

Theoretical Issues of Evolutionary Multiobjective Optimization: Selected Research Topics and Open Problems

Dimo Brockhoff

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Microsoft
Research



Dimo Brockhoff



study of CS (Dipl. inform.) in
Dortmund, Germany



ETH

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Dr. sc. ETH at
ETH Zurich, Switzerland



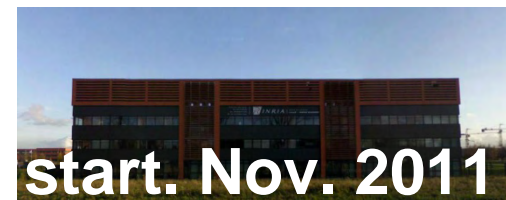
postdoc at
INRIA Saclay – Ile-de-France



postdoc at
Ecole Polytechnique



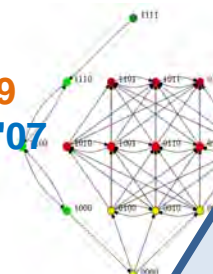
chargé de recherche (CR2)
INRIA Lille Nord-Europe



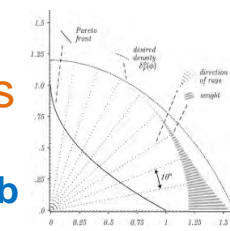
Contributions in EMO

Theory ^{chapter '10}

Runtime analyses
 IEEE-TEC '09
 ACM-GECCO '07

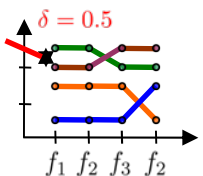


optimal μ -distributions
 ACM-FOGA '09
 ACM-GECCO '09b
 PPSN '10b
 TCS '11
 SEAL'10

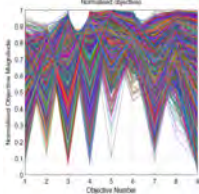


Runtime analyses
 PPSN '08

Many-Objective Optimization and Objective Reduction



ECJ '09
 chapter '07
 PPSN '06
 IEEE-CEC '07
 MOPGP '08
 OR '06

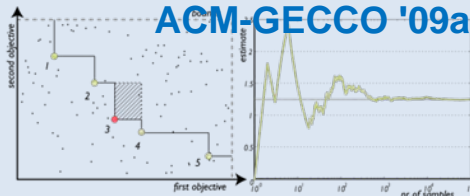


Hypervolume-Based Search

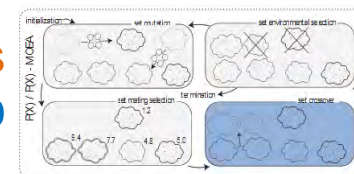
PhD thesis '09

Hypervolume Sampling

ACM-GECCO '09a

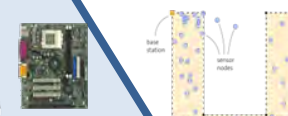


Set-Based EAs
 EMO '09



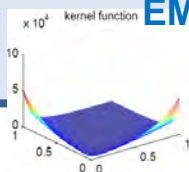
Network processor

ACM-GECCO '08



Weighted Hypervolume

EMO '07



Wireless Sensor Networks

MCDM '08



Hazmat Routing

CTW '11

Radar Waveforms



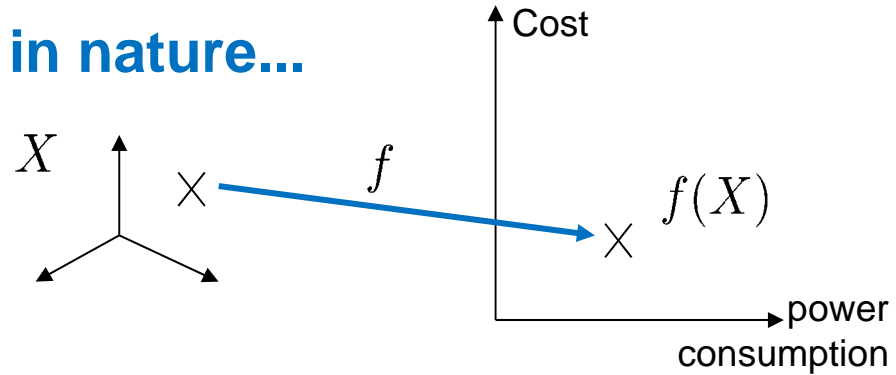
Algorithms

Applications

Blackbox Optimization

Most problems are multiobjective in nature...

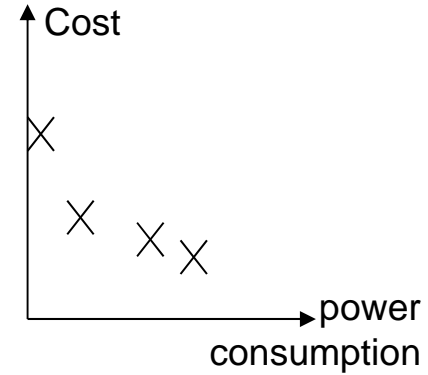
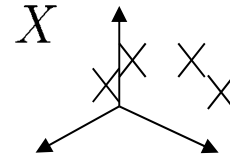
$$\min_{x \in X} f(x) = (f_1(x), \dots, f_k(x)) \in \mathbb{R}^k$$



Blackbox Optimization

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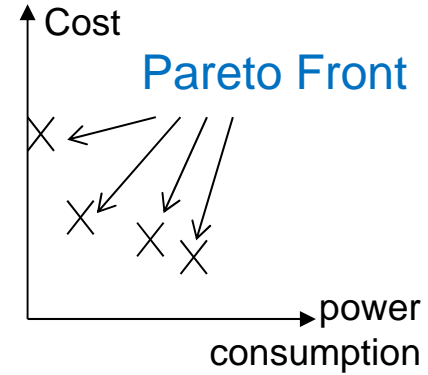
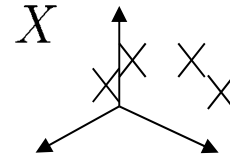
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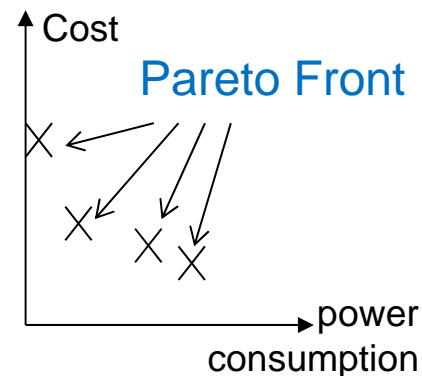
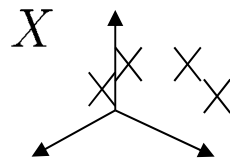
$$\min_{x \in X} f(x) = (f_1(x), \dots, f_k(x)) \in \mathbb{R}^k$$



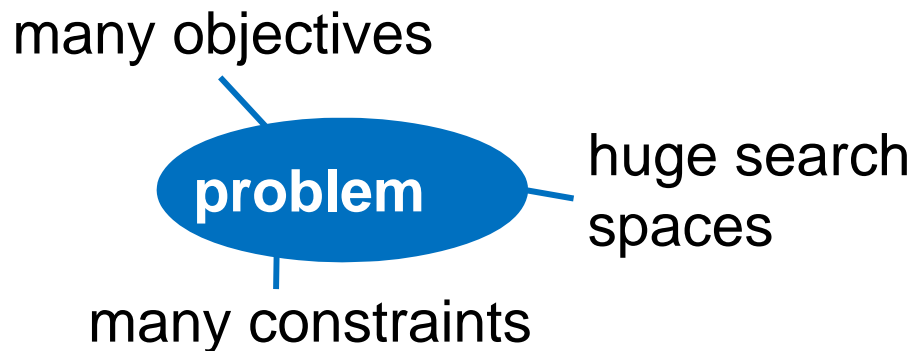
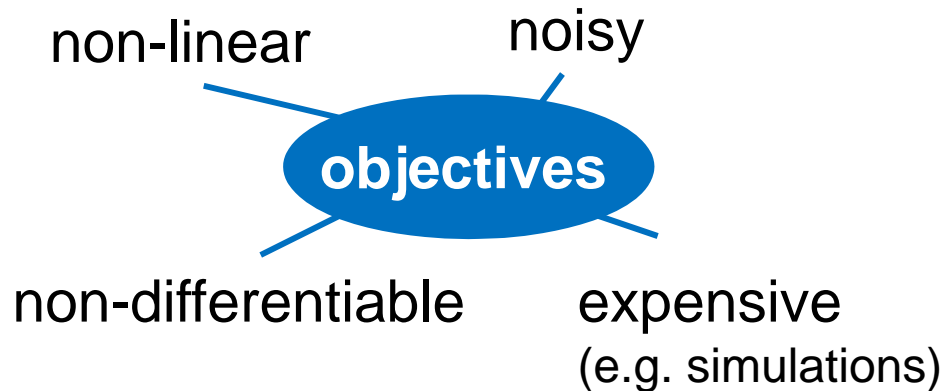
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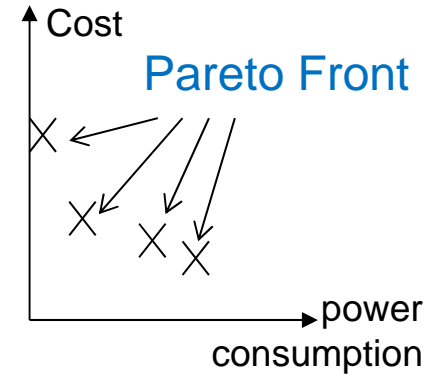
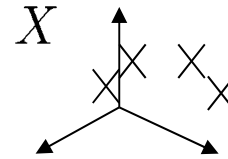
Issues:



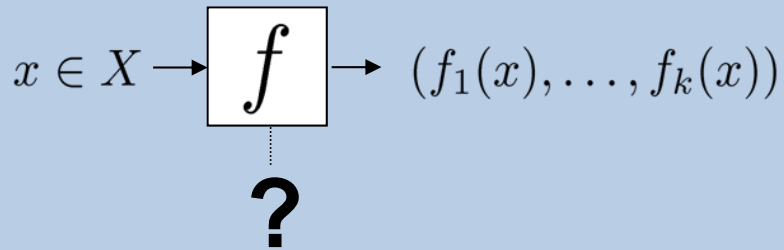
Blackbox Optimization

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Blackbox optimization



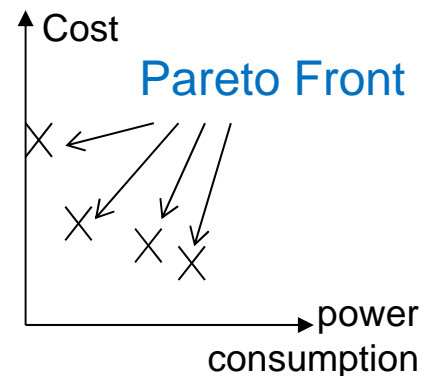
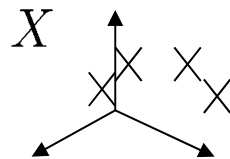
Features:

- function f used as an *oracle*
- only mild locality assumptions

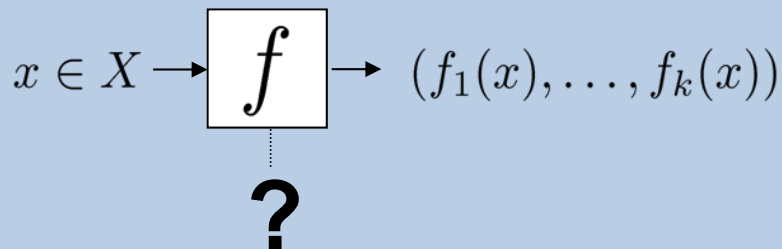
Blackbox Optimization

Most problems are multiobjective in nature...

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Blackbox optimization



Features:

- function f used as an *oracle*
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Evolutionary Multiobjective Optimization (EMO)

EMO =

randomized search heuristics
optimizing on solution **sets**

→ “**sampling**” the Pareto front
to **inform** decision maker

Main Purpose of My Talk

- Talk about some of my work
- A subjective list of “hot topics” in the theory of EMO
- Share interesting open questions and ideas

Why?

- build foundation for later discussions this week
- have content for possible collaborations/thesis topics

the GECCO deadline is soon ;-)

Benchmarking

“on how to compare sets of solutions”

Indicator-based Search and Preference Articulation

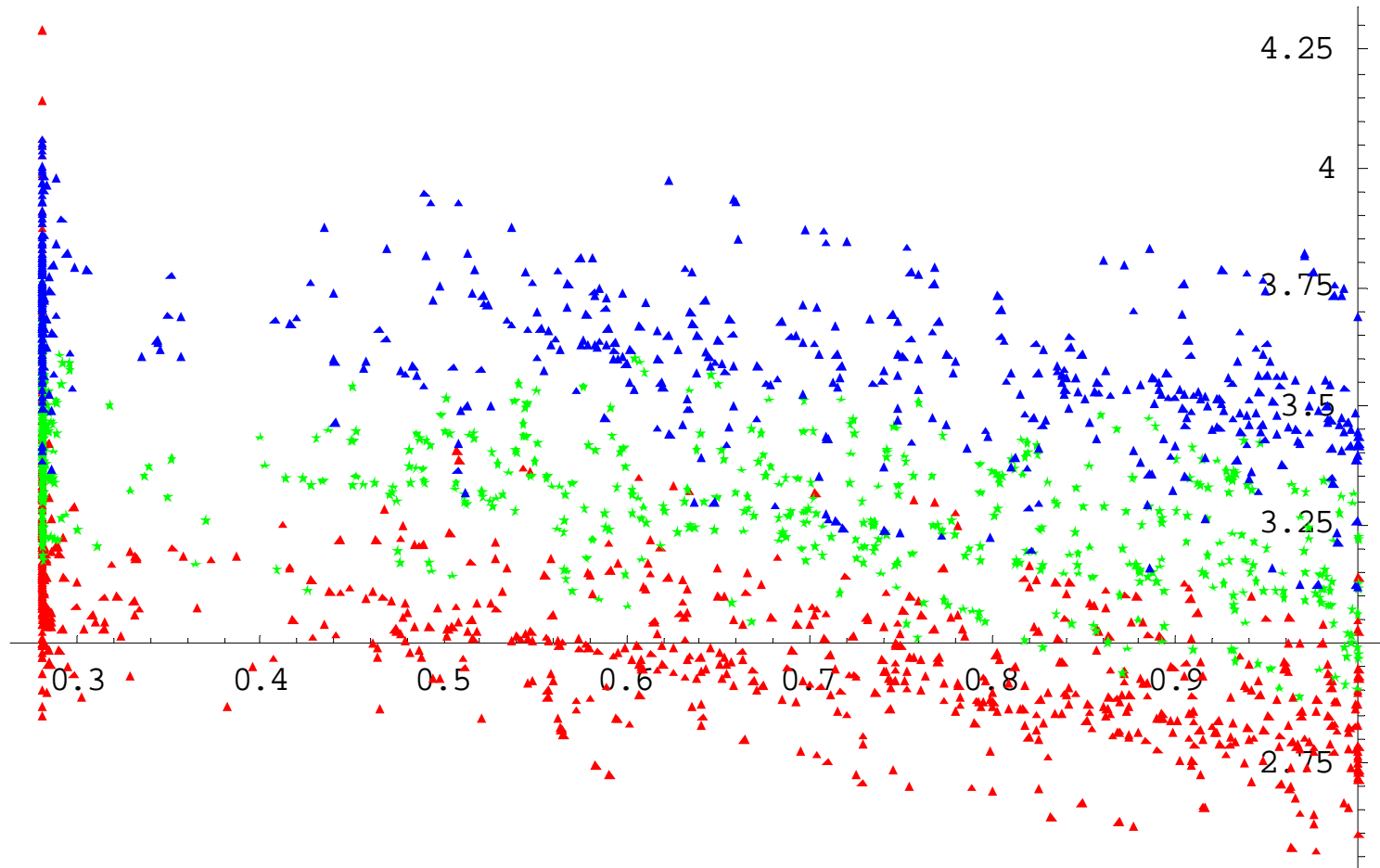
“on how to optimize and steer the search in many-objective problems”

Objective Reduction and Multiobjectivization

“on when to reduce and when to increase the number of objectives”

Once Upon a Time...

... multiobjective EAs were mainly compared visually:

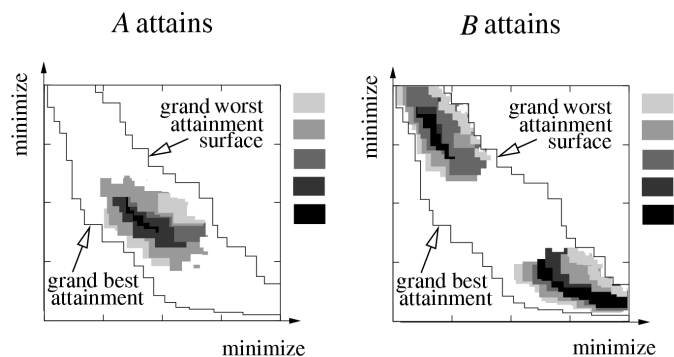


ZDT6 benchmark problem: **IBEA**, **SPEA2**, **NSGA-II**

Two Approaches for Empirical Studies

Attainment function approach:

- Applies statistical tests directly to the samples of approximation sets
- Gives detailed information about how and where performance differences occur



Quality indicator approach:

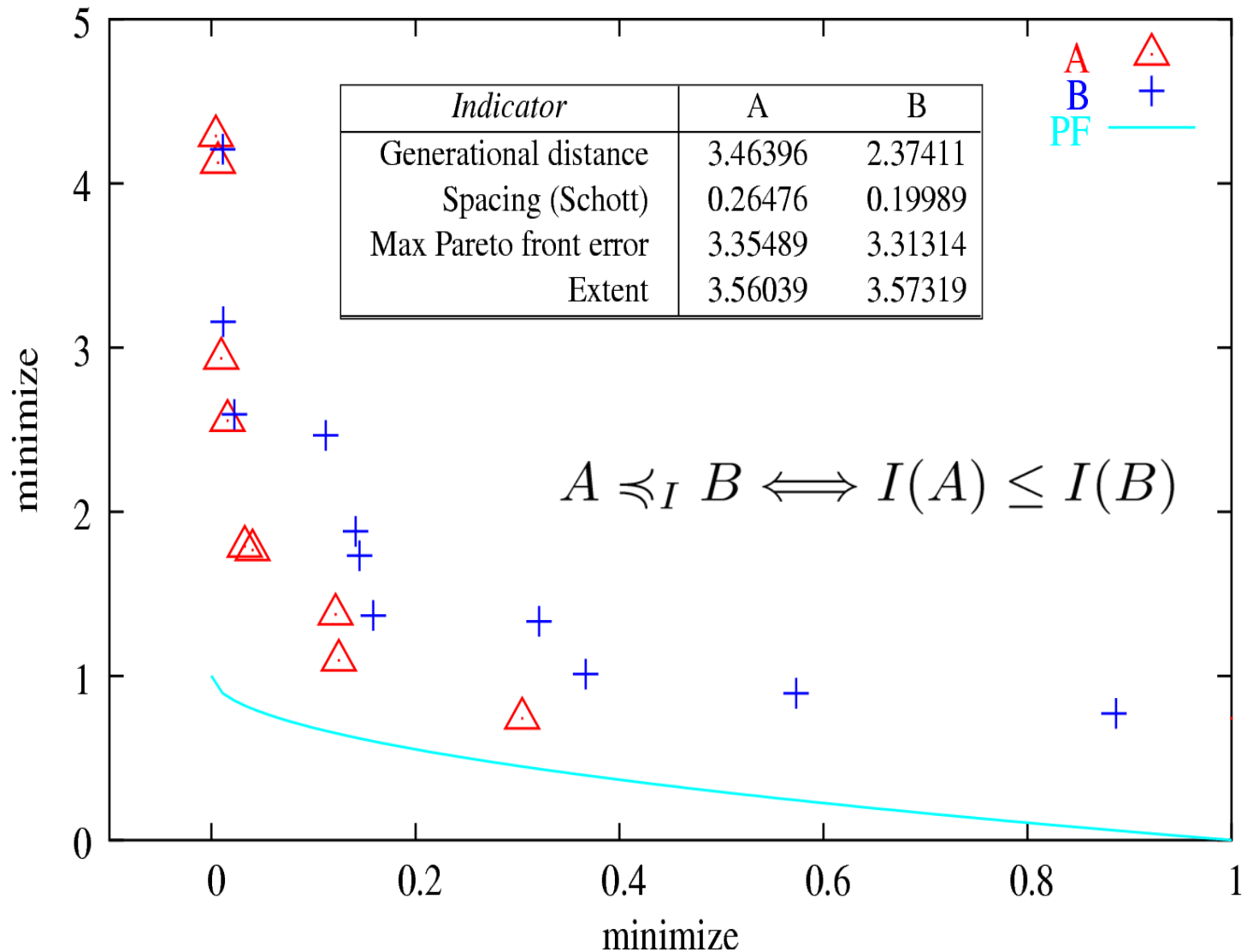
- First, reduces each approximation set to a single value of quality
- Applies statistical tests to the samples of quality values

<i>Indicator</i>	A	B
Hypervolume indicator	6.3431	7.1924
ϵ -indicator	1.2090	0.12722
R_2 indicator	0.2434	0.1643
R_3 indicator	0.6454	0.3475

see e.g. [\[Zitzler et al. 2003\]](#)

Problem With Arbitrary Quality Indicators

Don't use an arbitrary quality indicator, but a meaningful one...



Refinements

$\overset{\text{ref}}{\preceq}$ **refines** a preference relation \preceq iff

$$A \overset{\text{ref}}{\preceq} B \wedge B \not\preceq A \Rightarrow A \overset{\text{ref}}{\preceq} B \wedge B \overset{\text{ref}}{\not\preceq} A \quad (\text{better} \Rightarrow \text{better})$$

\Rightarrow fulfills requirement

...sought are total refinements!

(such as the hypervolume indicator)

Optimality in Indicator-Based Search

but still...

- difficult to interpret absolute numbers
- better: relative values: how far from the optimum (as in single-obj. opt.)

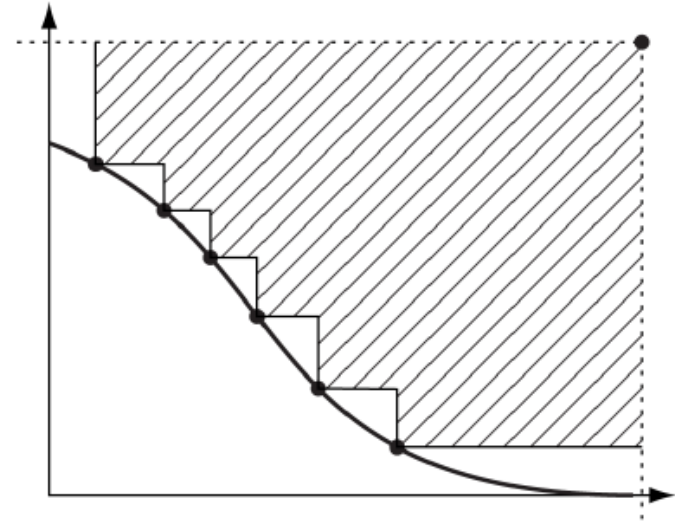
Question:

- what is the optimum?

Optimal μ -Distributions

When the goal is to maximize the hypervolume...

- this yields sets with only Pareto-optimal solutions
[Fleischer 2003]
- those sets, if unrestricted in size, cover the entire Pareto front
- many hypervolume-based EMO algorithms have a **population size μ** !



Optimal μ -Distribution:

A set of μ solutions that maximizes a certain (unary) indicator I among all sets of μ solutions is called **optimal μ -distribution for I** .

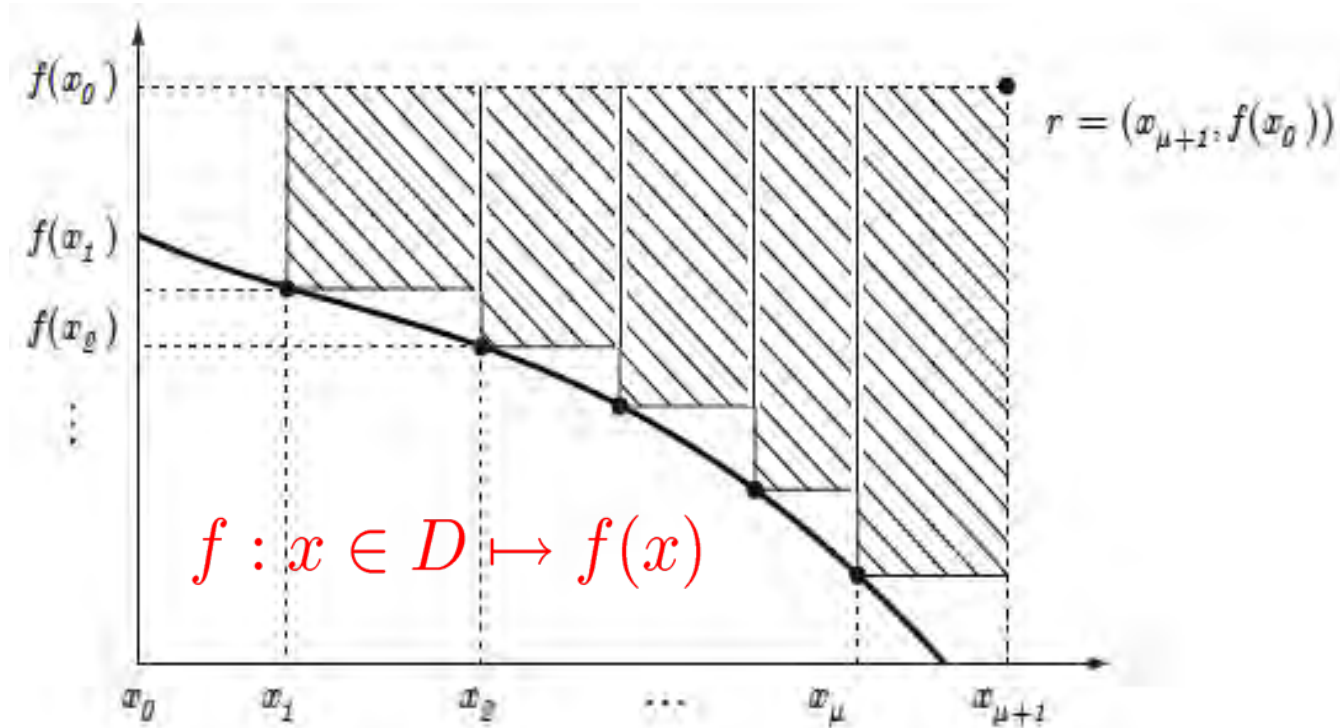
Optimal μ -Distributions

Questions:

- how are optimal μ -distributions characterized?
 - ▶ understand the bias of the indicator (influence on DM)
 - ▶ what is the influence of the indicator's parameters on optimal μ -distributions?
 - ▶ guidelines for practical usage
- do algorithms converge to optimal μ -distributions?

Notations for 2-Objective Case [Auger et al. 2009]

Results for 2 objectives only... (except [Auger et al. 2010])



hypervolume indicator:
$$I_H((x_1, \dots, x_\mu)) := \sum_{i=1}^{\mu} (x_{i+1} - x_i)(f(x_0) - f(x_i))$$

PROPOSITION 1. (Necessary condition for optimal μ -distributions) *If f is continuous, differentiable and $(x_1^\mu, \dots, x_\mu^\mu)$ denote the x -coordinates of a set of μ points maximizing the hypervolume indicator, then for all $x_{min} < x_i^\mu < x_{max}$*

$$f'(x_i^\mu) (x_{i+1}^\mu - x_i^\mu) = f(x_i^\mu) - f(x_{i-1}^\mu), i = 1 \dots \mu \quad (3)$$

where f' denotes the derivative of f , $f(x_0^\mu) = r_2$ and $x_{\mu+1}^\mu = r_1$.

$I_{H,w}$

Proof idea:

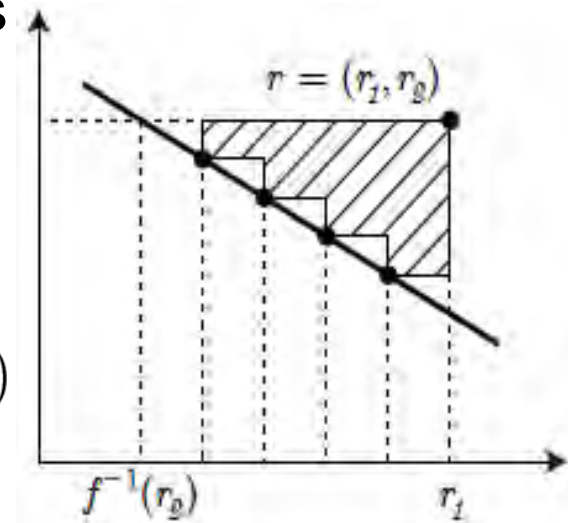
I_H max \Rightarrow derivative is 0 at each x_i^μ or x_i^μ is at the boundary of the domain

Interpretation of Necessary Condition

Example: equal distances (only) on linear fronts

$$f : x \in [x_{min}, x_{max}] \mapsto \alpha x + \beta$$

$$\alpha (x_{i+1}^\mu - x_i^\mu) = f(x_i^\mu) - f(x_{i-1}^\mu) = \alpha(x_i^\mu - x_{i-1}^\mu)$$



generalization of results in [Emmerich et al. 2005, Beume et al. 2007]

exact optimal μ -distribution for linear fronts and any choice of reference point
[Brockhoff 2010]

A Density Result: When μ Goes to Infinity

Observation:

general front shapes too difficult to investigate for finite μ

Question:

can we characterize optimal μ -distributions with respect to a

density $\delta(x) = \lim_{\substack{\mu \rightarrow \infty \\ h \rightarrow 0}} \left(\frac{1}{\mu h} \sum_{i=1}^{\mu} \mathbf{1}_{[x, x+h]}(x_i^{\mu}) \right) ?$

[Auger et al. 2009]

Result and Interpretation

The resulting density is

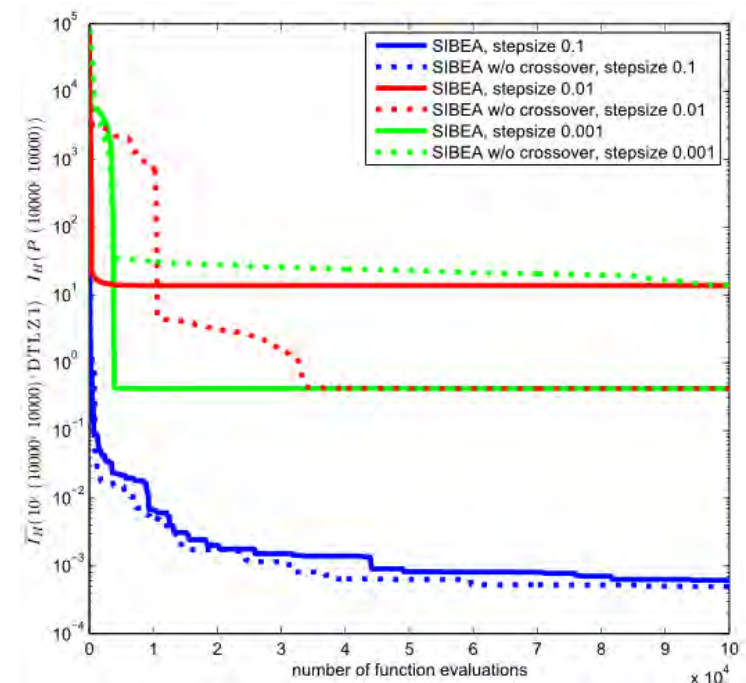
$$\delta(x) = \frac{\sqrt{-f'(x)}}{\int_0^{x_{max}} \sqrt{-f'(x)} dx}$$

How can we interpret this?

- bias only depends on slope of f *in contrast to* [Deb et al. 2005, Zitzler and Thiele 1998]
- density highest where slope = 45° *compliant to* [Beume et al. 2007]
- experimental results for finite and small μ support the result

Implications for Benchmarking

- now we can transform multiobjective benchmarking into a single-objective problem (where we sometimes know the optimum)
- we can use **exactly** the same methodology than for single-objective benchmarking:
 - horizontal view (i.e., fixing target values instead of runtime)
 - ERT
 - performance plots a la BBOB



Observation:

we are not as advanced in EMO as in single-objective optimization

Open Questions

Optimal μ -distributions

- uniqueness proofs
- other test problems & other indicators
- $>2D$
- efficient calculation/approximation
- ‘numbers’ for practical usage (on web page?)

Linear convergence speed

- what’s the problem in current algorithms?
- how to achieve it?

Others

- “good” test functions
- multiobjective BBOB
- effective restarts in EMO

Indicator-based Search and Preference Articulation

“on how to optimize and steer the search in many-objective problems”

Indicator-Based Search

Assume, we have chosen a total refinement and therefore an optimization goal

- how to achieve it as fast as possible?

Example: hypervolume indicator

- SMS-EMOA (changing the reference point might be bad?!)
- Even with fixed reference point, greedy selection might be bad
- HypE (?!)
- Something else?
- Isn't the variation operator even more important?

Needed:

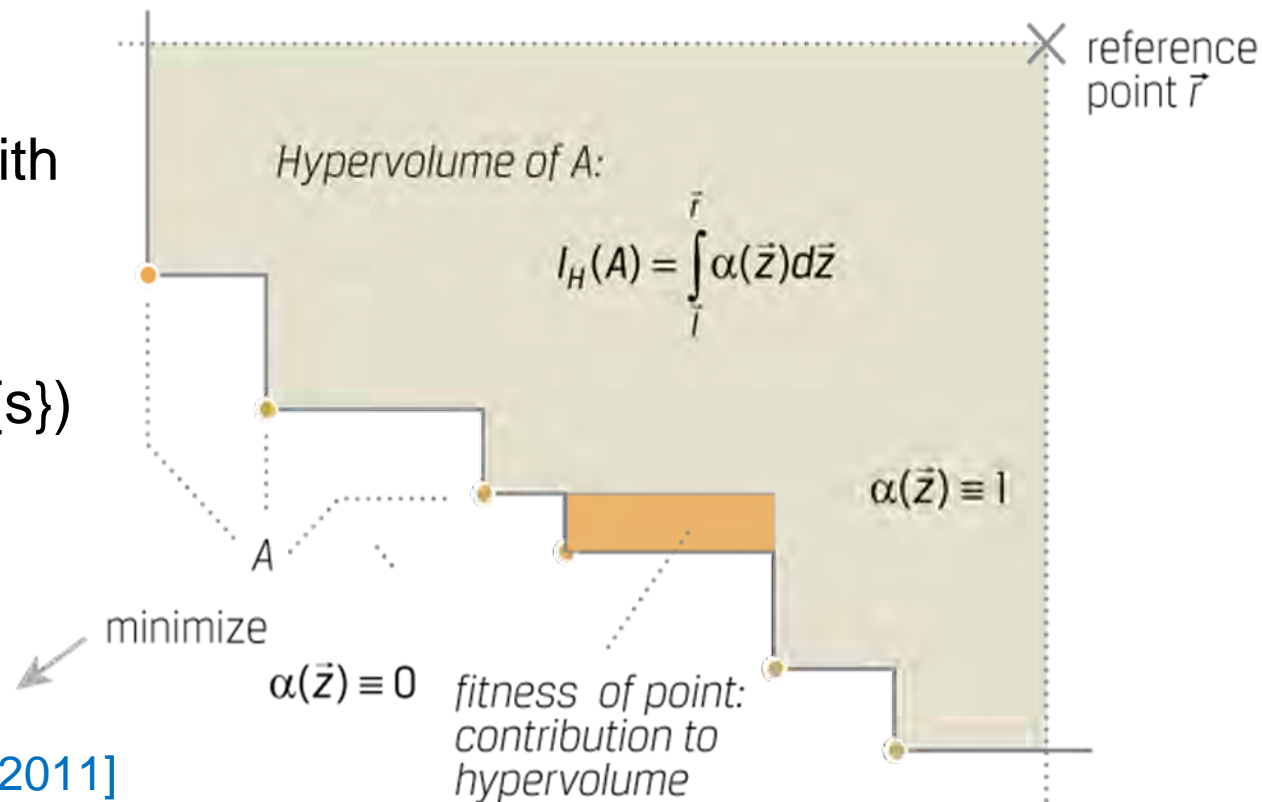
- better understanding of what's happening in search
- (first) examples of runtime analyses/convergence speed

Idea of Hypervolume-Based Selection

Main Idea (SMS-EMOA, MO-CMA-ES, HypE, ...)

use hypervolume indicator to guide the search: refinement!

Delete solutions with the smallest hypervolume loss
 $d(s) = I_H(P) - I_H(P \setminus \{s\})$
iteratively



But: can result

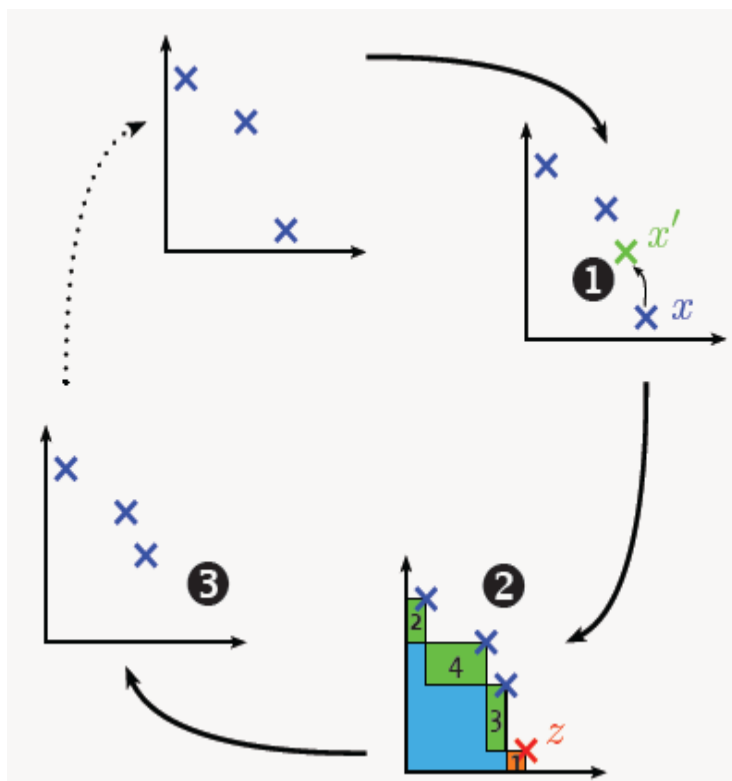
in cycles [Judt et al. 2011]

is expensive [Bringmann and Friedrich 2008]

and can result in arbitrarily bad sets compared to the optimal one

[Bringmann and Friedrich 2009]

A Simple Algorithm: SIBEA



Properties:

- No worsenings of I_H
- Duplicated solutions removed first
- Selection similar to SMS-EMOA [Emmerich et al. 2005] and MO-CMA-ES [Igel et al. 2007]

$(\mu+1)$ SIBEA

generate initial population $P \subseteq \{0, 1\}^n$ at random

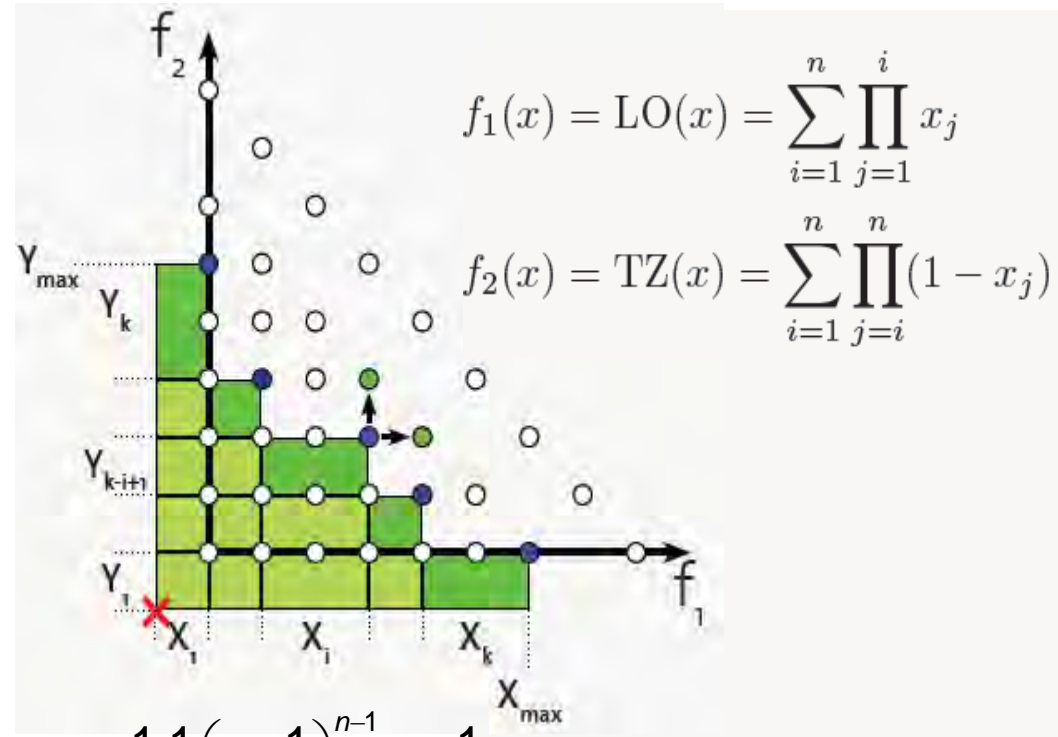
repeat:

- 1 mutate randomly selected $x \in P$ to x' by flipping each bit of x with probability $1/n$
 $P' = P \cup \{x'\}$
- 2 for all solutions $x \in P$, determine the hypervolume loss
 $d(x) = I_H(P') - I_H(P' \setminus \{x\})$
- 3 choose a $z \in P$ with smallest loss $d(z)$
 $P = P' \setminus \{z\}$

Runtime Analysis of SIBEA on LOTZ

Theorem [Brockhoff et al. 2008]:

If $\mu \geq n+1$, the $(\mu+1)$ SIBEA optimizes LOTZ in $O(\mu n^2)$ generations.



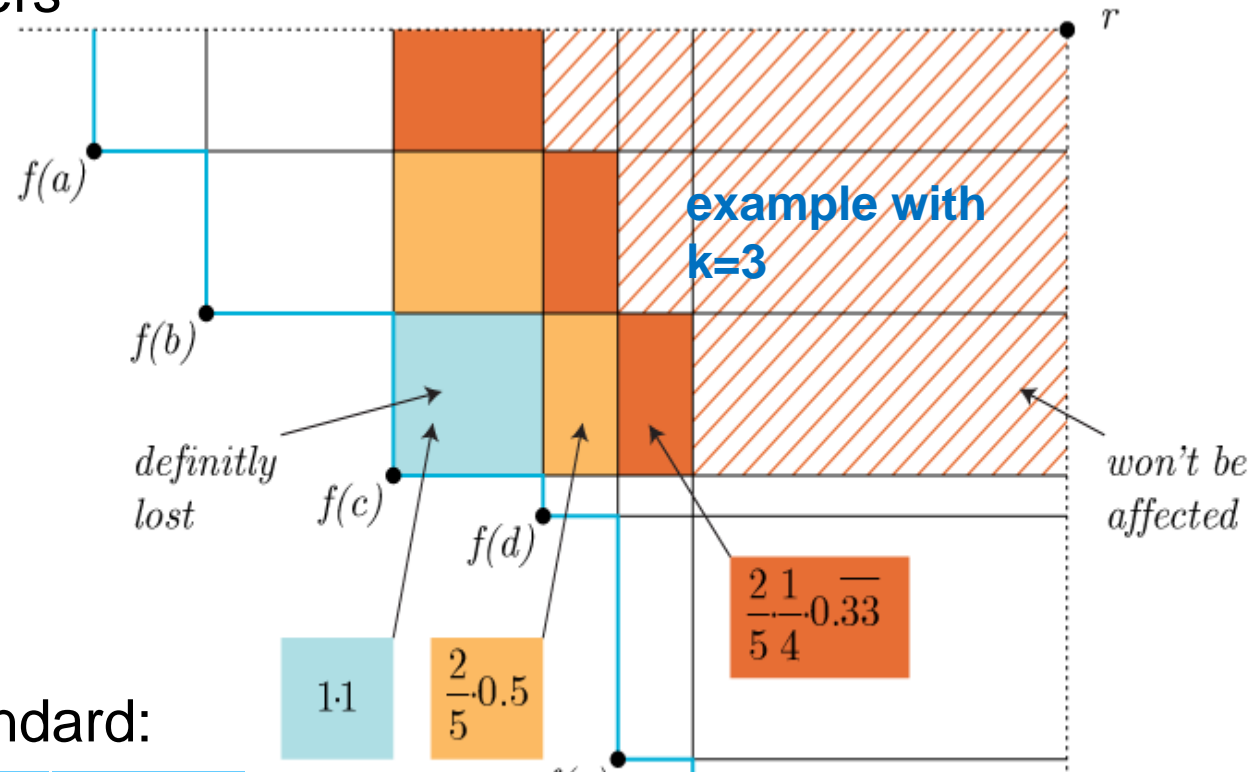
Sketch of Proof:

- $2k$ mutations increase I_H (prob. $\frac{1}{\mu} \frac{1}{n} \left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{1}{e\mu n}$)
- Total increase $\geq \max\{X_{\max}, Y_{\max}\} \geq \sqrt{X_{\max} \cdot Y_{\max}} \geq \sqrt{I_H}$
- Exp. increase for 1 mutation $\geq \sqrt{I_H} / 2k$; with Markov: i.e., in $8k$ good mutations $\sqrt{I_H}$ w.h.p.
- Exp. runtime for increase by $\sqrt{I_H}$ is $O\left(\frac{\mu n}{2k} \cdot 8k\right) = O(\mu n)$
- By induction, $O(n)$ such increases sufficient to reach front, then $O(\mu n)$ time enough to find all other n points

A More Involved Selection Scheme: HypE

Idea [Bader and Zitzler 2011]

Solution quality = expected loss, when removing the point and (randomly) $k-1$ others



Comparison HypE/standard:

	opt.	dist	better
new	59.7%	0.00109	30.2%
standard	44.5%	0.00261	3.2%

Question:

can we show the improvement also theoretically?

Articulating User Preferences

What if user wants something else than finding the optimal μ -distribution for the hypervolume indicator? E.g.

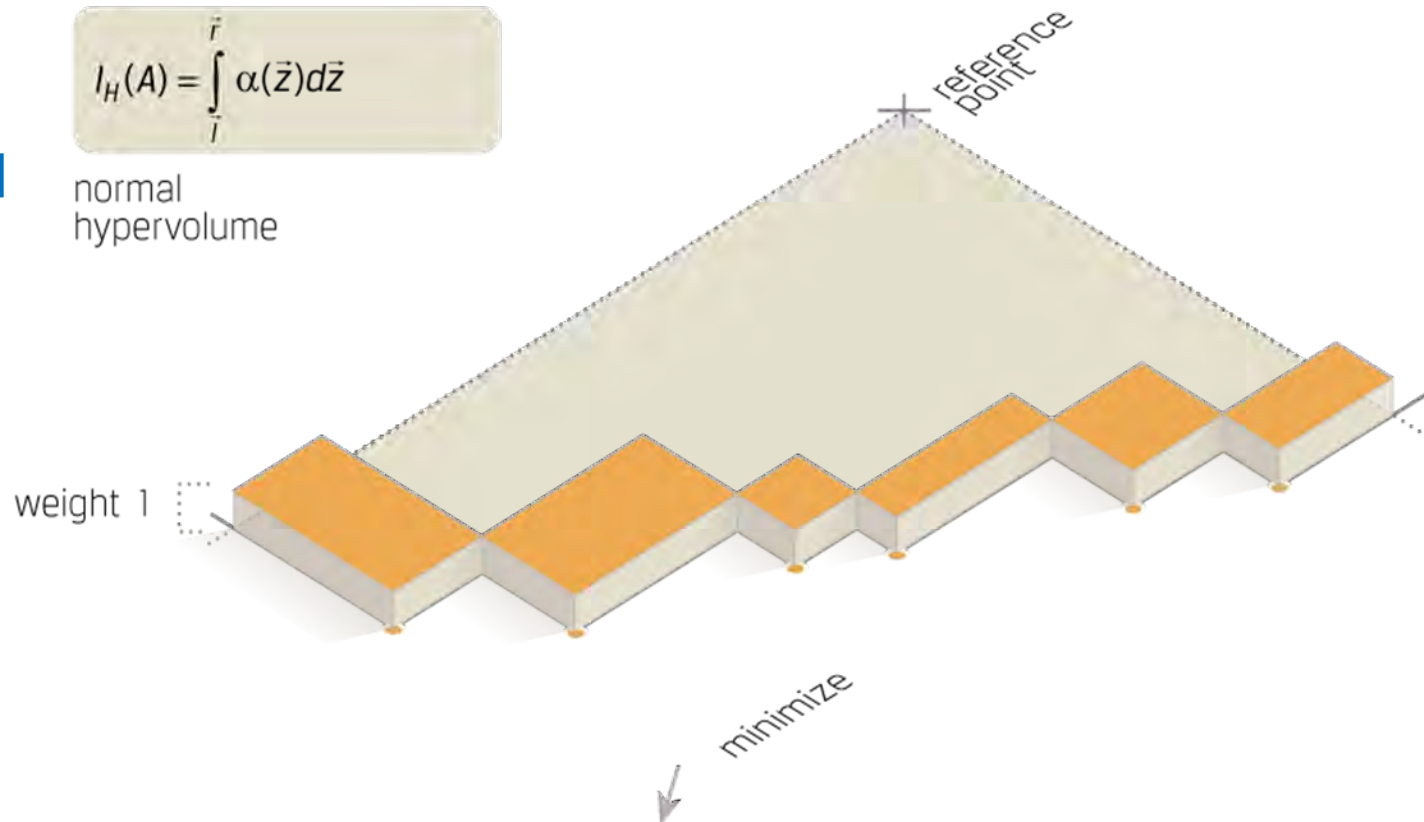
- (p)reference points
- stressing extremes
- simulate classical scalarizing function approaches

Idea:

[Zitzler et al. 2007]

$$I_H(A) = \int_{\vec{l}}^{\vec{r}} \alpha(\vec{z}) d\vec{z}$$

normal
hypervolume



Articulating User Preferences

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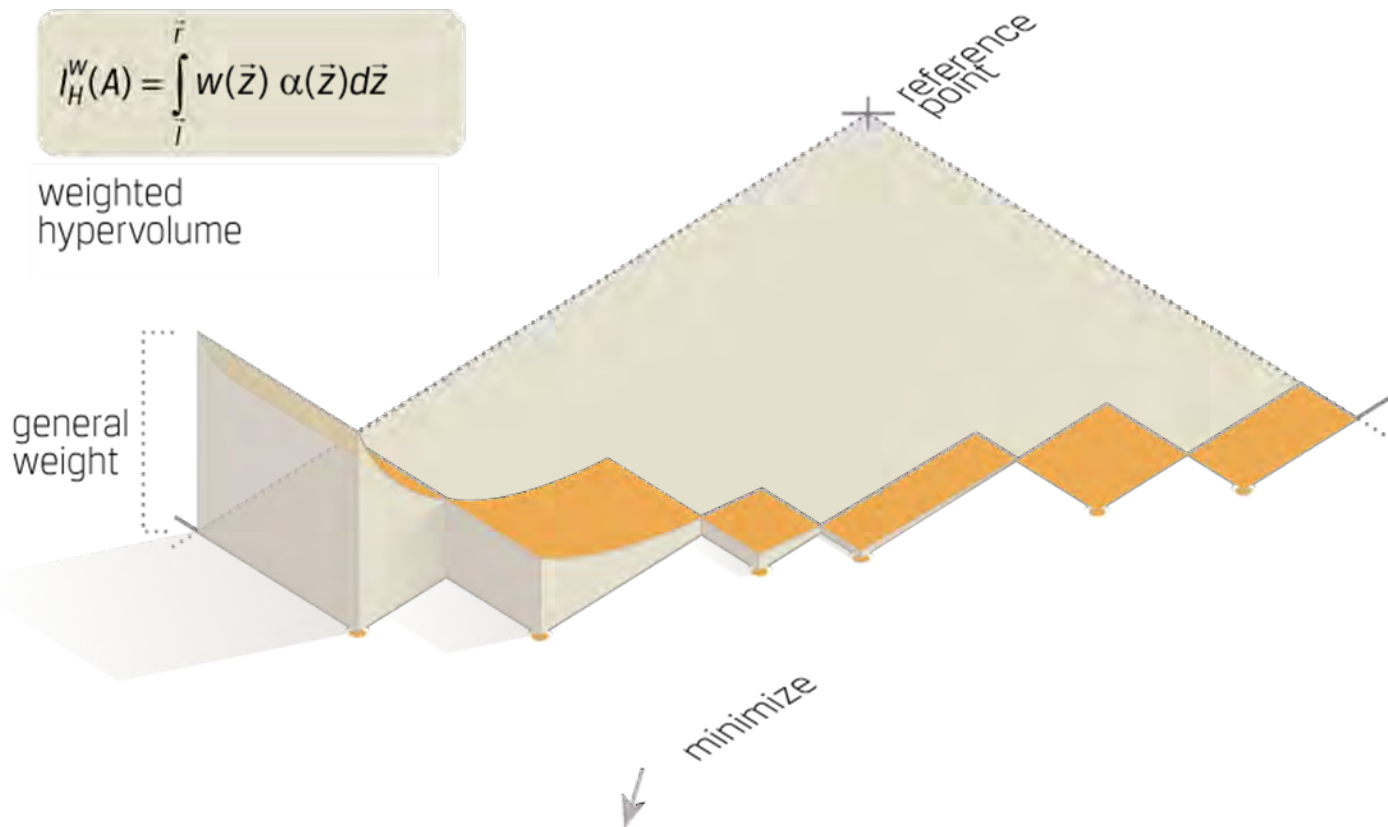
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Idea:

[Zitzler et al. 2007]

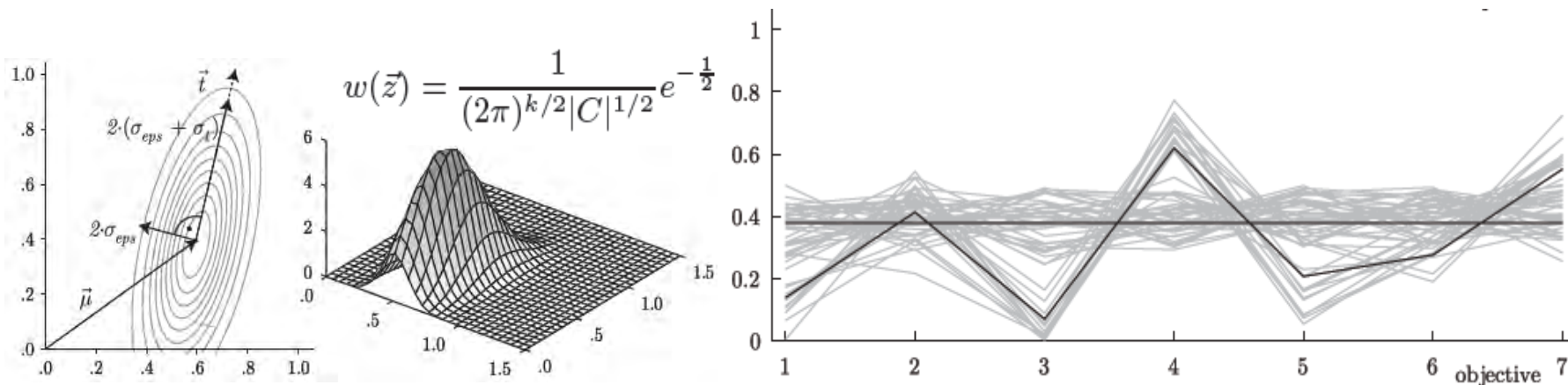
$$I_H^w(A) = \int_{\vec{l}}^{\vec{r}} w(\vec{z}) \alpha(\vec{z}) d\vec{z}$$

weighted
hypervolume

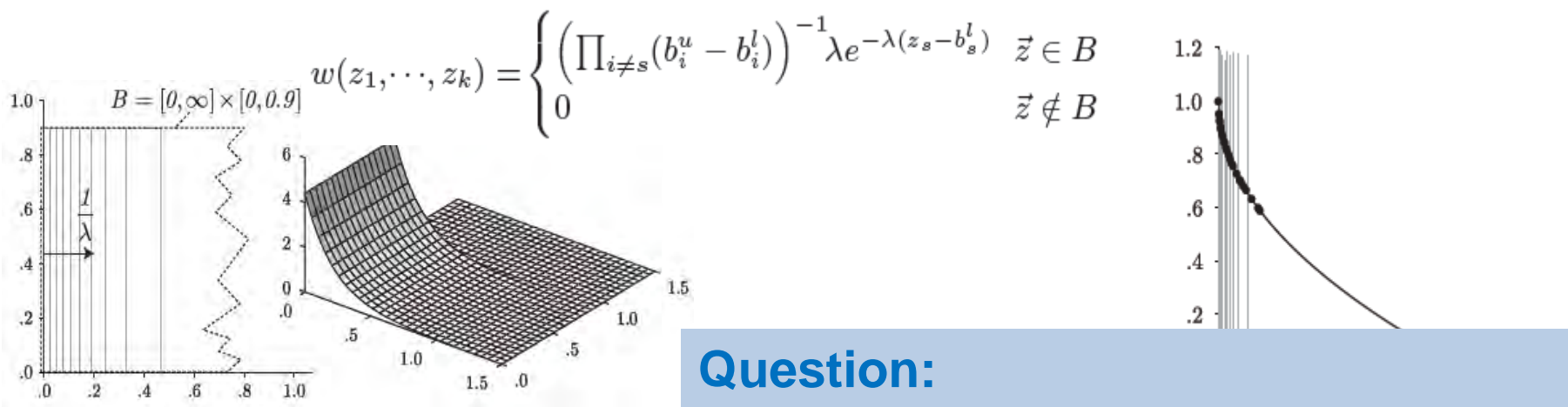


Examples of Weight Functions

preference point



stressing one objective

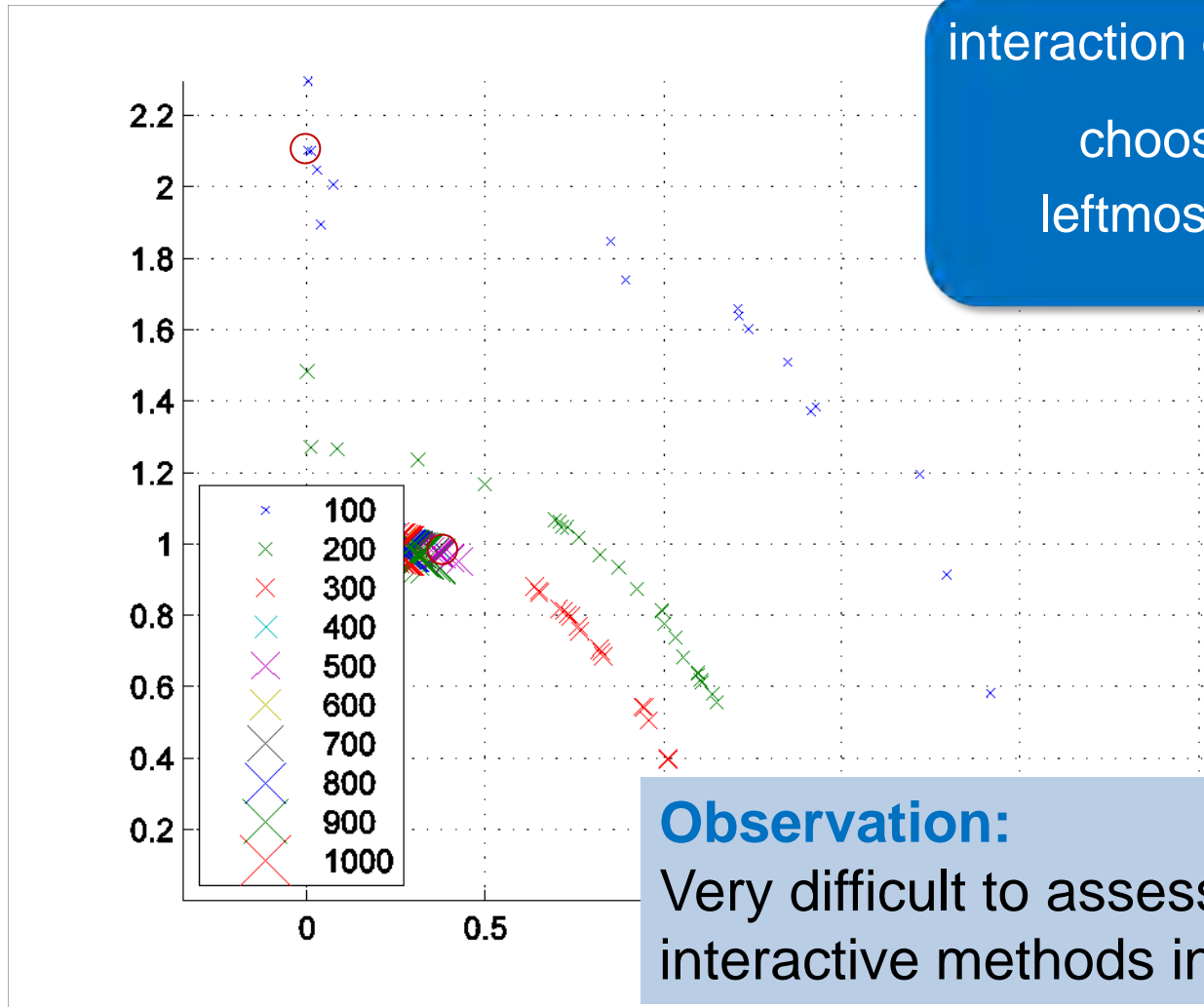


Question:

Does this work also interactively?

Some Experimental Results

Preliminary results shows yes:



Open Questions

HypE

- why is HypE better than normal HYP-based selection?
- and when? (Is there an example where it's provably better?)
- by how much (convergence speed?)
- greedy vs. oneShot: advantages and disadvantages
- a more advanced scheme than assuming uniform deletion?

SMS-EMOA: does algo becomes faster if HYP worsenings are not allowed (eg. by keeping old population if new one is worse)?

Convergence to optimal μ -distribution

- do other algorithms converge to optimal μ -distribution for other indicators?

Others

- more runtime analyses of indicator-based EMO
 - weighted hypervolume \rightarrow reduced pop size of SEMO?
- preferences: how to evaluate/compare algos objectively?

Objective Reduction and Multiobjectivization

“on when to reduce and when to increase the number of objectives”

Adding Objectives: Common Belief...

Statements are contradictory: some studies say that...

problems may become harder

- [Fonseca and Fleming 1995], [Deb 2001], [Coello et al. 2002], and others:
 - conflicts between objectives
 - Pareto front size
 - # incomparable solutions
- [Winkler 1985]:
 - *theoretical work for random objectives*

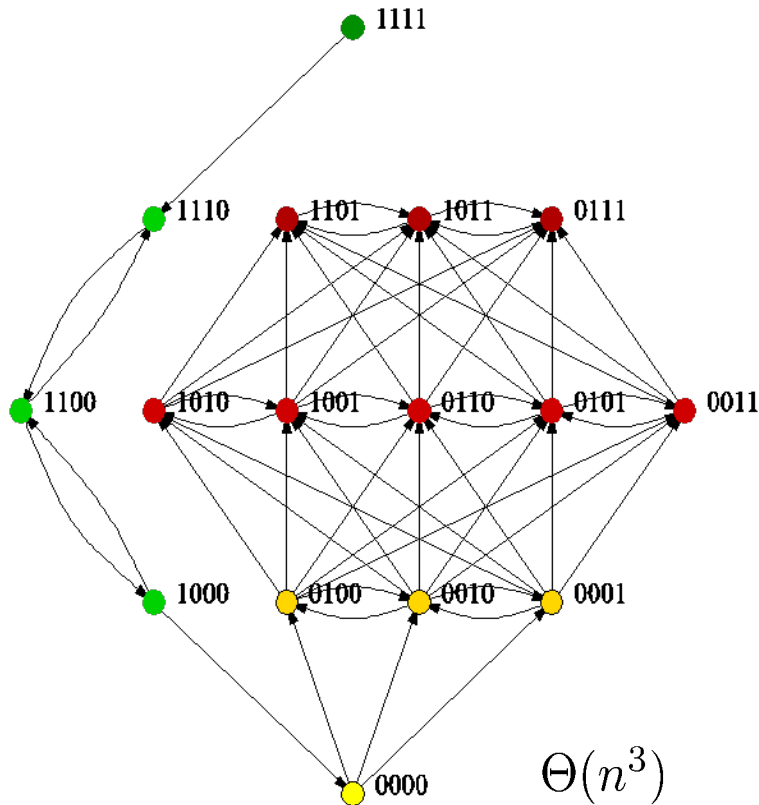
problems may become easier

- [Knowles et al. 2001]:
 - multiobjectivization
- [Jensen 2004]:
 - helper-objectives
- [Scharnow et al. 2002], [Neumann and Wegener 2006]:
 - theoretical investigations
 - 2D faster than 1D
 - decomposition

Adding Objectives: Runtime Analysis



$$\text{PLATEAU}_1(x) := \begin{cases} |x|_0 & : x \notin \{1^i 0^{n-i}, 1 \leq i \leq n\} \\ n + 1 & : x \in \{1^i 0^{n-i}, 1 \leq i < n\} \\ n + 2 & : x = 1^n. \end{cases}$$



Add ONEMAX(x)
Faster: $O(n^2 \log n)$



Add ZEROMAX(x)
Slower: exponential w.h.p.

Conclusions When Adding Objectives

Additional objectives can:

- turn a region with direction into a plateau of incomparable solutions
- add direction to a plateau of indifferent solutions

Contrary, removing objectives can do the opposite

- and therefore might also reduce the optimization time
- interesting: removing objectives results in a refinement !

Several works on **automated objective reduction**

- for reducing the runtime of hypervolume-based methods in many-objective optimization
- for giving insights into the problem for the decision maker

Open Questions

- faster aggregation heuristics
- what happens exactly when aggregating objectives?
 - which orders can be generated by e.g. a weighted sum?
- test problems with changing conflict
- GUI for decision support (incl. innovization?)
- online reduction:
 - when to delete, when to add objectives? (MAB problem)
- more examples of multiobjectivization:
 - both with runtime analysis + experimental

Conclusions

- Three aspects of Theory in EMO
 - benchmarking
 - indicator-based search and preference articulation
 - objective reduction and multiobjectivization
- Many open questions
- Lots of ideas for future work

...let's do it 😊

Announcement



French Summer School in Evolutionary Algorithms

June 12-15, 2012
Quiberon (Bretagne)

organizers: D. Brockhoff, L. Jourdan, A. Liefooghe, S. Verel



Questions?

References

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