Dynamic Models in Economics

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Basic models in dynamic economics

- The Ramsey models (1928)

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- The Overlapping Generations (OG) models (1947, 1958, 1965)

- The Ramsey model in continuous time

$$\max \int_0^\infty e^{-\theta t} u(c_t)\, dt$$

subject to the budget constraints (market economy)

$$\dot{k}_t + \delta k_t + c_t \leq r_t k_t + w_t l_t$$

or to the resource constraints (central planner)

$$\dot{k}_t + \delta k_t + c_t \leq f(k_t)$$

- The Ramsey model in discrete time

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the budget constraints (market economy)

$$k_{t+1} - k_t + \delta k_t + c_t \leq r_t k_t + w_t l_t$$

or to the resource constraints (central planner)

$$k_{t+1} - k_t + \delta k_t + c_t \leq f(k_t)$$

- The program

$$\max U(c_t, d_{t+1})$$

subject to the budget constraints (market economy)

$$\begin{aligned} \frac{K_{t+1}}{N_t} + c_t &\leq w_t l_t \\ d_{t+1} &\leq R_{t+1} \frac{K_{t+1}}{N_t} \end{aligned}$$

Extensions

- Ramsey + heterogeneity

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- Ramsey + heterogeneity
- Ramsey + imperfections

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- Ramsey + imperfections
- OG + heterogeneity

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Unified Field Theory

- Bridge Ramsey and OG

- Bridge Ramsey and OG
- Bridge Ramsey in CT and in DT

Bridging Ramsey and OG

- Introduce a degree of altruism

$$
\begin{aligned}
U_t &= u\left(c_t, d_{t+1}\right) + \alpha n_t U_{t+1} \\
&= \alpha^T U_{t+T} \prod_{s=0}^{T-1} n_{t+s} + u\left(c_t, d_{t+1}\right) + \sum_{s=1}^{T-1} \alpha^s u\left(c_{t+s}, d_{t+s+1}\right) \prod_{\tau=1}^{s} n_t
\end{aligned}
$$

Let

$$
n_t = n \text{ and } \lim_{T \to \infty} \alpha^T U_{t+T} \prod_{s=0}^{T-1} n_{t+s} = 0
$$

Ramsey-like objective:

$$
U_t = \sum_{s=0}^{\infty} (\alpha n)^s \, u\left(c_{t+s}, d_{t+s+1}\right)
$$

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- Equivalence of steady state and (local) dynamics ?
- How to pass from CT to DT?
- Is CT a pertinent representation for economic transaction?
- Is an economic variable in CT backward or forward?

Forward and backward variables: an example

- Taylor rules:

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\begin{aligned}
i &= \rho(\pi) \\
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- A similar formula holds forward.

Passing from CT to DT

- Delay equations.

Passing from CT to DT

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- Discretizations.

Difference-differential equations

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  - $i(z)$ represents investment at time z (vintage z).
  - Machines depreciate suddenly after $T > 0$ units of time.
  - The depreciation rate depends on delayed investment, which shows the vintage capital nature of the model.

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- How to discretize continuous-time models?
- Euler-Taylor discretizations.
- But equivalent models can behave differently.
- Other representations (other bases in the function space).

Our paper

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Our paper

- Focus on Euler.
- Focus on (local) bifurcations.
- Does the discretization (approximation) degree play a role in (local) dynamics?
- To plot a CT phase diagram, one usually uses Euler discretization.
- But, use this method to compare bifurcations.

Continuous time

- System:

$$\dot{x} = f(x)$$

Continuous time

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- Pick a regular sequence of time values:

$$\begin{aligned} (t_n)_{n=0}^{\infty} &= (nh)_{n=0}^{\infty} \\ x_n &\equiv x(t_n) \end{aligned}$$

where h is the discretization step.

Discretizations

- Reconstruct the path from x_n to x_{n+1} component by component

Discretizations

- Reconstruct the path from x_n to x_{n+1} component by component
- Focus on the ith component of the vector $x \in \mathbb{R}^m$ and integrate the time derivative on the right or on the left:

$$
\begin{aligned}
x_{in+1} - x_{in} &= \left.\int_{nh}^{nh+\sigma} \dot{x}_i \, dt\right|_{\sigma=h} = \left.\int_{nh}^{nh+\sigma} f_i\left(x\left(t\right)\right) dt\right|_{\sigma=h} \\
x_{in+1} - x_{in} &= \left.\int_{nh+\tau}^{nh+h} \dot{x}_i \, dt\right|_{\tau=0} = \left.\int_{nh+\tau}^{nh+h} f_i\left(x\left(t\right)\right) dt\right|_{\tau=0}
\end{aligned}
$$

Discretizations (cont.)

- Define

$$
\begin{aligned}
x_{in+1} - x_{in} &= \varphi_i(\sigma) \equiv \int_{nh}^{nh+\sigma} f_i(x(t))\,dt \\
x_{in+1} - x_{in} &= \psi_i(\tau) \equiv \int_{nh+\tau}^{nh+h} f_i(x(t))\,dt
\end{aligned}
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Discretizations (cont.)

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- Approximate φ and ψ with a polynomial (Taylor).

$$x_{in+1} - x_{in} = \varphi_i(h) \approx \sum_{p=0}^{q} \frac{(h-0)^p}{p!} \varphi_i^{(p)}(0)$$
$$x_{in+1} - x_{in} = \psi_i(0) \approx \sum_{p=0}^{q} \frac{(0-h)^p}{p!} \psi_i^{(p)}(h)$$

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- The most popular approximation is the Euler discretization : $q = 1$.

Backward-looking discretization

- First-order discretization.

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- First-order discretization.
- RHS derivative.

$$\begin{aligned}
\frac{x(t+h)-x(t)}{h} &\approx \dot{x}(t) \\
\frac{x(t_n+h)-x(t_n)}{h} &\approx f(x(t_n)) \\
\frac{x_{n+1}-x_n}{h} &\approx f(x_n) \\
x_{n+1} &\approx x_n + hf(x_n)
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\end{aligned}
$$

- The smaller the discretization step h, the more accurate the approximation.

Forward-looking discretization

- LHS derivative.

$$\begin{aligned} \frac{x(t)-x(t-h)}{h} &\approx \dot{x}(t) \\ \frac{x(t_n)-x(t_n-h)}{h} &\approx f(x(t_n)) \\ \frac{x_n-x_{n-1}}{h} &\approx f(x_n) \\ x_{n+1} &\approx x_n+hf(x_{n+1}) \end{aligned}$$

- 1th order (Euler): $x_{n+1} \approx x_n + f(x_n)\, h$.

One-dimensional Taylor discretizations (backward-looking)

- 1th order (Euler): $x_{n+1} \approx x_n + f(x_n)\, h$.
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- 3rd order: $x_{n+1} \approx x_n + f(x_n)\, h + f(x_n)\, f'(x_n)\, h^2/2 + \left[f(x_n)\, f'(x_n)^2 + f(x_n)^2\, f''(x_n)\right] h^3/6$.

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Hybrid discretization

- A hybrid discretization is backward-looking for some components of vector x and forward-looking for other components.

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  - 3rd: $x = x + hf(x) + \frac{h^2}{2} f(x) f'(x) + \frac{h^3}{6}\left[f(x) f'(x)^2 + f(x)^2 f''(x)\right] \Rightarrow f(x) = 0$.

Topological equivalence: stability issue

Theorem

(backward-looking discretization) Consider $h > 0$*.*
(1) Let the steady state be a sink in continuous time (figure 1).
(1.1) If $T_0^2 < 4D_0$*, then the steady state is a sink in discrete time if* $h < h_{H1}$ *and a source if* $h_{H1} < h$*.*
(1.2) If $T_0^2 > 4D_0$*, then the steady state is a sink if* $0 < h < h_{F1}$*, a saddle if* $h_{F1} < h < h_{F2}$ *and source if* $h_{F2} < h$*.*
(2) If the steady state is a saddle in continuous time, then the steady state is a saddle in discrete time if $0 < h < h_{F2}$ *and source if* $h_{F2} < h$ *(figure 2).*
(3) If the steady state is a source in continuous time, then the source property is preserved whatever $h > 0$ *(figure 3).The system generically undergoes a Hopf bifurcation at* h_{H1} *and flip bifurcations at* h_{Fi}*,* $i = 1, 2$*.*

Topological equivalence: stability issue

- Backward-looking discretization:

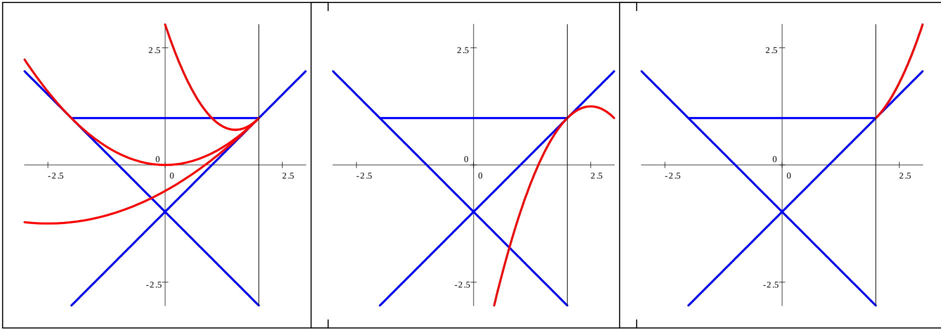

Fig. 1: Sink in CT   Fig. 2: Saddle in CT   Fig. 3: Source in CT

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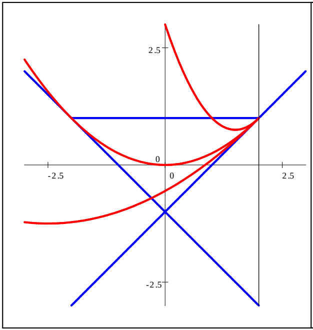

Fig. 1: Sink in CT

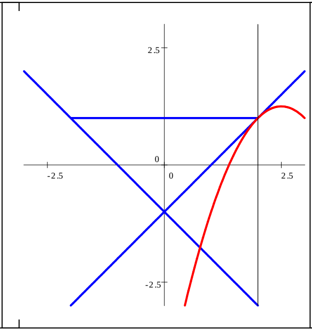

Fig. 2: Saddle in CT

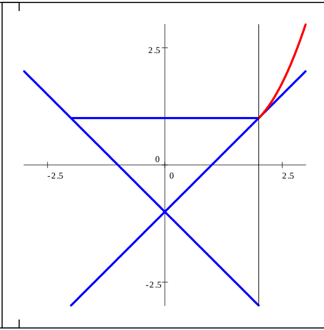

Fig. 3: Source in CT

- Similar results hold in the case of a forward-looking or hybrid discretization.

Topological equivalence: elementary bifurcations

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- CT: the eigenvalue real part crosses zero: $\mu(p^*) = \alpha(p^*) \pm i\beta(p^*)$ with $\alpha(p^*) = 0$.

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    - Saddle-node bifurcation when $\beta(p^*) = 0$.

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  - No room for flip.

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  - Hopf bifurcation when $\beta(p^*) \neq 0$.
  - No room for flip.
- One or two-dimensional analysis (Center Manifold Theorem).

Two-dimensional systems

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- Continuous time:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

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- Euler-Taylor discretization:

$$\begin{aligned} x_{1n+1} &\approx x_{1n} + hf_1(x_{1n}, x_{2n}) \equiv g_1(x_{1n}, x_{2n}) \\ x_{2n+1} &\approx x_{2n} + hf_2(x_{1n}, x_{2n}) \equiv g_2(x_{1n}, x_{2n}) \end{aligned}$$

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- Jacobian matrices.

$$J_0 \equiv \begin{bmatrix} \frac{\partial f_1}{\partial x_{1n}} & \frac{\partial f_1}{\partial x_{2n}} \\ \frac{\partial f_2}{\partial x_{1n}} & \frac{\partial f_2}{\partial x_{2n}} \end{bmatrix} \text{ and } J_1 \equiv \begin{bmatrix} \frac{\partial g_1}{\partial x_{1n}} & \frac{\partial g_1}{\partial x_{2n}} \\ \frac{\partial g_2}{\partial x_{1n}} & \frac{\partial g_2}{\partial x_{2n}} \end{bmatrix}$$

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- Linearity:

$$J_1 = I + hJ_0$$

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- Linearity:

$$J_1 = I + hJ_0$$

- I, identity, J_0 does not depend on h, J_1 depends linearly on h.

Local analysis

- Trace and determinant of $J_0 : (T_0, D_0)$.

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- CT and Euler-Taylor characteristic polynomials

$$
\begin{aligned}
P_0(\lambda) &\equiv \lambda^2 - T_0\lambda + D_0 \\
P_1(\lambda) &\equiv \lambda^2 - T_1\lambda + D_1
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- Links:

$$\begin{aligned} T_1 &= 2 + hT_0 \\ D_1 &= 1 + h(T_0 + hD_0) = T_1 - 1 + h^2 D_0 \end{aligned}$$

Saddle-node equivalence

Theorem

A saddle-node bifurcation generically occurs in CT if and only if it arises under a Euler-Taylor discretization, whatever the discretization step h.

- Sketch of the proof: $D_0 = 0 \Leftrightarrow D_1 = T_1 - 1$ whatever h.

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Theorem

A saddle-node bifurcation generically occurs in CT if and only if it arises under a Euler-Taylor discretization, whatever the discretization step h.

- Sketch of the proof: $D_0 = 0 \Leftrightarrow D_1 = T_1 - 1$ whatever h.
- Even a rough discretization (large h) preserves the bifurcation at the same parameter value.

Hopf equivalence

- Conditions for Hopf bifurcation in ET "tend" to conditions in CT as the distance h between the systems vanishes.

Theorem

Assume $f(x,p) \in C^2$ and $\lim_{h \to 0^+} \left|\det J_0 / H_p\right| < \infty$, where the ratio is evaluated in the steady state corresponding to a Hopf bifurcation value p_H in discrete time (under a positive discretization degree: $h > 0$). A Hopf bifurcation in continuous time generically requires $T_0 = 0$ and $D_0 \geq 0$. In discrete time, a Hopf bifurcation needs $T_0 = -hD_0$ and $D_0 \geq T_0^2/4$: these conditions "converge" to the corresponding conditions in continuous time (i.e. the right-hand sides $-hD_0$ and $T_0^2/4$ go to zero) as h goes to zero.

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- Saddle-node persists at the same parameter value.

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- Hopf persists in a neighborhood of the parameter value.
- Flip disappears in CT.

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  - Unbounded support.

Flip bifurcation

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  - Dimension one: $1 + hf_x(x(p), p) = -1$.
  - Dimension two: $D_1 = -T_1 - 1$ or $D_0 h^2 + 2T_0 h + 4 = 0$.
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Flip bifurcation

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  - Dimension one: $1 + hf_x(x(p), p) = -1$.
  - Dimension two: $D_1 = -T_1 - 1$ or $D_0 h^2 + 2T_0 h + 4 = 0$.
- Order two.
  - Dimension one:

$$-2 = hf_x(x(p), p) + \frac{h^2}{2}\left[f_x(x(p), p)^2 + f(x(p), p) f_{xx}(x(p), p)\right]$$

Discretization threshold (order one).

Theorem

Let $f \in C^1$ *and* $Y \equiv f_x(X(P))$, *where* $X(P)$ *is the graph of the steady states correspondence. If* $-\infty < \inf Y$, *there exists a nonempty discretization range* $(0, h^*)$ *with* $h^* \equiv |-2/\inf Y|$, *where no flip bifurcation arises.*

Corollary

Let $f \in C^1$ *and* $Z \equiv f_x(S \times P)$, *where* S *is the domain of* x. *If* $-\infty < \inf Z$, *then there exists a discretization range* $(0, h^*)$ *with* $h^* \equiv |-2/\inf Z|$ *with no flip bifurcation.*

- Lower boundedness required.

Discretization threshold (order two)

Theorem

Let $f \in C^3$ on $S \times P$ and f, f_x, f_{xx} be bounded over $X(P)$. Then there exists a nonempty discretization range $(0, h^*)$, where generically no flip bifurcation arises.

Theorem

If P is compact and, for every $(x, p) \in S \times P$, $f \in C^2$ and f_x is nonzero, then there are no flip bifurcations in the interval $(0, h^*)$.

- Hint. Compactness + IFT.

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Theorem

If P is compact and, for every $(x, p) \in S \times P$, $f \in C^2$ and f_x is nonzero, then there are no flip bifurcations in the interval $(0, h^*)$.

- Hint. Compactness + IFT.
- Order three: same lines.

Solow models

- Continuous time (original system):

$$\begin{aligned} \dot{K}_t &= sF(K_t, L_t) - \delta K_t \\ \dot{L}_t &= n_0 L_t \end{aligned}$$

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- Discrete time:

$$\begin{aligned} K_{t+1} - K_t &= sF(K_t, L_t) - \delta K_t \\ L_{t+1} - L_t &= n_1 L_t \end{aligned}$$

Solow: intensive laws

Solow	intensive law
continuous time	$\dot{k}_t = sf(k_t) - (\delta + n_0) k_t$
discrete time	$\boxed{k_{t+1} = \frac{1-\delta}{1+n_1} k_t + \frac{s}{1+n_1} f(k_t)}$
lin. discr. OS	$\boxed{k_{n+1} \approx \frac{1-h\delta}{1+hn_0} k_n + \frac{hs}{1+hn_0} f(k_n)}$
lin. discr. RF	$k_{n+1} \approx k_n + h[sf(k_n) - (\delta + n_0) k_n]$
quadr. discr. OS	$k_{n+1} \approx \frac{k_n + \left(h + \frac{h^2}{2}[sf'(k_n) - \delta]\right)[sf(k_n) - \delta k_n] + \frac{h^2}{2} sn_0 [f(k_n) - k_n f'(k_n)}{1 + hn_0 + \frac{(hn_0)^2}{2}}$
quadr. discr. RF	$k_{n+1} \approx k_n + [sf(k_n) - (\delta + n_0) k_n] \left(h + \frac{h^2}{2} [sf'(k_n) - (\right.$

Steady state in Solow models

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- is always given by

$$\frac{f(k)}{k} = \frac{n+\delta}{s}$$

(with $n_0 = n_1 \equiv n$).

Local bifurcations

Solow	bifurcations
continuous time	no
discrete time	$\boxed{\text{no}}$
linear discretization original system	$\boxed{h_F = \frac{2}{(1-a)(\delta+n_0)}}$
linear discretization intensive law	$h_F = \frac{2}{(1-a)(\delta+n_0)}$
quadratic discretization original system	$\boxed{\text{no flip}}$
quadratic discretization intensive law	no flip

Solow with negative externalities

- Richer dynamics (flip).

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$$\begin{aligned}\dot{K}_t &= sAK_t^a L_t^{1-\alpha}\left[m-(K_t/L_t)^{1-a}\right]-\delta K_t\\ \dot{L}_t &= n_0 L_t\end{aligned}$$

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  - Discrete time:

$$\begin{aligned} K_{t+1} &= K_t + sAK_t^a L_t^{1-\alpha} \left[m - (K_t / L_t)^{1-a} \right] - \delta K_t \\ L_{t+1} &= (1 + n_1) L_t \end{aligned}$$

Laws of motion

Solow + extern.	intensive law
continuous time	$\dot{k}_t \equiv sAk_t^a \left(m - k_t^{1-a}\right) - \left(\delta + n_0\right) k_t$
discrete time	$\boxed{k_{t+1} = \frac{1-(\delta+sA)}{1+n_1} k_t + \frac{sA}{1+n_1} mk_t^a}$
lin. discr. OS	$\boxed{k_{n+1} \approx \frac{1-h(\delta+sA)}{1+hn_0} k_n + \frac{hsA}{1+hn_0} mk_n^a}$
lin. discr. RF	$k_{n+1} \approx k_n + h\left[sAk_n^a\left(m - k_n^{1-a}\right) - \left(n_0 + \delta\right) k_n\right]$
quadr. discr. OS	$k_{n+1} = \frac{k_n + \left(h + \frac{h^2}{2}\left[asAk_n^{a-1}m - (\delta+sA)\right]\right)\left[sAk_n^a m - (\delta+sA)k_n\right] + \frac{h^2}{2} n_0}{1 + hn_0 + \frac{(hn_0)^2}{2}}$
quadr. discr. RF	$k_{n+1} \approx k_n + \left[msAk_n^a - \left(\delta + n_0 + sA\right) k_n\right] \left(h + \frac{h^2 \left[amsAk_n^{a} \right.}{} \right.$

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$$k_1^* = \left(\frac{sA}{n + \delta + sA} m \right)^{\frac{1}{1-a}}$$

Local bifurcations in Solow models with negative externalities

Solow + externalities	bifurcations
continuous time	no
discrete time	$\boxed{A_F \equiv \frac{1}{s}\left[\frac{2}{1-a}(1+n_1)-(n_1+\delta)\right]}$
linear discretization OS	$\boxed{A_F \equiv \frac{1}{s}\left[\frac{2}{1-a}\frac{1+hn_0}{h}-(n_0+\delta)\right]}$
linear discretization RF	$A_F = \frac{1}{s}\boxed{\frac{2}{(1-a)h}-n_0-\delta}$
quadratic discretization OS	$\boxed{\text{no flip}}$
quadratic discretization RF	no flip

Cass-Koopmans

- Continuous time: $\int_0^\infty \beta_t u\left(c_t\right) dt$.

Cass-Koopmans

- Continuous time: $\int_0^\infty \beta_t u(c_t)\, dt$.
- Discrete time: $\sum_{t=0}^\infty \beta_t u(c_t)$.

CK in continuous time

- System with multiplier:

$$\begin{aligned} \dot{k}_t &= f(k_t) - \delta k_t - c_t(\lambda_t) \\ \dot{\lambda}_t &= -\lambda_t \left[f'(k_t) - \delta \right] \end{aligned}$$

where $\lambda_t = \beta_t u'(c_t)$.

CK in discrete time

- System with consumption:

$$
\begin{aligned}
k_{t+1} - k_t &= f\left(k_t\right) - \delta k_t - c_t \\
\frac{u'\left(c_t\right)}{u'\left(c_{t+1}\right)} &= \frac{\beta_{t+1}}{\beta_t}\left[1 + f'\left(k_{t+1}\right) - \delta\right]
\end{aligned}
$$

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- The DT system is recovered under a hybrid discretization.

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- $h = 1 \Rightarrow$ discrete-time form.

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- $h = 1 \Rightarrow$ discrete-time form.
- The added value is Euler.

Discretizations of Euler equations

- Equivalent equations:

$$\begin{aligned}\dot{\lambda}_t &= -\lambda_t \left[f'(k_t) - \delta\right] \\ \dot{\mu}_t &= \mu_t \left[\rho_t + \delta - f'(k_t)\right]\end{aligned}$$

where $\mu_t = u'(c_t)$.

The equivalence issue

- Euler equations:

Euler	looking	discretization
λ-type	F	$\boxed{\dfrac{\mu_t}{\mu_{t+h}} \approx \dfrac{\beta_{t+h}}{\beta_t} \left(1 + h\left[f'\left(k_{t+h}\right) - \delta\right]\right)}$
	B	$\dfrac{\mu_t}{\mu_{t+h}} \approx \dfrac{\beta_{t+h}}{\beta_t} \dfrac{1}{1-h[f'(k_t)-\delta]}$
μ-type	F	$\dfrac{\mu_t}{\mu_{t+h}} \approx 1 + h\left[f'\left(k_{t+h}\right) - \delta\right] - \dfrac{\beta_t - \beta_{t+h}}{\beta_{t+h}}$
	B	$\dfrac{\mu_t}{\mu_{t+h}} \approx \dfrac{1}{1-h[f'(k_t)-\delta]+\frac{\beta_t-\beta_{t+h}}{\beta_t}}$

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μ-type	F	$\frac{\mu_t}{\mu_{t+h}} \approx 1 + h\left[f'\left(k_{t+h}\right) - \delta\right] - \frac{\beta_t - \beta_{t+h}}{\beta_{t+h}}$
	B	$\frac{\mu_t}{\mu_{t+h}} \approx \frac{1}{1-h[f'(k_t)-\delta]+\frac{\beta_t-\beta_{t+h}}{\beta_t}}$

- If $h = 1$: λ-type + forward-looking Euler-Taylor = discrete time.

- For simplicity, hybrid.

BL resource constraint + FL λ-type Euler.

$$
\begin{aligned}
k_{t+h} - k_t &\approx h\left[f\left(k_t\right) - \delta k_t - c_t\right] + h^2 \frac{\left[f'\left(k_t\right) - \delta\right]\left(f\left(k_t\right) - \delta k_t - c_t\left[1 + \varepsilon\left(c_t\right)\right]\right)}{2} \\
\frac{\mu_t}{\mu_{t+h}} &\approx \frac{\beta_{t+h}}{\beta_t}\left(1 + h\left[f'\left(k_{t+h}\right) - \delta\right] + \frac{h^2\left(\left[f'\left(k_{t+h}\right) - \delta\right]^2 - \left[f\left(k_{t+h}\right) - \delta k_{t+h} - c_{t}\right.\right.}{2}\right.
\end{aligned}
$$

Second-order discretization

- Could we think a DT second-order Cass-Koopmans model?

- Could we think a DT second-order Cass-Koopmans model?
- Set $h = 1$:

$$
\begin{aligned}
k_{t+1} - k_t &\approx f(k_t) - \delta k_t - c_t + \frac{[f'(k_t) - \delta](f(k_t) - \delta k_t - c_t[1 + \varepsilon(c_t)])}{2} \\
\frac{\mu_t}{\mu_{t+1}} &\approx \frac{\beta_{t+1}}{\beta_t}\left(1 + f'(k_{t+1}) - \delta + \frac{[f'(k_{t+1}) - \delta]^2 - [f(k_{t+1}) - \delta k_{t+1} - c_{t+1}]}{2}\right.
\end{aligned}
$$

Steady state

- MGR: $f'(k) = \rho + \delta$ under constant $\rho_t = -\ln \beta$.

Local dynamics

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Local dynamics

- At the SS.
  - μ_t stationary,
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- Focus on μ-type CT:

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- Utility functional

$$\int_0^\infty \left[u\left(c_t\right) - u\left(c\right)\right] dt$$

subject to the resource constraint

$$\dot{k}_t + \delta k_t + c_t \leq f\left(k_t\right)$$

given the initial endowment k_0.

CT Ramsey (1928) (cont.)

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$$\dot{k}_t = f(k_t) - \delta k_t - c(\mu_t) \quad (1)$$

$$\dot{\mu}_t = -\mu_t \left[f'(k_t) - \delta \right] \quad (2)$$

$+$ TC.

- The planner maximizes the intertemporal utility

$$\sum_{t=0}^{\infty} \left[u\left(c_t\right) - u\left(c\right) \right]$$

subject to the sequence of resource constraints:

$$k_{t+1} - k_t + \delta k_t + c_t \leq f\left(k_t\right)$$

where c is the bliss point defined above.

DT Ramsey (1928) (cont.)

- The optimal dynamical system is:

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c(\mu_t) \tag{3}$$

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- The discrete-time Ramsey model comes from a first-order hybrid Euler discretization of the continuous-time model, that is a backward-looking discretization of the resource constraint (1) and a forward-looking discretization of the Euler equation (2).

DT Ramsey (1928) (cont.)

- Backward-looking linear discretization of the continuous-time resource constraint (1):

$$k_{t+h} - k_t \approx h\left[f\left(k_t\right) - \delta k_t - c\left(\mu_t\right)\right]$$

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- But the intertemporal arbitrage can not be recovered in backward-looking. Focus on (2) and apply instead the forward-looking discretization (**??**): $\mu_{t+h} - \mu_t = -h\mu_{t+h}\left[f'\left(k_{t+h}\right) - \delta\right]$. We obtain

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- The forward-looking discretization of (2) is more adapted to capture the saving decision, the expected productivity (interest rate) matters in the current trade-off between consumption and saving.

DT Ramsey (1928) (cont.)

- For all the three dynamical systems (CT, DT, HD) the steady state is defined by:

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f'(k) &= \delta && (6) \\
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- The associated Jacobian matrix J_1 of the hybrid Euler discretization

$$J_1 \equiv \begin{bmatrix} 1 & 0 \\ h\mu f''(k) & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -h/u''(c) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & hAk/\mu \\ hB\mu/k & 1+ABh^2 \end{bmatrix}$$

with A and B defined as above

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- These examples illustrate the equivalence theorems.