Risk Measures and Optimal Risk Transfers

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Motivations

- Study of optimal risk transfer structures, **Natural question in Reinsurance**.
- Pricing of one example of these transfer contracts: Non proportional layer with reinstatements.
- Necessity of a **time** updating of the risk measures, with the arrival of new **Information**. Can be done through the particular case of BSDEs with jumps.

Outline

Static framework

- Risk Measures and Inf-convolution
- An application in Non proportional Reinsurance

Time dynamic framework

- An example using Quadratic BSDEs with Jumps

  Joint work with Dylan Possamai and Chao Zhou

Monetary risk measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space.

Key properties of a mapping $\rho : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$:

- If $X \geq Y$ $\mathbb{P}$-a.s. then $\rho(X) \geq \rho(Y)$. (Losses orientation)
- $\rho(X + m) = \rho(X) + m$, $m \in \mathbb{R}$. (Cash additivity property: Capital requirement)
- ρ is convex. (Diversification)

If X cannot be used as a hedge for Y (X and Y comonotone variables), then no possible diversification (comonotonic risk measures): $\rho(X + Y) = \rho(X) + \rho(Y)$.

Examples

- The *Average Value-at-Risk* at level $\alpha \in (0,1]$ is a coherent risk measure given by:

$$AVaR_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \bar{q}_X(u)du$$

  where $\bar{q}_X(u) := \inf\{x \in \mathbb{R} | \mathbb{P}(X > x) \leq u\},\ u \in (0,1)$.

- The *entropic risk measure* defined by:

$$e(X) = \frac{1}{\gamma} \ln \mathbb{E}_{\mathbb{P}}[\exp(\gamma X)],\ \gamma > 0.$$

  is a convex monetary risk measure.

- These are two examples of **law invariant risk measures**.

Monetary risk measures

Growing need of regulation professionals and VaR drawbacks conducted to an axiomatic analysis of required solvency capital.

- Artzner, Delbaen, Eber, and Heath (1999) (**Coherent case**)
- Frittelli, M. and Rosazza Gianin, E. (2002) (**Convex case**)
- Föllmer, H. and Schied, A. (2004) (**Monography**)
- Bion-Nadal, (2008-2009); Bion-Nadal and Kervarec (2010), Cheridito, Delbaen, and Kupper (2004) (**Dynamic case**)
- Acciaio (2007, 2009), Barrieu and El Karoui (2008), Jouini, Schachermayer and Touzi (2006,2008), Kervarec (2008) (**Inf-convolution**)

Many other references...

Robust representation of convex risk measures

Any convex risk measure ρ on $L^\infty(\mathbb{P})$, which is continuous from above, has the representation:

$$\rho(X) = \sup_{Q \in \mathcal{M}_1(\mathbb{P})} \{E_Q(X) - \alpha(Q)\},$$

where $\mathcal{M}_1(\mathbb{P})$ = set of $\mathbb{P}$-absolutely continuous probability measures on $\mathcal{F}$.

Key property: Comonotonicity

Denneberg (1994)

X and Y are comonotone if there exists a random variable Z such that X and Y can be written as nondecreasing functions of Z.

Examples (**typical reinsurance contracts**): $(\alpha X, (1-\alpha)X)$, $\alpha \in (0,1)$ or $(X \wedge k, (X-k)^+)$, $k \in \mathbb{R}$ are comonotone.

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Let $c : \mathcal{F} \to [0,1]$ be a normalized and monotone set function. The **Choquet integral** of a random variable X, with respect to c is defined by:

$$\int X d\, c := \int_{-\infty}^{0} \left(c(X > x) - 1\right) dx + \int_0^{\infty} c(X > x) dx$$

It is a comonotonic monetary risk measure.

Why using Choquet Integrals ?

- Greco (1977), Denneberg (1994), Föllmer and Schied (2004): A monetary risk measure defined on $L^\infty(\mathbb{P})$ is comonotone if and only if it is a Choquet integral.
- Many risk measures used in insurance: AVaR, Wang transform, PH-transform are examples of Choquet integrals.

Goal: Optimal risk transfer between agents using Choquet integrals as risk measures.

Focus on Choquet Integrals

G. Choquet, Theory of capacities, 1955:

- $\int X d\, c$ is convex iif c is submodular $\left[c(A \cup B) + c(A \cap B) \leq c(A) + c(B),\ \forall\ A,\ B \in \mathcal{F}\right]$, provided the probability space is atomless.
- c is called decreasing on $\mathcal{F}$ if for every decreasing sequence (A_n) of elements of $\mathcal{F}$, we have $c\left(\bigcap_n A_n\right) = \lim c(A_n)$. In that case $\int X d\, c$ is continuous from above.

Distortion Functions

- A non decreasing function $\psi : [0,1] \to [0,1]$ with $\psi(0) = 0$ and $\psi(1) = 1$ is called a *distortion function* (Rem: We do not need ψ to be càg or càd).
- We define a capacity c_ψ by

$$c_\psi(A) = \psi(\mathbb{P}(A)), \ \forall A \in \mathcal{F}.$$

- For $\psi(x) = x$, the Choquet integral $\int X d\, c_\psi$ is the expectation of X under the probability measure $\mathbb{P}$. The function ψ is used to distort the expectation operator $\mathbb{E}_\mathbb{P}$ into the non-linear functional ρ_ψ.
- The Choquet integral $\int X d\, c$ is **law invariant** under $\mathbb{P}$ if and only if c is a $\mathbb{P}$-**distortion** (Föllmer and Schied, 2004).

Inf-convolution

- Barrieu and El Karoui (2008): An agent minimizes his risk, **under the constraint that a transaction with the second agent takes place**. The cash-invariance property implies that the problem is equivalent to the inf-convolution of the agents risk measures.
- ρ_1 and ρ_2 risk measures. Inf-convolution defined by:

$$\rho_1 \Box \rho_2(X) := \inf_{F \in \mathcal{X}} \{\rho_1(X - F) + \rho_2(F)\}.$$

Inf-convolution of Choquet integrals

Theorem (K. 2012)

Let ρ_1 and ρ_2 be two Choquet integrals with respect to continuous set functions c_1 and c_2 verifying $\rho_1 \Box \rho_2(0) > -\infty$ and let X be a r.v. with no atoms.
We assume furthermore that the two agents "do not disagree too often". Then

$$\rho_1 \Box \rho_2(X) = \rho_1\left(X - Y^*\right) + \rho_2(Y^*)$$

where Y^* is given by:

$$Y^* = \sum_{p=0}^{N} (X - k_{2p})^+ - (X - k_{2p+1})^+,$$

where $\{k_n,\ n \leq N\}$ is a sequence of real numbers corresponding to quantile values of X.

Inf-convolution of Choquet integrals

- Similar result in the law invariant case proven by E. Jouini, W. Schachermayer and N. Touzi (2008), Optimal risk sharing for law invariant monetary utility functions.
- Means that the inf-convolution of comonotonic risk measures is given by a generalization of the Excess-of-Loss contract, with more treshold values. The domain of attainable losses is divided in "ranges", and each range is alternatively at the charge of one of the two agents.

Pricing Reinsurance Layers with Reinstatements

Once we have these non proportional contracts (layers), what are the possible pricing techniques ?

In particular, in the case of contracts with reinstatements.

Pricing Reinsurance Layers with Reinstatements

Motivations:

- Pricing in reinsurance, taking into account the cost of capital. Key issue within **Solvency II regulation framework**.
- Indifference pricing in this context: based on both a concave utility function and a convex risk measure.
- The pricing is possibly not satisfying, due to the presence of reinstatements.
- Goal: give easily computable bounds for the indifference price.
- Sundt (1991), Mata (2000), Wahlin and Paris (2001): **Pricing principles**.

  Albrecher and Haas (2011): **Ruin theory**.

The contract payoff

- Consider an XL reinsurance contract with retention l and limit m.
- Reinsurer's part: $Z_i = (X_i - l)^+ - (X_i - l - m)^+$.
- Total loss $Z = \sum_{i=1}^N Z_i$, $N =$ number of claims.
- Aggregate deductible L and limit M. In practice, M is expressed as a multiple of m, $M = (k+1)m$, we say the contract contains k reinstatements.
- Payoff: $\min\{(Z-L)^+, (k+1)m\}$.

Intuition: The insurance company can **reconstitute the layer** a limited number of times, by paying a price proportional to the initial price. So the total paid premium is **unknown**.

Indifference price

We say that p_0 is the indifference price of a given XL layer relatively to the pair (U, ρ), if p_0 solves the equation

$$U\left(R - \overline{c}\,\rho(R)\right) = U\left(R^{XL} - \overline{c}\,\rho(R^{XL})\right)$$

where $\overline{c}$ is a given cost of capital

and $R^{XL} := R + F - p_0(1 + \tilde{N})$.

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and $R^{XL} := R + F - p_0(1 + \tilde{N})$.

- Sundt (1991), Walhin and Paris (2001) gave conditions under which we can solve numerically the equation for different criteria.

Pricing bounds

Proposition (K., 2012)

If P_0 is the indifference price of a given XL layer relatively to the pair (U, ρ), then $p_1 \leq P_0 \leq p_2$,

$$\text{where } \; p_1 := \frac{A}{-\bar{U}(-1-\tilde{N})}, \quad p_2 := \frac{A}{\bar{U}(1+\tilde{N})}$$
$$\text{and } \quad A := \bar{U}\,(R+F) - \bar{U}(R).$$

$\tilde{N}$: Fraction of used reinstatements.
$\bar{U}(X) := U(X - \overline{c}\,\rho(X))$, correspond to a **cash-subadditive** utility.

Example

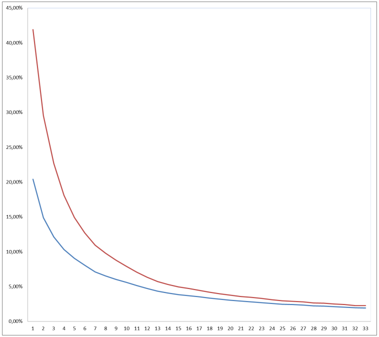

Figure : Semi-deviation utility function and $AVaR_\alpha$ risk measure

$k = 4$ possible reinstatements, $c_i = 100\%$, $AVaR_\alpha$ with $\alpha = 1/200$, semi-deviation utility with $\delta = 1/2$.

Quadratic BSDEs with Jumps

- We will now consider a time dynamic framework for the risk analysis.
- Study the arrival of new information and its impact on optimal risk transfer structures.
- The BSDE framework is convenient to do so: Barrieu and El Karoui (2008), Coquet, Hu, Mémin and Peng (2002), Quenez and Sulem (2012), Royer (2006).
- The quadratic case with jumps allows to consider more examples of risk measures (entropic) in an insurance framework.

Quadratic BSDEs with Jumps

Filtration: generated by a Brownian motion B and a Poisson random measure μ with compensator ν. The solution of the BSDE is rewritten as a triple (Y, Z, U) such that

$$dY_t = g_s(Y_s, Z_s, U_s)ds - Z_s dB_s - \int_{\mathbb{R}^d \setminus \{0\}} U_s(x)\widetilde{\mu}(dx, ds),$$
$$Y_T = \xi.$$

- Barles, Buckdahn and Pardoux (1997).

$U_t : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is a function, but plays a role analogous to Z.

Quadratic BSDEs with Jumps

Define the following function

$$j_t(u) := \int_E \left(e^{u(x)} - 1 - u(x) \right) \nu(dx)$$

and consider the following BSDE for $t \in [0, T]$ and $\mathbb{P} - a.s.$

$$y_t = \xi + \int_t^T \left(\frac{\gamma}{2} |z_s|^2 + \frac{1}{\gamma} j_s(\gamma u_s) \right) ds - \int_t^T z_s dB_s - \int_t^T \int_E u_s(x) \widetilde{\mu}(dx, ds).$$

An application of Itô's formula gives

$$y_t = \frac{1}{\gamma} \ln \left(\mathbb{E}_t^{\mathbb{P}} \left[e^{\gamma \xi} \right] \right), \ \ t \in [0, T], \mathbb{P} - a.s.$$

We recover the entropic risk measure.

Applications

g-expectation

Let $\xi \in \mathbb{L}^\infty$ and let g be such that the BSDE (g, ξ) has a unique solution and such that comparison holds. Then for every $t \in [0, T]$, we define the conditional g-expectation of ξ as follows

$$\mathcal{E}_t^g[\xi] := Y_t,$$

$\mathcal{E}$, thus defined, is

- Monotone and Time consistent
- Convex if g is convex in (y, z, u).
- Constant additive if g does not depend on y.
- We can define naturally a notion of g-submartingale.

Inf-convolution of g-expectations

Example: we want to calculate the inf-convolution of the two corresponding generators g^1 and g^2 given by

$$g_t^1(z,u) := \frac{1}{2\gamma}|z|^2 + \gamma \int_E \left(e^{\frac{u(x)}{\gamma}} - 1 - \frac{u(x)}{\gamma} \right) \nu(dx),$$

and

$$g_t^2(z,u) := \alpha z + \beta \int_E (1 \wedge |x|) u(x) \nu(dx),$$

where $(\gamma, \alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R} \times [-1+\delta, +\infty)$ for some $\delta > 0$.

Correspond to the **entropic risk measure** for the first agent and a **linear risk measure** for the second one.

Inf-convolution of g-expectations

Lemma (Possamai, Zhou, K., 2012)

We have, for any bounded $\mathcal{F}_T$-measurable random variable ξ_T,

$$(\mathcal{E}^{g^1} \square \mathcal{E}^{g^2})(\xi_T) = \mathcal{E}^{g^1}(F_T^{(1)}) + \mathcal{E}^{g^2}(F_T^{(2)}),$$

$$F_T^{(2)} = \xi_T + \frac{1}{2}\alpha^2\gamma T + \gamma \int_0^T \int_E (\beta(1 \wedge |x|) - \ln(1 + \beta(1 \wedge |x|)))\nu(dx)dt$$

$$- \alpha\gamma B_T - \gamma \int_0^T \int_E \ln(1 + \beta(1 \wedge |x|))\widetilde{\mu}(dt, dx),$$

This provides an example of risk sharing which is **neither proportional nor a layer**.

Thank you for your attention