QBSDEs with unbounded terminal data

KHALED BAHLALI

IMATH, Université de Toulon
& CNRS, I2M, Aix-marseille Université

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Previous results in BSDEs and Quadratic BSDEs

The BSDE under consideration is

$$Y_t = \xi + \int_t^T H(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \tag{1}$$

Usually, the BSDE $eq(\xi, H)$ is called quadratic if there exist $\alpha, \beta, \gamma > 0$ such that for every (t, y, z),

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- Barrieu-El Karoui, 2012-2013, with more or less similar conditions,
- Essaky-Hassani, 2011 and 2013, for stochastic quadratic BSDE and others conditions.

Our Aproach consists to derive the existence of solutions for the BSDE without reflection from solutions of a suitable 2-barriers Reflected BSDE.

To this end, we use the result of Essaky & Hassani which establishes the existence of solutions for reflected QBSDEs without assuming any integrability condition on the terminal datum.

For the self-contained, we state the Essaky & Hassani result of in the following theorem.

Theorem

(Essaky–Hassani (JDE 2013)). Let L and U be continuous processes and ξ be a $\mathcal{F}_T$ measurable random variable. Assume that

1) $L_T \leq \xi \leq U_T$.

2) there exists a semimartingale which passes between the barriers L and U.

3) H is continuous in (y, z) and satisfies for every (s, ω), every $y \in [L_s(\omega), U_s(\omega)]$ and every $z \in \mathbb{R}^d$.

$$|f(s, \omega, y, z)| \leq \eta_s(\omega) + \frac{C_s(\omega)}{2}|z|^2$$

where $\eta \in \mathbb{L}^1([0, T] \times \Omega)$ and C is a continuous process.

Then, the following RBSDE has a minimal and a maximal solution.

$$\begin{cases} (i) & Y_t = \xi + \int_t^T H(s, Y_s, Z_s)ds + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s \\ (ii) & \forall t \leq T,\ L_t \leq Y_t \leq U_t, \\ (iii) & \int_t^T (Y_t - L_t)dK_t^+ = \int_t^T (U_t - Y_t)dK_t^- = 0,\ a.s., \\ (iv) & K_0^+ = K_0^- = 0,\quad K^+, K^-,\ \text{are continuous nondecreasing.} \\ (v) & dK^+ \perp dK^- \end{cases} \tag{3}$$

Theorem

Let a and b be positive real numbers. Assume that ξ satifies

(HExp) $\mathbb{E}\exp[(e^{2bT}+1)|\xi|]<\infty$

Then, the BSDE

$$Y_t=\xi+\int_t^T(a+b|Y_s|+\frac{1}{2}|z|^2)ds-\int_t^T Z_s dW_s,\quad 0\leq t\leq T \tag{4}$$

has a unique solution (Y, Z) such that

$$\mathbb{E}(\sup_{0\leq s\leq T}\exp([e^{2bs}+1]|Y_s|))<\infty \quad \textit{and} \quad \mathbb{E}\int_0^T|Z_s|^2ds<\infty. \tag{5}$$

Proof simple QBSDE

Proof. We put $g(y,z) := a + b|y| + \frac{1}{2}|z|^2$. Let (Y,Z) be a solution to BSDE (4). By Itô's formula we have,

$$
\begin{aligned}
e^{Y_t} &= e^{Y_T} + \int_t^T e^{Y_s} g(s, Y_s, Z_s)ds - \int_t^T e^{Y_s} Z_s dW_s - \frac{1}{2}\int_t^T e^{Y_s}|Z_s|^2 ds \\
&= e^{\xi} + \int_t^T e^{Y_s}\left[(a + b|Y_s| + \frac{1}{2}|Z_s|^2) - \frac{1}{2}|Z_s|^2\right]ds - \int_t^T e^{Y_s} Z_s dW_s \\
&= e^{\xi} + \int_t^T e^{Y_s}(a + b|Y_s|)ds - \int_t^T e^{Y_s} Z_s dW_s
\end{aligned}
$$

We set, $\overline{Y_t} = e^{Y_t}, \overline{Z_t} = e^{Y_t} Z_t$ and $\bar{\xi} = e^{\xi}$.

It is not difficult to see that $\overline{Y} > 0$ and $(\overline{Y}, \overline{Z})$ satisfies the BSDE

$$
\overline{Y_t} = \bar{\xi} + \int_t^T (a\overline{Y_s} + b\overline{Y_s}|\ln \overline{Y}_s|)ds - \int_t^T \overline{Z_s} dW_s \tag{6}
$$

Hence,

according to Theorem on BSDEs with logarithmic nonlinearity (Slide 2), the previous equation has a unique solution which satisfy the inequalities

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|Y_t|^{e^{2c_0t}+1}\Big)<\infty \quad \textit{and} \quad \mathbb{E}\int_0^T|Z_s|^2ds<\infty \tag{7}$$

Therefore, the BSDE (4) has a unique solution too. Indeed,
Let (Y^1, Z^1) and (Y^2, Z^2) be two solutions to BSDE (4).

Then, for $i=1,2$, $\bar{Y}^i>0$ and $\bar{Y}^i_t=e^{Y^i_t}$ and $\bar{Z}^i_t=e^{Y^i_t}Z^i_t$ are solutions to the logarithmic BSDE (6) which satisfy inequalities (7).

Hence, according to Theorem on Logarithmic BSDEs ,
$(\bar{Y}^1, \bar{Z}^1)=(\bar{Y}^2, \bar{Z}^2)$ from which we deduce that $(Y^1, Z^1)=(Y^2, Z^2)$.

It remains to prove inequalities (5).

Proof, simple BSDE, continued 2

It is not difficult to show that $\mathbb{E}\big(\sup_{0\leq s\leq T}\exp([e^{2bs}+1]|Y_s|)\big)<\infty$.
To prove that Z belongs to $\mathcal{M}^2$, we need the following simple Lemma.

Lemma

The function $u(x):=e^x-x-1$ satisfies the differential equation, $u''(x)-u'(x)=1$ and has the following properties:

(i) u belongs to $\mathcal{C}^2(\mathbb{R})$,

(ii) for every $x\geq 0$, $u(x)\geq 0$ and $u'(x)\geq 0$.

(iii) The map $x\longmapsto v(x):=u(|x|)$ belongs to $\mathcal{C}^2(\mathbb{R})$, $v'(0)=0$ and $u'(|x|)\leq e^{|x|}$.

We now prove that Z belongs to $\mathcal{M}^2$.
For $N>0$, let $\tau_N:=\inf\{t>0:|Y_t|+\int_0^t|u'(|Y_s|)|^2|Z_s|^2ds\geq N\}\wedge T$.
Set $sgn(x)=1$ if $x\geq 0$ and $sgn(x)=-1$ if $x<0$.
Itô's formula gives,

Proof, simple BSDE, continued 3

$$u(|Y_{t\wedge\tau_N}|) = u(|Y_0|) + \int_0^{t\wedge\tau_N} \mathrm{sgn}(Y_s)u'(|Y_s|)Z_s dW_s$$
$$+ \int_0^{t\wedge\tau_N} \left[\frac{1}{2}u''(|Y_s|)|Z_s|^2 - \mathrm{sgn}(Y_s)u'(|Y_s|)g(s, Y_s, Z_s)\right] ds$$

Passing to expectation and using the previous simple Lemma, we get for any $N > 0$

Proof, simple BSDE, continued 4

$$
\begin{aligned}
\mathbb{E}u(|Y_0|) &= \mathbb{E}u(|Y_{t\wedge\tau_N}|) + \mathbb{E}\int_0^{t\wedge\tau_N} \mathrm{sgn}(Y_s)u'(|Y_s|)g(s,Y_s,Z_s)ds \\
&\quad - \frac{1}{2}\mathbb{E}\int_0^{t\wedge\tau_N} u''(|Y_s|)|Z_s|^2 ds \\
&\leq \mathbb{E}u(|Y_{t\wedge\tau_N}|) + \mathbb{E}\int_0^{t\wedge\tau_N} \left[u'(|Y_s|)(a+b|Y_s|) - \frac{1}{2}|Z_s|^2\right] ds \\
&\leq \mathbb{E}u(|Y_{t\wedge\tau_N}|) + \mathbb{E}\int_0^{t\wedge\tau_N} \left[u'(|Y_s|)(a+b|Y_s|) - \frac{1}{2}|Z_s|^2\right] ds \\
&\leq \mathbb{E}u(|Y_{t\wedge\tau_N}|) + \mathbb{E}\int_0^{t\wedge\tau_N} \left[u'(|Y_s|)(a+b|Y_s|) - \frac{1}{2}|Z_s|^2\right] ds
\end{aligned}
$$

Proof, simple BSDE, continued 5

We successively use the previous simple Lemma and the previous exponential integrability for Y to show that for every N

$$\begin{aligned}\frac{1}{2}\mathbb{E}\int_0^{t\wedge\tau_N}|Z_s|^2 ds &\leq u(|Y_0|) + \mathbb{E}\int_0^T \left[(a + b|Y_s|)u'(|Y_s|)\right] ds \\ &\leq \mathbb{E}e^{|Y_0|} + \mathbb{E}\int_0^T \left[(ae^{|Y_s|} + b|Y_s|e^{|Y_s|}\right] ds \\ &\leq K'\end{aligned}$$

where K' is a constant not depending on N.

Using Fatou's lemma, we deduce that $\mathbb{E}\int_0^T |Z_s|^2 ds < \infty$. The proof is complete.

Theorem

Theorem

Assume that there exist positive constants a and b such that ξ *satisfies* **(HExp)**. *Let the generator* $H(t, \omega, y, z)$ *be continuous in* (y, z) *for a.e* (t, ω) *and satisfies*

$$H(t, y, z) \leq a + b|y| + \frac{1}{2}|z|^2 \tag{8}$$

Then, the BSDE (ξ, H) *has at least one solution* (Y, Z) *which satisfies (5).*

Proof, H dominé

Proof. We put $g(t,y,z) := a + b|y| + \frac{1}{2}|z|^2$.

Let $\xi^+ := max(\xi, 0)$ and $\xi^- := min(\xi, 0)$.

According to the previous Theorem, the BSDE with the parameters (ξ^+, g) as well as the BSDE with the parameters $(-\xi^-, -g)$ have unique solutions satisfying inequalities (5).

We denote by (Y^g, Z^g) [resp. (Y^{-g}, Z^{-g})] the unique solution of $eq(\xi^+, g)$ [resp. $eq(-\xi^-, -g)$].

Using then the Essaky-Hassani result with

$L = Y^{-g}$, $U = Y^g$, $\eta_t = a + b(|Y_t^{-g}| + |Y_t^g|) + \frac{1}{2}c^2$, and $C_t = 1$,

we deduce the existence of solution (Y, Z, K^+, K^-) to the following Reflected BSDE, such that (Y, Z) belongs to $\mathcal{C} \times \mathcal{L}^2$.

Proof, H dominé, continued 1

$$
\begin{cases}
(i) & Y_t = \xi + \int_t^T H(s, Y_s, Z_s)ds + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dW_s, \\
(ii) & \forall\, t \leq T,\ Y_t^{-g} \leq Y_t \leq Y_t^g, \\
(iii) & \int_0^T (Y_t - Y_t^{-g})dK_t^+ = \int_0^T (Y_t^g - Y_t)dK_t^- = 0, \text{ a.s.}, \\
(iv) & K_0^+ = K_0^- = 0, \quad K^+, K^- \text{ are continuous nondecreasing.} \\
(v) & dK^+ \perp dK^-
\end{cases}
\tag{9}
$$

Now, if we show that $dK^+ = dK^- = 0$, then the proof is finished. We shall prove this property.

Since Y_t^g is a solution to BSDE $eq(\xi, g)$, then Tanaka's formula applied to $(Y_t^g - Y_t)^+$ shows that,

$$(Y_t^g - Y_t)^+ = (Y_0^g - Y_0)^+ + \int_0^t \mathbf{1}_{\{Y_s^g > Y_s\}}[f(s, Y_s, Z_s) - g(s, Y_s^g, Z_s^g)]ds$$
$$+ \int_0^t \mathbf{1}_{\{Y_s^g > Y_s\}}(dK_s^+ - dK_s^-) + \int_0^t \mathbf{1}_{\{Y_s^g > Y_s\}}(Z_s^g - Z_s)dW_s$$
$$+ L_t^0(Y^g - Y)$$

where $L_t^0(Y^g - Y)$ denotes the local time at time t and level 0 of the semimartingale $(Y^g - Y)$.

Since $Y^g \geq Y$, then $(Y_t^g - Y_t)^+ = (Y_t^g - Y_t)$.

Therefore, identifying the terms of $(Y_t^g - Y_t)^+$ with those of $(Y_t^g - Y_t)$ and using the fact that $\mathbf{1} - \mathbf{1}_{\{Y_s^g > Y_s\}} = \mathbf{1}_{\{Y_s^g \leq Y_s\}} = \mathbf{1}_{\{Y_s^g = Y_s\}}$, we show that $(Z - Z^g)\mathbf{1}_{\{Y_s^g = Y_s\}}$ $a.e.$

Using the previous equalities, we deduce that,

Proof, H dominé, continued 3

$$\int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}}(dK_s^+ - dK_s^-) = \int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}}[g(s, Y_s^g, Z_s^g) - f(s, Y_s, Z_s)]ds + L_t^0(Y^g - Y)$$

Since $\int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}} dK_s^+ = 0$, it holds that

$$0 \leq \int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}}[g(s, Y_s^g, Z_s^g) - f(s, Y_s, Z_s)]ds + L_t^0(Y^g - Y) \geq 0$$
$$= -\int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}} dK_s^- \leq 0$$

This shows that , $\int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}} dK_s^- = 0$, and hence $dK^- = 0$.

Arguing symmetrically, one can show that $dK^+ = 0$.

Therefore, (Y, Z) satisfies the initial non reflected BSDE.

Proof, H dominé, continued 4

Since both Y^g and Y^{-g} satisfies inequality (5), so it is for Y.

Arguing as in the previous Theorem, one can show that $\mathbb{E} \int_0^T |Z_s|^2 ds < \infty$. This complete the proof.

Proof, simple BSDE, continued

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