BSDEs with continuous generator.
I. Sublinear growth and Logarithmic growth

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Definitions and notations

The BSDE under consideration is

$$Y_t = \xi + \int_t^T H(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \tag{1}$$

It will be referred as $eq(\xi, H)$.

- $\mathcal{F}_t := \mathcal{F}_t^W$.
- $\mathcal{C}$:= the space of continuous and $\mathcal{F}_t$ –adapted processes.
- $\mathcal{S}^2$:= $\{Y$ which is $\mathcal{F}_t$-adapted and $\mathbb{E} \sup_{0 \le t \le T} |Y_t|^2 < \infty\}$
- $\mathcal{M}^2$:= $\{Z$ which is $\mathcal{F}_t$-adapted and $\mathbb{E} \int_0^T |Z_s|^2 ds < \infty\}$
- $\mathcal{L}^2$:= $\{Z$ which is $\mathcal{F}_t$-adapted and $\int_0^T |Z_s|^2 ds < \infty \mathbb{P}$-a.s.$\}$

Definition

A solution of the BSDE $eq(\xi, H)$ is an $\mathcal{F}_t$–adapted processes (Y, Z) which satisfy BSDE $eq(\xi, H)$ for each $t \in [0, T]$ and such that Y is continuous and $\int_0^T |Z_s|^2 ds < \infty$ $\mathbb{P} - a.s.$, that is $(Y, Z) \in \mathcal{C} \times \mathcal{L}^2$.

Our Approach consists to derive the existence of solutions for the BSDE without reflection from solutions of a suitable 2-barriers Reflected BSDE.

To this end, we use the result of Essaky & Hassani which establishes the existence of solutions for reflected QBSDEs without assuming any integrability condition on the terminal datum.

For the self-contained, we state the Essaky & Hassani result in the following theorem.

Theorem

(Essaky–Hassani (JDE 2013)). Let L and U be continuous processes and ξ be a $\mathcal{F}_T$ measurable random variable. Assume that

1) $L_T \leq \xi \leq U_T$.

2) there exists a semimartingale which passes between the barriers L and U.

3) H is continuous in (y,z) and satisfies for every (s,ω), every $y \in [L_s(\omega), U_s(\omega)]$ and every $z \in \mathbb{R}^d$.

$$|f(s,\omega,y,z)| \leq \eta_s(\omega) + \frac{C_s(\omega)}{2}|z|^2$$

where $\eta \in \mathbb{L}^1([0,T] \times \Omega)$ and C is a continuous process. Then, the following RBSDE has a minimal and a maximal solution.

$$\begin{cases} (i) & Y_t = \xi + \int_t^T H(s,Y_s,Z_s)ds + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s \\ (ii) & \forall t \leq T,\ L_t \leq Y_t \leq U_t, \\ (iii) & \int_t^T (Y_t - L_t)dK_t^+ = \int_t^T (U_t - Y_t)dK_t^- = 0,\ a.s., \\ (iv) & K_0^+ = K_0^- = 0,\quad K^+, K^-,\ \text{are continuous nondecreasing.} \\ (v) & dK^+ \perp dK^- \end{cases} \tag{2}$$

Theorem

Theorem

Assume that there exist positive constants a and b such that ξ *satisfies*

(HL2) $\quad \mathbb{E}(|\xi|^2) < \infty$.

Let the generator $H(t, \omega, y, z)$ *be continuous in* (y, z) *for a.e* (t, ω) *and satisfies*

$$H(t, y, z) \leq a + b|y| + c|z| \tag{3}$$

for some positive constants a, b and c.
Then, the BSDE (ξ, H) *has at least one solution* (Y, Z) *which belongs to* $\mathcal{S}^2 \times \mathcal{M}^2$.

Proof, H sublinear

Proof. We put $g(t,y,z) := a + b|y| + c|z|$.

Let $\xi^+ := max(\xi, 0)$ and $\xi^- := min(\xi, 0)$.

According to Pardoux-Peng Theorem, the BSDE with the parameters (ξ^+, g) as well as the BSDE with the parameters $(-\xi^-, -g)$ have unique solutions in $\mathcal{S}^2 \times \mathcal{M}^2$.

We denote by (Y^g, Z^g) [resp. (Y^{-g}, Z^{-g})] the unique solution of $eq(\xi^+, g)$ [resp. $eq(-\xi^-, -g)$].

Using then the Essaky-Hassani result with

$L = Y^{-g}$, $U = Y^g$, $\eta_t = a + b(|Y_t^{-g}| + |Y_t^g|) + \frac{1}{2}c^2$, and $C_t = 1$,

we deduce the existence of solution (Y, Z, K^+, K^-) to the following Reflected BSDE, s.t. (Y, Z) belongs to $\mathcal{C} \times \mathcal{L}^2$.

$$
\begin{cases}
(i) & Y_t = \xi + \int_t^T H(s, Y_s, Z_s)ds + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dW_s \,, \\
(ii) & \forall\ t \leq T,\ Y_t^{-g} \leq Y_t \leq Y_t^g, \\
(iii) & \int_0^T (Y_t - Y_t^{-g})dK_t^+ = \int_0^T (Y_t^g - Y_t)dK_t^- = 0,\ \text{a.s.}, \\
(iv) & K_0^+ = K_0^- = 0, \quad K^+, K^- \text{ are continuous nondecreasing.} \\
(v) & dK^+ \perp dK^-
\end{cases}
\tag{4}
$$

Now, if we show that $dK^+ = dK^- = 0$, then the proof is finished. We shall prove this property.

Since Y_t^g is a solution to BSDE $eq(\xi, g)$, then Tanaka's formula applied to $(Y_t^g - Y_t)^+$ shows that,

Proof, H sublinear, continued 2

$$
\begin{aligned}
(Y_t^g - Y_t)^+ = (Y_0^g - Y_0)^+ &+ \int_0^t \mathbf{1}_{\{Y_s^g > Y_s\}}[f(s, Y_s, Z_s) - g(s, Y_s^g, Z_s^g)]ds \\
&+ \int_0^t \mathbf{1}_{\{Y_s^g > Y_s\}}(dK_s^+ - dK_s^-) + \int_0^t \mathbf{1}_{\{Y_s^g > Y_s\}}(Z_s^g - Z_s)dW_s \\
&+ L_t^0(Y^g - Y)
\end{aligned}
$$

where $L_t^0(Y^g - Y)$ denotes the local time at time t and level 0 of the semimartingale $(Y^g - Y)$.

Since $Y^g \geq Y$, then $(Y_t^g - Y_t)^+ = (Y_t^g - Y_t)$.

Therefore, identifying the terms of $(Y_t^g - Y_t)^+$ with those of $(Y_t^g - Y_t)$ and using the fact that $\mathbf{1} - \mathbf{1}_{\{Y_s^g > Y_s\}} = \mathbf{1}_{\{Y_s^g \leq Y_s\}} = \mathbf{1}_{\{Y_s^g = Y_s\}}$, we show that $(Z - Z^g)\mathbf{1}_{\{Y_s^g = Y_s\}} = 0$.

Using the previous equalities, we deduce that,

Proof, H sublinear, continued 3

$$\int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}}(dK_s^+ - dK_s^-) = \int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}}[g(s, Y_s^g, Z_s^g) - f(s, Y_s, Z_s)]ds \\ + L_t^0(Y^g - Y)$$

Since $\int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}} dK_s^+ = 0$, it holds that

$$0 \leq \int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}}[g(s, Y_s^g, Z_s^g) - f(s, Y_s, Z_s)]ds + L_t^0(Y^g - Y)$$

$$= -\int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}} dK_s^- \leq 0 \qquad \geq 0$$

This shows that , $\int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}} dK_s^- = 0$, and hence $dK^- = 0$.

Arguing symmetrically, one can show that $dK^+ = 0$.

Therefore, (Y, Z) satisfies the initial non reflected BSDE.

Proof, H sublinear, continued 4

Since both Y^g and Y^{-g} belong to $\mathcal{S}^2$, so it is for Y.

Using standard arguments in BSDEs, we show that $\mathbb{E}\int_0^T |Z_s|^2 ds < \infty$. This complete the proof.

Logarithmic nonlinearity

Theorem

Let a, b, c_0 be positive real numbers, and assume that,

(H.1) $\quad \mathbb{E}[|\xi|^{e^{2c_0T}+1}] < +\infty.$

Then, the BSDE

$$Y_t = \xi + \int_t^T (a + b|Y_s| + c_0|Y_s||\ln|Y_s||)ds - \int_t^T Z_s dW_s \qquad (5)$$

has a unique solution such that

$$\mathbb{E}\big(\sup_{t\in[0,T]} |Y_t|^{e^{2c_0t}+1}\big) < \infty \ \ \textit{and} \ \ \mathbb{E}\int_0^T |Z_s|^2 ds < \infty \qquad (6)$$

To prove this theorem we need some a priori estimates and approximations. This will be given in the following Lemmas.

Estimate of Y

Lemma

Let $C \geq 2c_0$. Let (Y, Z) be a solution to BSDE (5) such that $\mathbb{E}\sup_{t\in[0,T]} |Y_t|^{e^{Ct}+1} < \infty$ and $\mathbb{E}\int_0^T |Z_s|^2 ds < \infty$. Then, there exists a constant $K = K_T$ such that:

$$\mathbb{E} \sup_{t\in[0,T]} |Y_t|^{e^{Ct}+1} \leq K_T \left(\mathbb{E}|Y_T|^{(e^{CT}+1)} + a(e^{CT}+1)T \exp[b(e^{CT}+1)T\right)$$

Idea of the Proof. Let $f(y) := a + b|y| + c_0|y||\ln|y||$. Itô's formula gives Let u be $\mathcal{C}^{1,2}$ function. Itô's formula gives,

$$\begin{aligned} u(t, Y_t) = & u(T, Y_T) - \int_t^T Y_s \partial_y u(s, Y_s) Z_s dW_s \\ & + \int_t^T \left[Y_s f(s, Y_s) \partial_y u(s, Y_s) - \partial_s u(s, Y_s) \right] ds \\ & - \frac{1}{2} \int_t^T \partial^2_{yy} u(s, Y_s) |Z_s|^2 ds \end{aligned}$$

If we can find a $\mathcal{C}^{1,2}$ and positive function u such that

$$\left[Y_s f(s, Y_s) \partial_y u(s, Y_s) - \partial_s u(s, Y_s) \right] \ \leq \ a_1 + b_1 u(s, Y_s), \qquad (7)$$

then Gronwall Lemma gives the result.
The function $u(t, y) = |y|^{e^{2c_0 t}+1}$ satisfies these properties.

Proof.

Itô's formula gives

$$|Y_t|^{e^{Ct}+1} = |Y_T|^{e^{CT}+1} - \int_t^T (e^{Cs}+1)|Y_s|^{(e^{Cs})} sgn(Y_s) Z_s dW_s$$
$$- \int_t^T (Ce^{Cs}\ln(|Y_s|)|Y_s|^{e^{Cs}+1} - (e^{Cs}+1)|Y_s|^{(e^{Cs})} sgn(Y_s) f(Y_s))ds$$
$$- \frac{1}{2}\int_t^T |Z_s|^2 (e^{Cs}+1)(e^{Cs})|Y_s|^{(Cs-1)} ds$$

Proof estimate of Y, continued

Since $C \geq 2c_0$, we deduce that,

$$|Y_t|^{e^{Ct}+1} \leq |Y_T|^{(e^{CT}+1)} + \int_t^T (e^{Cs}+1)|Y_s|^{(e^{Cs})}(a+b|Y_s|)ds$$
$$- \int_t^T (e^{Cs}+1)|Y_s|^{(e^{Cs})} sgn(Y_s) Z_s dW_s.$$

Hence,

$$\mathbb{E}|Y_t|^{e^{CT}+1} \leq \mathbb{E}|Y_T|^{(e^{CT}+1)} + \mathbb{E}\int_t^T (e^{Cs}+1)|Y_s|^{(e^{Cs})}(a+b|Y_s|)ds$$

Using Gronwall Lemma, we obtain

$$\mathbb{E}|Y_t|^{e^{CT}+1} \leq \left(\mathbb{E}|Y_T|^{(e^{CT})+1)} + a(e^{CT}+1)T\right)\exp[b(e^{CT}+1)T].$$

Using the BDG inequality, we complete the proof of Lemma 5. ■

Estimate of Z

Lemma

Let (Y, Z) be a solution of BSDE (5) such that $\mathbb{E}\sup_{t\in[0,T]}|Y_t|^{e^{Ct}+1} < \infty$ *and* $\mathbb{E}\int_0^T |Z_s|2ds < \infty$. *Then, there exits a positive constant* $C_1 = C_1(C, T, K, c_0)$ *such that*

$$\mathbb{E}\int_0^T |Z_s|^2 ds \leq C_1 \mathbb{E}\left(1 + |\xi|^2 + \int_0^T |(a + b|Y_s|)|^2 ds + \sup_{s\leq T}|Y_s|^{e^{Cs}+1}\right) \tag{8}$$

Proof By Itô's formula, we have:

$$|Y_0|^2 + \int_0^T |Z_s|^2 ds = |\xi|^2 + 2\int_0^T Y_s f(s, Y_s, Z_s) ds - 2\int_0^T Y_s Z_s dW_s$$

$$\leq |\xi|^2 + 2\int_0^T |Y_s|(a + b|Y_s|) + 2c_0|Y_s||\ln|Y_s||)ds - 2\int_0^T Y_s Z_s dW_s.$$

Since for every y and every $\gamma > 0$, we have $|y||\ln|y|| \leq 1 + \frac{1}{\gamma}|y|^{1+\gamma}$, we use standard arguments in BSDEs to get,

$$
\begin{aligned}
\frac{1}{2}\int_0^T |Z_s|^2 ds \leq |\xi|^2 + 2c_0 T \sup_{s\leq T}|Y_s|^2 + \int_0^T (a + b|Y_s|)^2 ds \\
+ 2c_0 T\big(1 + \frac{K(C,T)}{(e^{2c_0T}-1)^+}\big) \\
+ 2c_0 T \sup_{s\leq T}(|Y_s|^{(e^{2c_0T}+1)}) - 2\int_0^T Y_s Z_s dW_s
\end{aligned}
$$

The result follows by passing to expectation. ■

Estimate for $|YLn|Y||$

Lemma

Let $f(y) := a + b|y| + c_0|y||\ln|y||$. Assume that ξ satisfies **(H.1)**. *Let (Y, Z) be a solution to BSDE (5) satisfying Lemma 5 and Lemma 6. Then, there exists a positive constant $\bar{\alpha}$ such that*

$$\mathbb{E}\int_0^T |f(Y_s)|^{\bar{\alpha}} ds \;\leq\; K\left[1 + \mathbb{E}\int_0^T |Y_s|^2 ds + \mathbb{E}\int_0^T |Z_s|^2 ds\right]$$

where K is a positive constant which depends on c_0 and T.

Proof

It is not difficult to show that for every $\varepsilon > 0$, $|y \ln y| \leq 1 + \frac{1}{\varepsilon}|y|^{1+\varepsilon}$.
We put $\alpha := 1 + \varepsilon$ and $\bar{\alpha} = \frac{2}{\alpha}$. A simple computation gives

Proof estimate for $|YLn|Y||$, continued

$$
\begin{aligned}
\mathbb{E}\int_0^T |f(Y_s)|^{\bar\alpha} ds &\leq \mathbb{E}\int_0^T ((a+b|Y_s|)+c_0|y||\ln|y||)^{\bar\alpha} ds \\
&\leq \mathbb{E}\int_0^T ((a+b|Y_s|)+c_0(1+\frac{1}{\alpha-1}|Y_s|^{\alpha})^{\bar\alpha} ds \\
&\leq (1+c_0^{\bar\alpha})\mathbb{E}\int_0^T ((a+b|Y_s|)^{\bar\alpha}+(1+\frac{1}{\alpha-1}|Y_s|)^{\alpha\bar\alpha}) ds \\
&\leq (1+c_0^{\bar\alpha})\mathbb{E}\int_0^T ((1+(a+b|Y_s|))^2+(1+\frac{1}{\alpha-1}|Y_s|)^2 ds \\
&\leq (1+c_0^{\bar\alpha})\big(4T+\mathbb{E}\int_0^T ((a+b|Y_s|)^2+\frac{1}{(\alpha-1)^2})|Y_s|^2 ds\big)
\end{aligned}
$$

■

Approximation for f

Lemma

Let $f(y) := a + b|y| + c_0|y||\ln|y||$. There exists a sequence of functions (f_n) such that,

(a) *For each n, f_n is bounded and globally Lipschitz in (y, z) a.e. t and P-a.s.ω.*

(b) $\sup_n |f_n(y)| \leq a + b|y| + c_0|y||\ln|y||$, *$P$-a.s., a.e. $t \in [0, T]$.*

(c) *For every N, $\rho_N(f_n - f) \longrightarrow 0$ as $n \longrightarrow \infty$,*

where $\rho_N(f) := E\int_0^T \sup_{|y|,|z|\leq N} |f(s, y, z)|ds$.

Proof Let $\varepsilon_n : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a sequence of regularization functions which satisfy $\int \varepsilon_n(u)du = 1$.

Let ψ_n from $\mathbb{R}$ to $\mathbb{R}_+$ be a sequence of smooth functions such that $0 \leq |\psi_n| \leq 1$, $\psi_n(u) := 1$ for $|u| \leq n$ and $\psi_n(u) := 0$ for $|u| \geq n+1$.

We put, $\varepsilon_{q,n}(y) := \int f(y-u)\alpha_q(u)du\psi_n(y)$.

For $n \in \mathbb{N}^*$, let $q(n)$ be an integer such that $q(n) \geq n + n^\alpha$.

It is not difficult to see that the sequence $f_n := \varepsilon_{q(n),n}$ satisfies all the assertions (a)-(c). ■

Approximation for Y, Z, f

Using the previous estimates for Y, Z and f, and standard arguments of BSDEs, we prove the following estimates.

Lemma

Let f and ξ be as in Theorem 4. Let $\xi_n := \xi \wedge n$ and (f_n) be the sequence of functions associated to f by the previous Lemma. Denote by (Y^{f_n}, Z^{f_n}) the solution of BSDE (ξ_n, f_n). Then, there exit constants K_1, K_2, K_3 and a universal constant ℓ such that

$a)$ $\sup_n \mathbb{E} \int_0^T |Z_s^{f_n}|^2 ds \leq K_1$

$b)$ $\sup_n \mathbb{E} \sup_{0 \leq t \leq T} (|Y_t^{f_n}|^2) \leq \ell K_1 := K_2$

$c)$ $\sup_n \mathbb{E} \int_0^T |f_n(Y_s^{f_n})|^{\bar{\alpha}} ds \leq K_3$

where $\bar{\alpha}$ is a constant defined on the proof of Lemma 7.

From this Lemma we deduce the following limits.

Approximation for Y, Z, f, continued

After extracting a subsequence, if necessary, we have

Corollary

There are $Y \in \mathbb{L}^2(\Omega, L^\infty[0,T])$, $Z \in \mathbb{L}^2(\Omega \times [0,T])$ *and* $\Gamma \in \mathbb{L}^{\bar{\alpha}}(\Omega \times [0,T])$ *such that*

$$\begin{aligned} &Y^{f_n} \rightharpoonup Y, \text{ weakly star in } \mathbb{L}^2(\Omega, L^\infty[0,T]) \\ &Z^{f_n} \rightharpoonup Z, \text{ weakly in } \mathbb{L}^2(\Omega \times [0,T]) \\ &f_n(., Y^{f_n}, Z^{f_n}) \rightharpoonup \Gamma_. \text{ weakly in } \mathbb{L}^{\bar{\alpha}}(\Omega \times [0,T]). \end{aligned}$$

Moreover

$$Y_t = \xi + \int_t^T \Gamma_s ds - \int_t^T Z_s dW_s, \ \forall t \in [0,T]. \tag{9}$$

Estimate between the (Y^{f_n}, Z^{f_n})

The key estimate is,

Lemma

Let $\bar{\alpha}$ be the positive constant defined on the proof of the previous Lemma, and put $M_2 := 3c_0$. For every $R \in \mathbb{N}$, $\beta \in]1, \min(3 - \frac{2}{\bar{\alpha}}, 2)[$, $\delta' < \frac{(\beta-1)}{(2\ell+\beta-1)M_2} \min(\frac{1}{2}, \frac{\kappa}{\beta}, \kappa := 3 - \frac{2}{\bar{\alpha}} - \beta)$ and $\varepsilon > 0$, there exists $N_0 > R$ such that for all $N > N_0$ and $T' \leq T$:

$$\limsup_{n,m\to+\infty} E \sup_{(T'-\delta')^+ \leq t \leq T'} |Y_t^{f_n} - Y_t^{f_m}|^\beta + E \int_{(T'-\delta')^+}^{T'} \frac{\left|Z_s^{f_n} - Z_s^{f_m}\right|^2}{\left(|Y_s^{f_n} - Y_s^{f_m}|^2 + \nu_R\right)^{\frac{2-\beta}{2}}} d$$

$$\leq \varepsilon + \frac{\ell}{\beta - 1} e^{C_N \delta'} \limsup_{n,m\to+\infty} E|Y_{T'}^{f_n} - Y_{T'}^{f_m}|^\beta.$$

where $C_N = \frac{2M_2^2\beta}{(\beta-1)} \log N$, $\nu_R := \sup\{\frac{1}{N}, N \geq R\}$ and ℓ is the BDG constant

Let $0 < T' \leq T$. It follows from Itô's formula that for all $t \leq T'$,

$$\left|Y_t^{f_n} - Y_t^{f_m}\right|^2 + \int_t^{T'} \left|Z_s^{f_n} - Z_s^{f_m}\right|^2 ds$$
$$= \left|Y_{T'}^{f_n} - Y_{T'}^{f_m}\right|^2 + 2\int_t^{T'} \left(Y_s^{f_n} - Y_s^{f_m}\right)\left(f_n(s, Y_s^{f_n}) - f_m(s, Y_s^{f_m})\right) ds$$
$$- 2\int_t^{T'} \langle Y_s^{f_n} - Y_s^{f_m}, \quad (Z_s^{f_n} - Z_s^{f_m}) dW_s \rangle.$$

For $N \in \mathbb{N}^\star$ we set, $\Delta_t := \left|Y_t^{f_n} - Y_t^{f_m}\right|^2 + \frac{1}{N}$.

Let $C > 0$ and $1 < \beta < \min\{(3 - \frac{2}{\bar{\alpha}}), 2\}$. Itô's formula applied to $e^t.(\Delta_t)^{\frac{\beta}{2}}$ shows that,

Proof Estimate between the (Y^{f_n}, Z^{f_n}), continued 2

$$
\begin{aligned}
& e^{Ct}\Delta_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}} ds \\
& = e^{CT'}\Delta_{T'}^{\frac{\beta}{2}} + \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left(Y_s^{f_n} - Y_s^{f_m}\right)\left(f_n(s, Y_s^{f_n}) - f_m(s, Y_s^{f_m})\right) ds \\
& - \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left|Z_s^{f_n} - Z_s^{f_m}\right|^2 ds \\
& - \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^{f_n} - Y_s^{f_m}, \quad \left(Z_s^{f_n} - Z_s^{f_m}\right) dW_s\rangle \\
& - \beta(\frac{\beta}{2} - 1)\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-2}\left((Y_s^{f_n} - Y_s^{f_m})(Z_s^{f_n} - Z_s^{f_m})\right)^2 ds
\end{aligned}
$$

If we put $\Phi(s) = |Y_s^{f_n}| + |Y_s^{f_m}|$, then the previous formula becomes,

$$
\begin{aligned}
& e^{Ct}\Delta_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}} ds \\
& = e^{CT'}\Delta_{T'}^{\frac{\beta}{2}} - \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^{f_n} - Y_s^{f_m}, \quad \left(Z_s^{f_n} - Z_s^{f_m}\right) dW_s\rangle \\
& - \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left|Z_s^{f_n} - Z_s^{f_m}\right|^2 ds \\
& + \beta\frac{(2-\beta)}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-2}\left((Y_s^{f_n} - Y_s^{f_m})(Z_s^{f_n} - Z_s^{f_m})\right)^2 ds \\
& + J_1 + J_2 + J_3 + J_4,
\end{aligned}
$$

where

$$J_1 := \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left(Y_s^{f_n} - Y_s^{f_m}\right)\left(f_n(s, Y_s^{f_n}) - f_m(s, Y_s^{f_m})\right)1\!\!1_{\{\Phi(s)>N\}} ds.$$

$$J_2 := \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left(Y_s^{f_n} - Y_s^{f_m}\right)\left(f_n(s, Y_s^{f_n}) - f(s, Y_s^{f_n})\right)1\!\!1_{\{\Phi(s)\leq N\}} ds.$$

$$J_3 := \beta \int_t^{T'} e^{Cs} \Delta_s^{\frac{\beta}{2}-1} \left(Y_s^{f_n} - Y_s^{f_m}\right)\left(f(s, Y_s^{f_n}) - f(s, Y_s^{f_m})\right) 1\!\!1_{\{\Phi(s)\le N\}} ds.$$

$$J_4 := \beta \int_t^{T'} e^{Cs} \Delta_s^{\frac{\beta}{2}-1} \left(Y_s^{f_n} - Y_s^{f_m}\right)\left(f(s, Y_s^{f_m}) - f_m(s, Y_s^{f_m})\right) 1\!\!1_{\{\Phi(s)\le N\}} ds.$$

Proof Estimate between the (Y^{f_n}, Z^{f_n}), continued 5

Since $|Y_s^{f_n} - Y_s^{f_m}|^2 \leq \Delta_s$, we have

$$
\begin{aligned}
& e^{Ct}\Delta_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}} ds \\
& \leq e^{CT'}\Delta_{T'}^{\frac{\beta}{2}} - \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^{f_n} - Y_s^{f_m}, \quad \left(Z_s^{f_n} - Z_s^{f_m}\right) dW_s\rangle \\
& - \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left|Z_s^{f_n} - Z_s^{f_m}\right|^2 ds \\
& + \beta\frac{(2-\beta)}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}|Z_s^{f_n} - Z_s^{f_m}|^2 ds \\
& + J_1 + J_2 + J_3 + J_4,
\end{aligned}
$$

Passing to expectation, we obtain for every t,

$$
\begin{aligned}
& \mathbb{E}\Big(e^{Ct}\Delta_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}} ds + \beta\frac{(\beta-1)}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}|Z_s^{f_n} - Z_s^{f_m}|^2 ds\Big) \\
& \leq \mathbb{E}(J_1 + J_2 + J_3 + J_4)
\end{aligned}
$$

Proof Estimate between the (Y^{f_n}, Z^{f_n}), continued 6

According to BDG inequality, we deduce that

$$
\begin{aligned}
& E \sup_{(T'-\delta)^+ \le t \le T'} e^{Ct} \Delta_t^{\frac{\beta}{2}} + C \int_{(T'-\delta)^+}^{T'} e^{Cs} \Delta_s^{\frac{\beta}{2}} ds \\
& + \beta \frac{(\beta-1)}{2} \int_{(T'-\delta)^+}^{T'} e^{Cs} \Delta_s^{\frac{\beta}{2}-1} |Z_s^{f_n} - Z_s^{f_m}|^2 ds \\
& \le e^{CT'} \Delta_{T'}^{\frac{\beta}{2}} + \frac{2\ell}{\beta-1} E(J_1 + J_2 + J_3 + J_4) + \mathbb{E}(J_1 + J_2 + J_3 + J_4) \\
& = e^{CT'} \Delta_{T'}^{\frac{\beta}{2}} + (\frac{2\ell + \beta - 1}{\beta - 1}) E(J_1 + J_2 + J_3 + J_4) \qquad (10)
\end{aligned}
$$

It remains now to estimate $\mathbb{E}(J_i)$ for $i = 1, 2, 3, 4$.

Let $\kappa = 3 - \frac{2}{\bar{\alpha}} - \beta$. Since $\frac{(\beta-1)}{2} + \frac{\kappa}{2} + \frac{1}{\bar{\alpha}} = 1$, we use Hölder inequality to obtain,

$$\mathbb{E}(J_1) \leq \beta e^{CT'} \frac{1}{N^{\kappa}} \left[\mathbb{E}\int_t^{T'} \Delta_s ds\right]^{\frac{\beta-1}{2}} \left[\mathbb{E}\int_t^{T'} \Phi(s)^2 ds\right]^{\frac{\kappa}{2}}$$
$$\times \left[\mathbb{E}\int_t^{T'} |f_n(s, Y_s^{f_n}) - f_m(s, Y_s^{f_m})|^{\bar{\alpha}} ds\right]^{\frac{1}{\bar{\alpha}}}.$$

Since $|Y_s^{f_n} - Y_s^{f_m}| \leq \Delta_s^{\frac{1}{2}}$, it easy to see that

$$J_2 + J_4 \leq 2\beta e^{CT'} [2N^2 + \nu_1]^{\frac{\beta-1}{2}} \left[\int_t^{T'} \sup_{|y|,|z|\leq N} |f_n(y) - f(y)| ds\right.$$
$$\left. + \int_t^{T'} \sup_{|y|,|z|\leq N} |f_m(y) - f(y)| ds\right].$$

Since $\Delta_t := \left|Y_t^{f_n} - Y_t^{f_m}\right|^2 + \frac{1}{N}$ and the generator $f(y) := a + b|y| + c_0|y||\ln|y||$ satisfies for every $N > max(e, e^b)$ and every $|y| \leq N$

$$|f(y) - f(y')| \leq 3c_0(\ln N)|y - y'| + 3c_0 \frac{\ln N}{N}.$$

we then deduce [by putting $M_2 := 3c_0$] that,

$$J_3 \leq \beta M_2 \int_t^{T'} e^{Cs} \Delta_s^{\frac{\beta}{2}-1} \left[|Y_s^{f_n} - Y_s^{f_m}|^2 \ln N + \frac{\ln N}{N} \right] 1\!\!1_{\{\Phi(s) \leq N\}} ds$$
$$\leq \beta M_2 \int_t^{T'} e^{Cs} \Delta_s^{\frac{\beta}{2}-1} \left[\Delta_s \ln N \right] 1\!\!1_{\{\Phi(s) \leq N\}} ds.$$

If we choose $C = C_N = \beta\left(\frac{2\ell+\beta-1}{\beta-1}\right) M_2 \ln N$, then we obtain

$$
\begin{aligned}
&E \sup_{(T'-\delta)^+\le t\le T'} e^{C_N t}\Delta_t^{\frac{\beta}{2}} + \beta\frac{(\beta-1)}{2}\int_{(T'-\delta)^+}^{T'} e^{C_N s}\Delta_s^{\frac{\beta}{2}-1}|Z_s^{f_n}-Z_s^{f_m}|^2 ds \\
&\le e^{C_N T'}\Delta_{T'}^{\frac{\beta}{2}} \\
&+ \left(\frac{2\ell+\beta-1}{\beta-1}\right)\left\{\beta e^{C_N T'}\frac{1}{N^\kappa}\left[\mathbb{E}\int_{(T'-\delta)^+}^{T'}\Delta_s ds\right]^{\frac{\beta-1}{2}}\left[\mathbb{E}\int_{(T'-\delta)^+}^{T'}\Phi(s)^2 ds\right]^{\frac{\kappa}{2}}\right. \\
&\quad\times\left[\mathbb{E}\int_{(T'-\delta)^+}^{T'}|f_n(s,Y_s^{f_n})-f_m(s,Y_s^{f_m})|^{\bar\alpha}ds\right]^{\frac{1}{\bar\alpha}} \\
&+ 2\beta e^{C_N T'}[2N^2+\nu_1]^{\frac{\beta-1}{2}}\left[\int_{(T'-\delta)^+}^{T'}\sup_{|y|\le N}|f_n(y)-f(y)|ds\right. \\
&\left.\left.+\int_{(T'-\delta)^+}^{T'}\sup_{|y|\le N}|f_m(y)-f(y)|ds\right]\right\} \qquad (11)
\end{aligned}
$$

Proof Estimate between the (Y^{f_n}, Z^{f_n}), continued 10

Since $T' - (T' - \delta')^+ \leq \delta'$, we use the estimates for Y and Z to show that there exists a positive constant $K' = K'(K_1, K_2, \bar{\alpha}, \kappa, \beta, \ell)$ such that,

$$\mathbb{E} \sup_{(T'-\delta')^+ \leq t \leq T'} |Y_t^{f_n} - Y_t^{f_m}|^\beta + \beta\frac{(\beta-1)}{2}\mathbb{E}\int_{(T'-\delta')^+}^{T'} \frac{\left|Z_s^{f_n} - Z_s^{f_m}\right|^2}{\left(|Y_s^{f_n} - Y_s^{f_m}|^2 + \nu_R\right)^{\frac{2-\beta}{2}}}$$

$$\leq e^{C_N\delta'}\left(\mathbb{E}|Y_{T'}^{f_n} - Y_{T'}^{f_m}|^\beta\right) + \frac{e^{C_N\delta'}}{N^{\frac{\beta}{2}}} + K'\frac{e^{C_N\delta'}}{N^\kappa}$$

$$+ \left(\frac{2\ell+\beta-1}{\beta-1}\right)e^{C_N\delta'}\left\{\beta[2N^2+\nu_1]^{\frac{\beta-1}{2}}\left[\rho_N(f_n - f) + \rho_N(f_m - f)\right]\right\}$$

Since for $\delta' < \frac{(\beta-1)}{(2\ell+\beta-1)M_2}\min\left(\frac{1}{2}, \frac{\kappa}{\beta}\right)$, the quantities $\frac{e^{C_N\delta'}}{N^{\frac{\beta}{2}}}$ and $\frac{e^{C_N\delta'}}{N^\kappa}$ tend to 0 as N tends to ∞, we pass to the limits on n, m to complete the proof. ∎

Proof of the existence

We now pass to the Proof of the existence. Passing to the limit first on n, m and next on N, then taking successively $T' = T$, $T' = (T-\delta')^+$, $T' = (T-2\delta')^+$... in the previous inequality, we obtain, for every $\beta \in]1, \quad \min\left(3 - \frac{2}{\bar{\alpha}}, 2\right)[$

$$\lim_{n,m\to+\infty}\left(\mathbb{E}\sup_{0\le t\le T}|Y_t^{f_n} - Y_t^{f_m}|^{\beta} + \mathbb{E}\int_0^T \frac{\left|Z_s^{f_n} - Z_s^{f_m}\right|^2}{\left(|Y_s^{f_n} - Y_s^{f_m}|^2 + \nu_R\right)^{\frac{2-\beta}{2}}}ds\right) = 0.$$

But, by Schwarz inequality we have

$$\mathbb{E}\int_0^T |Z_s^{f_n} - Z_s^{f_m}|ds \le \left(\mathbb{E}\int_0^T \frac{\left|Z_s^{f_n} - Z_s^{f_m}\right|^2}{\left(|Y_s^{f_n} - Y_s^{f_m}|^2 + \nu_R\right)^{\frac{2-\beta}{2}}}ds\right)^{\frac{1}{2}}$$

$$\times\left(\mathbb{E}\int_0^T \left(|Y_s^{f_n} - Y_s^{f_m}|^2 + \nu_R\right)^{\frac{2-\beta}{2}}ds\right)^{\frac{1}{2}}$$

Hence

Proof Existence, continued 1

$$\lim_{n\to+\infty}\left(\mathbb{E}\sup_{0\le t\le T}|Y_t^{f_n}-Y_t|^\beta+\mathbb{E}\int_0^T|Z_s^{f_n}-Z_s|ds\right)=0.$$

This allows to show that $\int_0^T f_n(s,Y_s^{f_n})ds$ tends to $\int_0^T f(s,Y_s)|ds$ as n tends to ∞. And hence to deduce that (Y,Z) is a solution to the BSDE (5). Using Fatou's Lemma and our a priori estimates, one can show that (Y,Z) satisfies inequality (6).
Indeed, the last inequality shows that there exists a subsequence, which we still denote (Y^{f_n},Z^{f_n}), such that

$$\lim_{n\to+\infty}\left(|Y_t^{f_n}-Y_t|+|Z_t^{f_n}-Z_t|\right)=0 \quad a.e.\ (t,\omega). \tag{12}$$

We have,

$$\mathbb{E}\int_0^T|f_n(s,Y_s^{f_n})-f(s,Y_s)|ds\le I_1^n+I_2^n$$

where

Proof Existence, continued 2

where

$$I_1^n := \mathbb{E}\int_0^T |f_n(s, Y_s^{f_n}) - f(s, Y_s^{f_n})| ds$$

$$I_2^n := \mathbb{E}\int_0^T |f(s, Y_s^{f_n}) - f(s, Y_s)| ds$$

we have

$$\begin{aligned} I_1^n &\leq \mathbb{E}\int_0^T |f_n(s, Y_s^{f_n}) - f(s, Y_s^{f_n})| \mathbb{1}_{\{|Y_s^{f_n}| \leq N\}} ds \\ &+ \mathbb{E}\int_0^T |f_n(s, Y_s^{f_n}) - f(s, Y_s^{f_n})| \frac{(|Y_s^{f_n}|)^{(2-\frac{2}{\bar{\alpha}})}}{N^{(2-\frac{2}{\bar{\alpha}})}} \mathbb{1}_{\{|Y_s^{f_n}|| \geq N\}} ds \\ &\leq \rho_N(f_n - f) + \frac{2K_3^{\frac{1}{\bar{\alpha}}} \left[TK_2 + K_1\right]^{1-\frac{1}{\bar{\alpha}}}}{N^{(2-\frac{2}{\bar{\alpha}})}}. \end{aligned}$$

Passing to the limit on n and N, we show that I_1^n tends to 0. We use the continuity of f and the uniform integrability to show that I_2^n tends to 0.

Uniqueness. Let (Y, Z) and (Y', Z') be two solutions. Arguing as previously one can show (with $M_2 := 3c_0$) that: for every $R > 2$, $\beta \in]1, \min\left(3 - \frac{2}{\bar{\alpha}}, 2\right)[$, $\delta' < \frac{(\beta-1)}{(2\ell+\beta-1)M_2} \min\left(\frac{1}{2}, \frac{\kappa}{\beta}\right)$ and $\varepsilon > 0$, there exists $N_0 > R$ such that for all $N > N_0$ and $T' \leq T$:

$$\mathbb{E} \sup_{(T'-\delta')^+ \leq t \leq T'} |Y_t - Y_t'|^\beta + \mathbb{E} \int_{(T'-\delta')^+}^{T'} \frac{\left|Z_s - Z_s'\right|^2}{\left(|Y_s - Y_s'|^2 + \nu_R\right)^{\frac{2-\beta}{2}}} ds$$
$$\leq \varepsilon + \frac{\ell}{\beta - 1} e^{C_N \delta'} \mathbb{E}|Y_{T'} - Y_{T'}'|^\beta.$$

Again, taking successively $T' = T$, $T' = (T - \delta')^+$, $T' = (T - 2\delta')^+$..., we establish the uniqueness of solution. Theorem 4 is proved. ■

The previous Theorem can be extended, with the an analogue integrability condition on ξ, to the case where

$$H(t,y,z) = a + b|y| + c|z| + d|y||\ln|y|| + e|z|\sqrt{|\ln|z||} \qquad (13)$$

More generally, using Essaky-Hassan result one can establish the following theorem.

Theorem

Assume that,

(H.1) *H is continuous on $(y,\ z)$ and, there exit positive real numbers a, b, c, d and e such that for every $t,\ y,\ z$*

$$|H(t,y,z)| \leq a + b|y| + c_1|z| + c_0|y||\ln|y|| + e|z|\sqrt{|\ln|z||}$$

(H.2) *There exists a positive constant C (large enough) such that* $\mathbb{E}[|\xi|^{e^{CT}+1}] < +\infty$.

Then, the BSDE with parameters $(\xi,\ H)$ has a unique solution such that

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|Y_t|^{e^{Ct}+1}\Big) < \infty \ \ and \ \ \mathbb{E}\int_0^T |Z_s|^2 ds < \infty \qquad (14)$$