how to do the transformation, it seems to be a good idea for both exercise parts to come up with (few and small) example graphs in order to see a pattern between the solutions to both problems in the requested reduction.

2 Part II: VERTEX COVER $\leq_p$ CLIQUE

A vertex cover of a graph $G = (V, E)$ is a subset $S \subseteq V$ of its vertices such that for each edge $e = (v, w)$ in E, either v or w (or both) is in S. If S is a vertex cover of G, we also say that the vertices in S "cover" the edges of G. The decision problem VERTEX COVER asks for a given graph $G = (V, E)$ and a threshold $k \in \mathbb{N}$ whether G contains a vertex cover with at most k vertices.

A clique in a graph $G = (V, E)$ is a subset $S \subseteq V$ of its vertices such that all possible edges between vertices in S are contained in the original graph. In other words a clique is a subgraph, which is complete. The CLIQUE decision problem, given a graph $G = (V, E)$ and a threshold $k \in \mathbb{N}$ asks whether a clique of at least k vertices exists in G.

Please prove the polynomial reduction VERTEX COVER $\leq_p$ CLIQUE. Note that both problems are known to be NP-complete as well.