Quadratic BSDEs with L^2 terminal data. Some existence results, Krylov's inequality and Itô-Krylov's formula

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Definitions and notations

The BSDE under consideration is

$$Y_t = \xi + \int_t^T H(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \tag{1}$$

It will be referred as $eq(\xi, H)$.

- $\mathcal{F}_t := \mathcal{F}_t^W$.
- $\mathcal{S}^2 := \{Y$ which is $\mathcal{F}_t$-adapted and $\mathbb{E} \sup\limits_{0\le t\le T} |Y_t|^2 < \infty\}$
- $\mathcal{M}^2 := \{Z$ which is $\mathcal{F}_t$-adapted and $\mathbb{E} \int_0^T |Z_s|^2 ds < \infty\}$
- $\mathcal{L}^2 := \{Z$ which is $\mathcal{F}_t$-adapted and $\int_0^T |Z_s|^2 ds < \infty \mathbb{P}$-a.s.$\}$
- $L_t^a(Y)$:= the local time of Y at time t at the level a.

Definition

A solution of the BSDE $eq(\xi, H)$ is an $\mathcal{F}_t$–adapted processes (Y, Z) which satisfy BSDE $eq(\xi, H)$ for each $t \in [0, T]$ and such that Y is continuous and $\int_0^T |Z_s|^2 ds < \infty$ $\mathbb{P} - a.s.$, that is $(Y, Z) \in \mathcal{C} \times \mathcal{L}^2$

Previous results in BSDEs and Quadratic BSDEs

Usually, the BSDE $eq(\xi, H)$ is called quadratic if there exist $\alpha, \beta, \gamma > 0$ such that for avery (t, y, z),

$$|H(t, y, z)| \leq \alpha + \beta|y| + \gamma|z|^2. \tag{2}$$

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- Barrieu-El Karoui, 2012-2013, with more or less similar conditions,
- Essaky-Hassani, 2011 and 2013, for stochastic quadratic BSDE and others conditions.

An example of Quadratic BSDEs

The following simple example shows that the exponential integrability of the terminal condition is not necessary for the existence of solutions.

- Let f be continuous function with compact support, consider the following simple BSDE

$$Y_t = \xi + \int_t^T f(Y_s) Z_s^2 ds - \int_t^T Z_s dW_s \tag{3}$$

We shall refer to this equation as $eq(\xi, f(y)z^2)$

- Let $u \in \mathcal{C}^2(\mathbb{R})$. Itô's formula gives

$$u(Y_t) = u(\xi) - \int_t^T u'(Y_s)Z_s dW_s + \int_t^T \left(u'(Y_s)f(Y_s) - \frac{1}{2}u''(Y_s) \right) Z_s^2 ds$$

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- A solution to remove the last (quadratic) term is

$$u(x) := \int_0^x \exp\left(2 \int_0^y f(t) dt \right) dy,$$

$u \in \mathcal{C}^2(\mathbb{R})$ moreover u' and u'' are bounded.

- Put, $\bar{Y}_t := u(Y_t)$, $\bar{Z}_t := u'(Y_t)Z_t$, our BSDE becomes

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- But u is a quasi-isometry: $\exists$ m and M s.t.

$$\forall\, x, y \in \mathbb{R}, \quad m\,|x-y| \leq |u(x) - u(y)| \leq M\,|x-y|\,.$$

Therefore $eq(\xi, f(y)Z^2)$ has a unique solution if and only if $eq(u(\xi), 0)$ has a unique solution.

But, we know that $eq(u(\xi), 0)$ has a unique solution if $u(\xi) \in L^2(\Omega)$.

This is equivalent to $\xi \in L^2(\Omega)$ since u is a quasi-isometry.

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- **Remark:** Neither the local Lipschitz on y nor the convexity condition on z are needed for the uniqueness.
- **The following examples also are covered by our result.**

$$f(y) = \mathbf{1}_{[a,b]}(y), \;\; f(y) = \mathbf{1}_{[a,b]}(y) - \mathbf{1}_{[c,d]}(y), \;\; f(y) := \frac{1}{(1+y^2)\sqrt{|y|}}$$

Removing the quadratic term

Lemma

Let f belongs to $L^1(\mathbb{R})$. The function

$$u(x) := \int_0^x \exp\left(2\int_0^y f(t)dt\right) dy$$

satisfies following properties

1. $\frac{1}{2}u''(x) - f(x)u'(x) = 0.$

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2. *u is a one to one function from $\mathbb{R}$ onto $\mathbb{R}$.*

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1. $\frac{1}{2}u''(x) - f(x)u'(x) = 0.$
2. *u is a one to one function from $\mathbb{R}$ onto $\mathbb{R}$.*
3. *Both u and its inverse u^{-1} belong to $\mathcal{C}^1(\mathbb{R}) \cap W^2_{1,loc}(\mathbb{R})$.*
   where $W^2_{p,\,loc} := \{u : \mathbb{R} \mapsto \mathbb{R}\,;\; u, u', u'' \in L^p_{loc}(\mathbb{R})\}$.

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4. *u is a quasi-isometry: there exist m and M s.t. for any $x, y \in \mathbb{R}$,*
   $m\,|x-y| \leq |u(x)-u(y)| \leq M\,|x-y|$

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4. *u is a quasi-isometry: there exist m and M s.t. for any $x, y \in \mathbb{R}$,*
   $$m|x-y| \leq |u(x) - u(y)| \leq M|x-y|$$
5. *If in addition f is continuous then both u and u^{-1} are of class $\mathcal{C}^2$.*

Theorem

(Essaky–Hassani (JDE 2013)). Let L and U be continuous processes and ξ be a $\mathcal{F}_T$ measurable random variable. Assume that

1) $L_T \le \xi \le U_T$.

2) there exists a semimartingale which passes between the barriers L and U.

3) H is continuous in (y,z) and satisfies for every (s,ω), every $y \in [L_s(\omega), U_s(\omega)]$ and every $z \in \mathbb{R}^d$.

$$|f(s,\omega,y,z)| \le \eta_s(\omega) + \frac{C_s(\omega)}{2}|z|^2$$

where $\eta \in \mathbb{L}^1([0,T]\times\Omega)$ and C is a continuous process.

Then, the following RBSDE has at least one solution.

$$\begin{cases} (i) & Y_t = \xi + \int_t^T H(s,Y_s,Z_s)ds + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s \\ (ii) & \forall t \le T,\ L_t \le Y_t \le U_t, \\ (iii) & \int_t^T (Y_t - L_t)dK_t^+ = \int_t^T (U_t - Y_t)dK_t^- = 0,\ a.s., \\ (iv) & K_0^+ = K_0^- = 0,\quad K^+, K^-,\ \text{are continuous nondecreasing.} \\ (v) & dK^+ \perp dK^- \end{cases} \tag{4}$$

The equation $eq(\xi, a + by + f(y)z^2)$

The BSDE under consideration in this subsection is,

$$Y_t = \xi + \int_t^T (a + b|Y_s| + f(Y_s)|Z_s|^2)ds - \int_t^T Z_s dW_s \qquad (5)$$

where $a, b \in \mathbb{R}_+$ and $f : \mathbb{R} \longmapsto \mathbb{R}_+$.

Lemma

Assume that $\xi \in L^2(\Omega)$ and f be continuous and belongs to $L^1(\mathbb{R})$. Then, the BSDE (5) has solution in $\mathcal{S}^2 \times \mathcal{M}^2$ has a minimal and a maximal solution.

The result remains valid when $H(y, z) := a + b|y| + c|z| + f(|y|)|z|^2)$.

Proof. Itô's formula applied to the transformation u defined in Lemma 2 shows that

$$u(Y_t) = u(\xi) + \int_t^T u'(Y_s)(a+b|Y_s|)ds - \int_t^T u'(Y_s)Z_s dW_s \qquad (6)$$

We put,

$$Y' := u(Y),\ Z' := u'(Y)Z \quad \text{and} \quad \xi' := u(\xi)$$

(Y', Z') satisfies then the BSDE,

$$Y'_t = \xi' + \int_t^T G(Y'_s)ds - \int_t^T Z'_s dW_s \qquad (7)$$

where $\quad G(x) := u'(u^{-1}(x)(a+b|u^{-1}(x)|))$.

Proof, continued. Using Lemma 2, one can show that equation (5) has a (unique) solution if and only if equation (7) has a (unique) solution.

From Lemma 2, we deduce that the coefficient G of BSDE (7) is continuous and with linear growth. Moreover, the terminal condition $\xi' := u(\xi)$ is square integrable if and only if ξ is square integrable.

Therefore, according to Lepeltier & San-Martin 97-98, the BSDE (7) has a minimal and a maximal solution.

The equation $eq(\xi, H)$ with $H(t,y,z) \le a + b|y| + f(|y|)z^2)$

Theorem

Assume that

(H1) $\xi \in L^2(\Omega)$

(H3) $|H(s,y,z)| \le a + b|y| + f(|y|)|z|^2 := g(y,z)$ *with f continuous and in* $L^1(\mathbb{R})$*.*

(H4) *H is continuous in* (y,z)*.*

Then, the BSDE $eq(\xi, H)$ *has at least one solution in* $\mathcal{S}^2 \times \mathcal{M}^2$*.*

The result remains valid when $H(y,z)$ is dominated by $g(y,z) := a + b|y| + c|z| + f(|y|)|z|^2)$.

Proof.
Let (Y^g, Z^g) be the maximal solution of BSDE $eq(\xi^+, g)$ and $(Y^{-g}, Z^{-g}))$ be the minimal solution of BSDE $eq(-\xi^-, -g)$.

Using the Essaky & Hassani result, with $L = Y^{-g}$, $U = Y^g$,
$\eta_t = a + b(|Y_t^{-g}| + |Y_t^g| + c^2$,
$C_t = 1 + \sup_{s\leq t} \sup_{\alpha\in[0,1]} |f(\alpha Y_s^{-g} + (1-\alpha) Y_s^g|)$, we deduce the existence of a minimal and a maximal solution to the RBSDE,

$$\begin{cases} (j) & Y_t = \xi + \int_t^T H(s, Y_s, Z_s)ds + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s \,, \\ (jj) & \forall t \leq T,\ Y_t^{-g} \leq Y_t \leq Y_t^g, \\ (jjj) & \int_t^T (Y_t - Y_t^{-g}) dK_t^+ = \int_t^T (Y_t^g - Y_t) dK_t^- = 0,\ \text{a.s.}, \\ (iv) & K_0^+ = K_0^- = 0,\quad K^+, K^-,\ \text{are continuous nondecreasing} \end{cases}$$

Proof, H dominé continued 1

- We shall prove that $dK^- = dK^+ = 0$.

Since Y_t^g and Y_t^{-g} are solutions for BSDEs, we proceed as follows :

1. we apply Tanaka's formula to $(Y_t^g - Y_t)^+$,
2. from (jj), we have $(Y_t^g - Y_t)^+ = (Y_t^g - Y_t)$
3. we identify the terms of $(Y_t^g - Y_t)^+$ with those of $(Y_t^g - Y_t)$, then we use properties (jjj) and (v) of the previous RBSDE, to prove that $dK^- = 0$.
4. Symmetrically, one can show that $dK^+ = 0$.

Proof, H dominé continued 2

- We shall prove the previous claim 3.

Using the fact that: $\mathbf{1} - \mathbf{1}_{\{Y_s^g > Y_s\}} = \mathbf{1}_{\{Y_s^g \leq Y_s\}} = \mathbf{1}_{\{Y_s^g = Y_s\}}$, we first show that $(Z = Z^g)\mathbf{1}_{\{Y_s^g = Y_s\}}$. And next we get

$$\int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}}(dK_s^+ - dK_s^-) = \int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}}[g(s, Y_s^g, Z_s^g) - H(s, Y_s, Z_s)]ds + L_t^0(Y^g - Y)$$

Since $\int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}} dK_s^+ = 0$, it holds that

$$L_t^0 + \int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}}[g(s, Y_s^g, Z_s^g) - H(s, Y_s, Z_s)]ds + \int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}} dK_s^- = 0$$

Since the the three terms of the left hand side are positive, we deduce that, $\int_0^t \mathbf{1}_{\{Y_s^g = Y_s\}} dK_s^- = 0$, which implies that $dK^- = 0$.

- It now remains to show that $Z \in \mathcal{M}^2$.

To this end, we need the following lemma

Zvonkin 1 for the Proof $Z \in \mathcal{M}^2$

Lemma

Let f belongs to $L^1(\mathbb{R})$ *and put* $K(y) := \int_0^y \exp(-2\int_0^x f(r)dr)dx$. *The function* $u(x) := \int_0^x K(y)\exp(2\int_0^y f(t)dt)dy$ *satisfies following properties*

(i) *u belongs to* $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}^2_{1,loc}(\mathbb{R})$, *and,* $u(x) \geq 0$ *and* $u'(x) \geq 0$ *for* $x \geq 0$. *Moreover,* $\frac{1}{2}u''(x) - f(x)u'(x) = \frac{1}{2}$ *for a.e x.*

(ii) *The map* $x \longmapsto v(x) := u(|x|)$ *belongs to* $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}^2_{1,loc}(\mathbb{R})$, *and* $v'(0) = 0$.

(iii) *There exists a positive constant c such that for every* $x \in \mathbb{R}$, $u(|x|) \leq c|x|^2$ *and* $u'(|x|) \leq c(|x|)$.

(iv) *If moreover, f is continuous, then the function* $v(x) := u(|x|)$ *belongs to* $\mathcal{C}^2(\mathbb{R})$.

Proof H dominé continued 3

For $N > 0$, let $\tau_N := \inf\{t > 0 : |Y_t| + \int_0^t |v'(Y_s)|^2 |Z_s|^2 ds \geq N\} \wedge T$. Applying Itô-Krylov's formula to the function v (defined in the previous Lemma), it holds that for every $t \in [0, T]$,

$$
\begin{aligned}
u(|Y_0|) &= u(|Y_{t\wedge\tau_N}|) + \int_0^{t\wedge\tau_N} sign(Y_s) u'(|Y_s|) Z_s dW_s \\
&\quad + \int_0^{t\wedge\tau_N} \left(sign(Y_s) u'(|Y_s|) H(s, Y_s, Z_s) ds - \frac{1}{2} u''(|Y_s|) Z_s^2 \right) ds \\
&\leq u(|Y_{t\wedge\tau_N}|) + \int_0^{t\wedge\tau_N} sign(Y_s) u'(|Y_s|) Z_s dW_s \\
&\quad + \int_0^{t\wedge\tau_N} (u'(|Y_s|)(a + b|Y_s|) ds \\
&\quad \int_0^{t\wedge\tau_N} \left(u'(|Y_s|) f(|Y_s|) - \frac{1}{2} u''(|Y_s|) Z_s^2 \right) ds
\end{aligned}
$$

Since, $u'(x)f(x) - \frac{1}{2}u''(x) = -\frac{1}{2}$, then passing to expectation and using the previous Lemma, we get for any $N > 0$

$$\frac{1}{2}\mathbb{E}\int_0^{t\wedge\tau_N}|Z_s|^2ds \leq 2\mathbb{E}\Big(\sup_{0\leq s\leq T} u(|Y_s|)\Big) + \mathbb{E}\int_0^T \big(u'(|Y_s|)(a+b|Y_s|)\big)ds$$

We successively use assertion (iii) of the previous Lemma, the fact that the process Y belongs to $\mathcal{S}^2$ and Fatou's Lemma, to show that,

$$\mathbb{E}\int_0^T |Z_s|^2ds < \infty.$$

.

Krylov's estimates.

Assumptions.

(H2) $\exists \eta_t(\omega)$, *and* $f(y)$; $\forall (t, \omega, y, z)$, $|H(t, y, z)| \leq \eta_t + f(y)|z|^2$.

For simplicity, we assume in the sequel that $\eta_t = 0$.

Krylov estimate. The case f integrable

Lemma

Let $(Y,Z) \in \mathcal{S}^2 \times \mathcal{L}^2$ *be a solution of the BSDE eq*(ξ, H). *Assume that* H *satisfies* ***(H2)*** *with* $f \in L^1(\mathbb{R})$. *Then,*

-(i)- there exists a postive constant C *depending on* T, $\|\xi\|_{L^1(\Omega)}$ *and* $\|f\|_{L^1(\mathbb{R})}$ *such that for any nonnegative measurable function* ψ,

$$\mathbb{E}\int_0^T \psi(Y_s)Z_s^2 ds \leq C\,\|\psi\|_{L^1(\mathbb{R})}.$$

-(ii)- There exists a positive constant C *depending on* T, $\|\xi\|_{L^1(\Omega)}$ *and* $\|f\|_{L^1(\mathbb{R})}$ *such that for any nonnegative measurable function* ψ,

$$\mathbb{E}\int_0^{T\wedge\tau_R} \psi(Y_s)Z_s^2 ds \leq C\,\|\psi\|_{L^1([-R,R])},$$

where $\tau_R := \inf\{t > 0, |Y_t| \geq R\}$

Proof of the Lemma

For simplicity we assume that $H(s, y,) = f(y)z^2$. The general case follows by localization. Let a be a real number. By Tanaka's formula

$$(Y_t - a)^- = (Y_0 - a)^- - \int_0^t \mathbf{1}_{\{Y_s<a\}} dY_s + \frac{1}{2} L_t^a(Y)$$

$$= (Y_0 - a)^- - \int_0^t \mathbf{1}_{\{Y_s<a\}} H(Y_s, Z_s) ds + \int_0^t \mathbf{1}_{\{Y_s<a\}} Z_s dW_s + \frac{1}{2} L_t^a(Y)$$

We successively use the fact that the map x^- is Lipschitz and the density occupation formula to get,

$$\begin{aligned}
\frac{1}{2} L_t^a(Y) &\leq |Y_t - Y_0| + \int_0^t \mathbf{1}_{\{Y_s<a\}} f(Y_s) Z_s^2 ds + \int_0^t \mathbf{1}_{\{Y_s<a\}} Z_s dW_s \\
&= |Y_t - Y_0| + \int_0^t \mathbf{1}_{\{Y_s<a\}} f(Y_s) d\langle Y \rangle_s + \int_0^t \mathbf{1}_{\{Y_s<a\}} Z_s dW_s \\
&= |Y_t - Y_0| + \int_{-\infty}^a f(x) L_t^x(Y) dx + \int_0^t \mathbf{1}_{\{Y_s<a\}} Z_s dW_s
\end{aligned}$$

Proof of the Lemma

Since $\mathbb{E}\int_{-\infty}^{+\infty} f(x)L_t^x(Y)dx < \infty$, then passing to expectation to obtain,

$$\mathbb{E}\left[L_t^a(Y)\right] \leq 2\mathbb{E}|Y_t - Y_0| + \int_{-\infty}^{a} 2f(x)\mathbb{E}\left[L_t^x(Y)\right]dx$$

Using Gronwall's lemma we get

$$\sup_{a\in\mathbb{R}} \mathbb{E}\left[L_T^a(Y)\right] \leq 2\mathbb{E}\sup_{0\leq s\leq T}|Y_s| \exp\left(2||f||_{L^1(\mathbb{R})}\right) = C_T \exp\left(2||f||_{L^1(\mathbb{R})}\right).$$

Now, let $\psi \in L^1_+(\mathbb{R})$. We use the previous inequality to get,

$$\begin{aligned} \mathbb{E}\int_0^T \psi(Y_s)Z_s^2 ds &= \mathbb{E}\int_0^T \psi(Y_s)d\langle Y\rangle_s \\ &\leq \int_{-\infty}^{+\infty} \psi(a)\mathbb{E}L_T^a(Y)da \\ &\leq C_T\,\|\psi\|_{L^1(\mathbb{R})} \end{aligned}$$

The Lemma is proved.

Krylov's estimate. The case where f is locally integrable

Lemma

Let $(Y, Z) \in \mathcal{S}^2 \times \mathcal{L}^2$ be a solution of the BSDE eq(ξ, H). Assume that H satisfies **(H2)** *with f locally integrable. Assume moreover that,*

$$\int_0^T |H(s, Y_s, Z_s)| ds < \infty \ \ \mathbb{P} - a.s$$

Let $\tau_R := \inf\{t > 0, |Y_t| \geq R\}$.

Then, there exists C depending on T, R, $\|\xi\|_{L^1(\Omega)}$ and $\|f\|_{L^1([-R,R])}$ such that for any nonnegative measurable function $\psi \in L^1_{loc}$,

$$\mathbb{E} \int_0^{T \wedge \tau_R} \psi(Y_s) Z_s^2 ds \leq C \, \|\psi\|_{L^1([-R,R])} \, ,$$

Proof of the Lemma

Let $\tau'_N := \inf\{t > 0 : \int_0^t |Z_s|^2 ds \geq N\}$,
$\tau''_M := \inf\{t > 0 : \int_0^t |H(s, Y_s, Z_s)| ds \geq M\}$, and put $\tau := \tau_R \wedge \tau'_N \wedge \tau''_M$.
By Tanaka's formula we have,

$$(Y_{T\wedge\tau} - a)^- = (Y_0 - a)^- - \int_0^{T\wedge\tau} \mathbf{1}_{\{Y_s<a\}} dY_s + \frac{1}{2} L^a_{T\wedge\tau}(Y)$$

Since the map $y \mapsto (y-a)^-$ is Lipschitz, we obtain

$$\begin{aligned} \frac{1}{2} L^a_{T\wedge\tau}(Y) \leq\ & |Y_{T\wedge\tau} - Y_0| + \int_0^{T\wedge\tau} \mathbf{1}_{\{Y_s<a\}} |H(s, Y_s, Z_s)| ds \qquad (9) \\ & - \int_0^{T\wedge\tau} \mathbf{1}_{\{Y_s<a\}} Z_s dW_s \end{aligned}$$

Passing to expectation in inequality (9), and since $\tau := \tau_R \wedge \tau'_N \wedge \tau''_M$, we get $\mathbb{E}\left[L^a_{T\wedge\tau}(Y)\right] \leq 4R + 2M$. Since a is an arbitrary real number, we obtain

$$\sup_a \mathbb{E}\left[L^a_{T\wedge\tau}(Y)\right] \leq 4R + 2M. \qquad (10)$$

Proof of the Lemma continued 1

Passing to expectation in inequality (9) and use the occupation density formula, we obtain,

$$\begin{aligned}\mathbb{E}\left[L^a_{T\wedge\tau}(Y)\right] &\leq \mathbb{E}|Y_{T\wedge\tau} - Y_0| + \int_0^{T\wedge\tau} \mathbf{1}_{\{Y_s<a\}} f(Y_s) Z_s^2 ds \\ &\leq \mathbb{E}|Y_{T\wedge\tau} - Y_0| + \int_0^{T\wedge\tau} \mathbf{1}_{\{Y_s<a\}} f(Y_s) d\langle Y\rangle_s \\ &\leq \mathbb{E}|Y_{T\wedge\tau} - Y_0| + \int_{-R}^{a} 2|f(x)|\mathbb{E}\left[L^x_{T\wedge\tau}(Y)\right] dx \qquad (11)\end{aligned}$$

Using inequality (10) and Gronwall's lemma, we get

$$\mathbb{E}\left[L^a_{T\wedge\tau_R}(Y)\right] \leq 2\mathbb{E}\sup_{0\leq t\leq T}|Y_t| \exp\left(2||f||_{L^1([-R,R])}\right)$$

Let now $\psi \in L^1_+(\mathbb{R})$. Using the previous bound and the occupation density formula, we get

$$\begin{aligned}
\mathbb{E}\int_0^{T\wedge\tau} \psi(Y_s) Z_s^2 ds &= \mathbb{E}\int_0^{T\wedge\tau} \mathbf{1}_{\{Y_s<a\}} f(Y_s) d\langle Y\rangle_s \\
&\leq \int_{-R}^{R} \psi(a)\mathbb{E}\left[L^a_{T\wedge\tau_R}(Y)\right] da \\
&\leq C(T,R)\,\|\psi\|_{L^1([-R,R])}
\end{aligned}$$

Passing to the limit on N and M, having in mind that $\tau := \tau_R \wedge \tau'_N \wedge \tau''_M$, we get the desired estimate. The Lemma is proved.

Itô-Krylov formula

Theorem

Let (Y, Z) be a solution for BSDE $eq(\xi, H)$. Assume that **(H1)** *holds with f locally integrable. Then, for any function u belonging to $\mathcal{C}^1(\mathbb{R}) \cap W^2_{1,loc}(\mathbb{R})$*

$$u(Y_{t\wedge\tau_R}) = u(Y_0) + \int_0^{t\wedge\tau_R} u'(Y_s)dY_s + \frac{1}{2}\int_0^{t\wedge\tau_R} u''(Y_s)Z_s^2 ds$$

where $\tau_R := \inf\{t > 0, |Y_t| \geq R\}$.

Proof.

- Using Lemma 7 (local Krylov'estimate), the term $\int_0^{t\wedge\tau_R} u''(Y_s)Z_s^2 ds$ is well defined.

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- To establish Itô's formula, we approximate u by a suitable sequence of smooth functions u_n for which Itô's formula is valid.
- We use, once again, Lemma 7 and pass to the limit.

The equation $eq(\xi, f(y)z^2)$

The following proposition shows that neither the exponential moment of ξ nor the continuity of the generator are needed to obtain the existence and uniqueness of the solution to quadratic BSDEs.

Theorem

Assume that $\xi \in L^2(\Omega)$ and $f \in L^1(\mathbb{R})$. Then the BSDE $eq(\xi, f(y)z^2)$ has a unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$.

The equation $eq(\xi, f(y)z^2)$ continued

Proof. Let (Y, Z) be a solution of equation $eq(\xi, f(y)z^2)$. Since $u(x) = \int_0^x \exp(\int_0^y f(t)dt)dy$ belongs to $\mathcal{C}^1(\mathbb{R}) \cap W^2_{1,loc}(\mathbb{R})$, then Itô-Krylov's formula (Lemma 9) shows that,

$$u(Y_t) = u(\xi) - \int_t^T u'(Y_s)Z_s dW_s$$

The function u satisfies : u is invertible and

$$0 < \exp(-\|f\|_{L^1(\mathbb{R})}) \leq u' \leq \exp(\|f\|_{L^1(\mathbb{R})}).$$

Therefore

(Y_t, Z_t) is the unique solution of the BSDE $eq(\xi, f(y)z^2)$ if and only if

$(u(Y_t), u'(Y_t)Z_t)$ is the unique solution to the BSDE $eq(u(\xi), 0)$.

Since, $|u(\xi)| \leq |\xi| \exp(\|f\|_{L^1(\mathbb{R})})$, we deduce that equation $eq(u(\xi), 0)$ has a unique solution if ξ belongs to $L^2(\Omega)$.

Comparison theorem for $eq(\xi, f(y)z^2)$

This is a comparison theorem for measurable generators.

Theorem

Let ξ_1, ξ_2 be $\mathcal{F}_T$–measurable and satisfy assumption **(H1)***. Let f, g be in $L^1(\mathbb{R})$. Let (Y^f, Z^f), (Y^g, Z^g) be respectively the solution of the BSDEs $eq(\xi_1, f(y)|z|^2)$ and $eq(\xi_2, g(y)|z|^2)$. Assume that $\xi_1 \leq \xi_2$ a.s. and $f \leq g$ a.e. Then $Y_t^f \leq Y_t^g$ for all t, $\mathbb{P}$–a.s.*

Corollary

Let ξ be $\mathcal{F}_T$–measurable and satisfies assumption **(H1)***. Let f, g be in $L^1(\mathbb{R})$. Let (Y^f, Z^f), (Y^g, Z^g) be respectively the solution of the BSDEs $eq(\xi, f(y)|z|^2)$ and $eq(\xi, g(y)|z|^2)$. Assume that $f = g$ a.e. Then $Y_t^f = Y_t^g$ for all t, $\mathbb{P}$–a.s.*

Proof of comparison.

Proof. According to Theorem 10, the solutions (Y^f, Z^f) and (Y^g, Z^g) belong to $\mathcal{S}^2 \times \mathcal{M}^2$. For a given function h, we put

$$u_h(x) := \int_0^x \exp\left(2\int_0^y h(t)dt\right) dy$$

The idea consists to apply suitably the Ito-Krylov formula to $u_f(Y_T^g)$. This gives,

$$\begin{aligned} u_f(Y_T^g) &= u_f(Y_t^g) + \int_t^T u_f'(Y_s^g)dY_s^g + \frac{1}{2}\int_t^T u_f''(Y_s^g)d\langle Y_\cdot^g\rangle_s \\ &= u_f(Y_t^g) + M_T - M_t - \int_t^T u_f'(Y_s^g)g(Y_s^g)|Z_s^g|^2 ds \\ &+ \frac{1}{2}\int_t^T u_f''(Y_s^g)|Z_s^g|^2 ds \end{aligned}$$

where $(M_t)_{0\le t\le T}$ is a martingale.

Proof of comparison continued.

Proof, continued Since $u_g''(x) - 2g(x)u_g'(x) = 0$, $u_f''(x) - 2f(x)u_f'(x) = 0$ and $u_f'(x) \geq 0$, then

$$u_f(Y_T^g) = u_f(Y_t^g) + M_T - M_t - \int_t^T u_f'(Y_s^g)\left[g(Y_s^g) - f(Y_s^g)\right]|Z_s^g|^2 ds$$

Since, $\int_t^T u_f'(Y_s^g)\left[g(Y_s^g) - f(Y_s^g)\right]|Z_s^g|^2 ds \geq 0$, then

$$u_f\left(Y_t^g\right) \geq u_f(Y_T^g) + M_T - M_t$$

Passing to conditional expectation, and since u_f is increasing and $\xi_2 \geq \xi_1$, we get

$$\begin{aligned} u_f\left(Y_t^g\right) &\geq \mathbb{E}\left[u_f(Y_T^g)\,/\mathcal{F}_t\right] \\ &= \mathbb{E}\left[u_f(\xi_2)\,/\mathcal{F}_t\right] \\ &\geq \mathbb{E}\left[u_f(\xi_1)\,/\mathcal{F}_t\right] \\ &= u_f\left(Y_t^f\right) \end{aligned}$$

Passing to u_f^{-1}, we get $Y_t^g \geq Y_t^f$. The proof finished.

The equation $eq(\xi, a + by + f(y)z^2)$

The BSDE under consideration in this subsection is,

$$Y_t = \xi + \int_t^T (a + b|Y_s| + f(Y_s)|Z_s|^2)ds - \int_t^T Z_s dW_s \qquad (12)$$

where $a, b \in \mathbb{R}_+$ and $f : \mathbb{R} \longmapsto \mathbb{R}_+$.

Lemma

Assume that $\xi \in L^2(\Omega)$ and $f \in L^1(\mathbb{R})$. Then, the BSDE (12) has solution in $\mathcal{S}^2 \times \mathcal{M}^2$ has a minimal and a maximal solution.

The result remains valid when $H(y, z) := a + b|y| + c|z| + f(|y|)|z|^2)$.

The equation $eq(\xi,\ a+by+f(y)z^2)$

Proof. Itô–Krylov formula applied to the transformation u defined in Lemma 2 shows that

$$u(Y_t) = u(\xi) + \int_t^T u'(Y_s)(a+b|Y_s|)ds - \int_t^T u'(Y_s)Z_s dW_s \qquad (13)$$

We put,

$$Y' := u(Y),\ Z' := u'(Y)Z \quad \text{and} \quad \xi' := u(\xi)$$

Y' and Z' satisfies then the BSDE,

$$Y'_t = \xi' + \int_t^T G(Y'_s)ds - \int_t^T Z'_s dW_s \qquad (14)$$

where $G(x) := u'(u^{-1}(x)(a+b|u^{-1}(x)|))$.

Proof for equation $eq(\xi,\ a + by + f(y)z^2)$, continued

Proof, continued. Using Lemma 2, one can show that equation (12) has a (unique) solution if and only if equation (14) has a (unique) solution.

From Lemma 2, we deduce that the coefficient G of BSDE (14) is continuous and with linear growth. Moreover, the terminal condition $\xi' := u(\xi)$ is square integrable if and only if ξ is square integrable.

Therefore, according to Lepeltier & San-Martin 97-98, the BSDE (14) has a minimal and a maximal solution.

Assumptions.

Existence of a viscosity solution.

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(H5) $[\sigma$, b are uniformly Lipschitz $]$

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Theorem

Assume (H3)-(H7) hold. Then, $u(t,x) := Y_t^{(t,x)}$ is a viscosity solution for the PDE associated to $\sigma(x), b(x), f(u)(\nabla(u))^2$ with the terminal condition ψ.

Touching property.

To prove the existence of viscosity solution, we need the following touching property.

Lemma

Let $(\xi_t)_{0\leq t\leq T}$ be a continuous adapted process such that

$$d\xi_t = \beta(t)dt + \alpha(t)dW_t,$$

where β and α are continuous adapted processes such that b,$|\sigma|^2$ are integrable. If $\xi_t \geq 0$ a.s.for all t, then for all t,

$$1_{\{\xi_t=0\}}\alpha(t) = 0 \quad a.s.,$$

$$1_{\{\xi_t=0\}}\beta(t) \geq 0 \quad a.s.,$$

Proof viscosity solution.

We assume, $H(x,y,z) := f(y)z^2$ with some integrable f.
We denote $(X_s, Y_s, Z_s) := (X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$
Since X is a Markov diffusion and $u(t,x) = Y_t^{(t,x)}$, then

$$\forall s \in [0,T], \quad u(s, X_s) = Y_s \tag{15}$$

Let $\phi \in C^{1,2}$. Let (t,x) be a local Maximum of $u - \phi$, which we suppose global and equal to 0, that is
$\phi(t,x) = u(t,x)$ and $\phi(\bar{t},\bar{x}) \geq u(\bar{t},\bar{x})$ for all $(\bar{s},\bar{x})$.
This and equality (15) imply that

$$\phi(s, X_s) \geq Y_s \tag{16}$$

Proof viscosity solution, continued.

We now show that u satisfies the comparison property. We have

$$Y_t = Y_s + \int_t^s H(X_s, Y_r, Z_r)dr - \int_t^s Z_r dW_r$$

$$\phi(s, X_s) = \phi(t, X_t) + \int_t^s (\frac{\partial \phi}{\partial r} + L\phi)(r, X_r)dr + \int_t^s \sigma D\phi(r, X_r)dW_r$$

As $\phi(s, X_s) \geq Y_s$, the touching property shows that, for each s,

$$1\!\!1_{\{\phi(s,X_s)=Y_s\}}(\frac{\partial \phi}{\partial t} + L\phi)(s, X_s) + H(X_s, Y_s, Z_s) \geq 0 \quad a.s.,$$

$$1\!\!1_{\{\phi(s,X_s)=Y_s\}}|-Z_s + \sigma^T D\phi(s, X_s)| \geq 0 \quad a.s.$$

As $\phi(t, x) := \phi(t, X_t) = Y_t := u(t, x)$ for $s = t$, the second equation gives $Z_t = \sigma D\phi(t, X_t) := \sigma D\phi(t, x)$, and the first inequality gives the expected result .

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