Transport equation on a network of circles with a persistently excited damping

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joint work with Yacine Chitour and Mario Sigalotti

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Geometric, stochastic and PDE control
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Outline

Introduction
PE systems in finite dimension

- We consider:
$$\dot{x} = Ax + \alpha(t)Bu.$$
- $x \in \mathbb{R}^d$: state; $u \in \mathbb{R}^m$: control; $\alpha \in \mathcal{G} \subset L^\infty(\mathbb{R}_+; [0, 1])$.
- Linear time-invariant control system: $\dot{x} = Ax + Bu$.
- $\alpha(t)$: activity of the control $u(t)$ at time t.
- If $\alpha(t) \in \{0, 1\}$: switched system between
$$\dot{x} = Ax \quad \text{and} \quad \dot{x} = Ax + Bu.$$

Introduction

PE systems in finite dimension

$$\dot{x} = Ax + \alpha(t)Bu, \qquad \alpha \in \mathcal{G}$$

- If $\alpha(t) \equiv 0$, there is no action of the control on the system.
- The class $\mathcal{G}$ should ensure a sufficient amount of action of the control on the system.
- Persistently exciting (PE) signals: for $T \geq \mu > 0$, we say that $\alpha \in \mathcal{G}(T, \mu)$ if $\alpha \in L^\infty(\mathbb{R}_+; [0, 1])$ and

$$\forall t \in \mathbb{R}_+, \quad \int_t^{t+T} \alpha(s) ds \geq \mu.$$

- Persistently excited (PE) system: system with $\alpha \in \mathcal{G}(T, \mu)$.

Introduction

PE systems in finite dimension

Theorem (A. Chaillet, Y. Chitour, A. Loría, M. Sigalotti, 2008)

Suppose that the pair (A, B) is controllable and that the matrix A is skew-symmetric. Then, for every $T \geq \mu > 0$, there exists constants $C \geq 1$, $\gamma > 0$ such that, for every $x_0 \in \mathbb{R}^d$ and every $\alpha \in \mathcal{G}(T, \mu)$, the corresponding solution of

$$\dot{x} = (A - \alpha(t) B B^{\mathsf{T}}) x$$

satisfies

$$\left\| x(t) \right\| \leq C e^{-\gamma t} \left\| x_0 \right\|.$$

- $u = -B^{\mathsf{T}} x$ is a feedback that stabilizes the system.
- More information on switched systems, PE systems and their stability: course by Yacine Chitour on Saturday and Sunday.

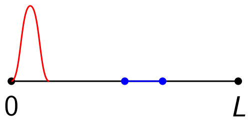
Introduction
PE systems in infinite dimension

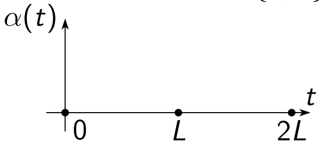
$$\dot{z} = Az + \alpha(t) Bu, \qquad z \in \mathsf{X},\ u \in \mathsf{U},\ \alpha \in \mathcal{G}(T, \mu).$$

X, U Banach spaces.

The previous theorem does not hold in this case.

$$\begin{cases} \partial_{tt}^2 u(t,x) = \partial_{xx}^2 u(t,x) - \alpha(t)\chi(x)\partial_t u(t,x), & x \in [0, L], \\ u(t,x) = 0, & x \in \{0, L\}. \end{cases}$$





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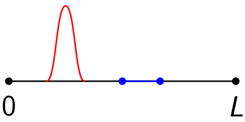

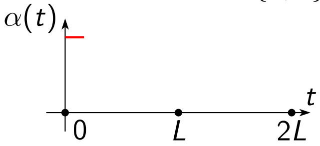

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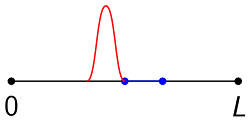

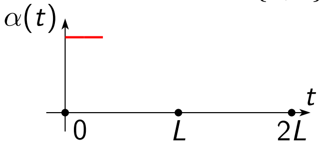

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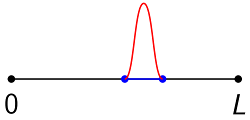

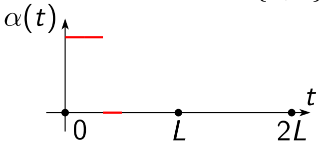

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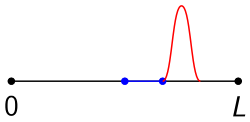

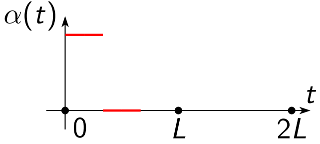

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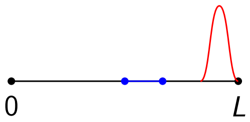

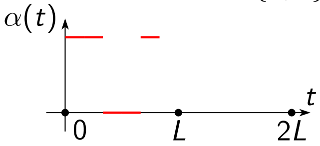

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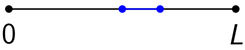
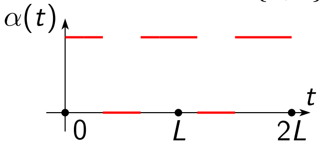




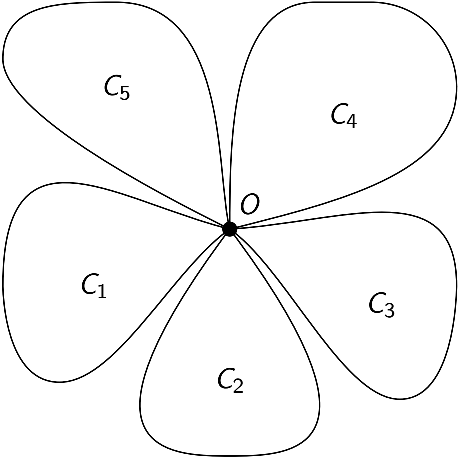
Introduction

PE systems in infinite dimension

- Few results are known concerning the stability and the stabilizability of PE systems in infinite dimension.
- [F. Hante, M. Sigalotti, M. Tucsnak, 2012]: generalized observability inequality and unique continuation principle for stability analysis.
- It would be useful to have a "toy model" to understand the effects of PE signals in infinite dimensional systems.

Transport equation on a network of circles

The model



Transport equation on a network of circles

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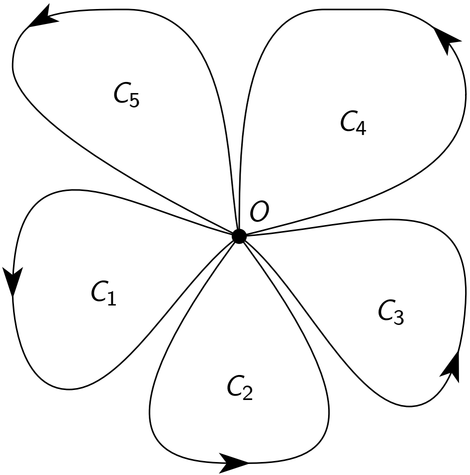

Transport equation on a network of circles
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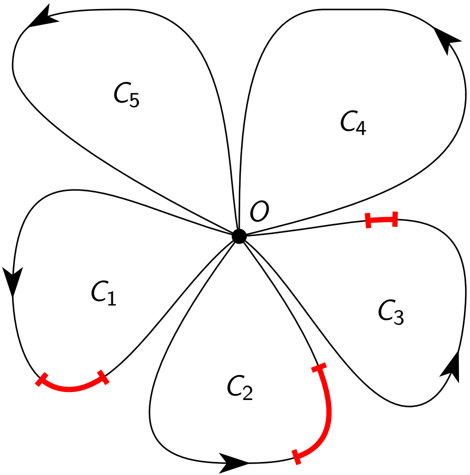


Transport equation on a network of circles

The model

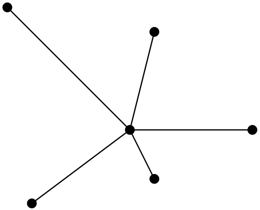
$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) \\ \quad + \alpha_i(t)\chi_i(x) u_i(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0, L_i],\ i \in [\![1, N_d]\!], \\ \partial_t u_i(t,x) + \partial_x u_i(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0, L_i],\ i \in [\![N_d + 1, N]\!], \\ u_i(t,0) = \displaystyle\sum_{j=1}^N m_{ij} u_j(t, L_j), & t \in \mathbb{R}_+,\ i \in [\![1, N]\!], \\ u_i(0,x) = u_{i,0}(x), & x \in [0, L_i],\ i \in [\![1, N]\!]. \end{cases}$$

- $\alpha_i \in \mathcal{G}(T, \mu)$ for $i \in [\![1, N_d]\!]$.
- χ_i: characteristic function of an interval $[a_i, b_i] \subset [0, L_i]$.
- $M = (m_{ij})_{1 \leq i,j \leq N}$: transmission matrix.

Transport equation on a network of circles

Motivation

- Understand the effects of PE signals in infinite dimensional systems.
- Inspired by the wave equation on a star-shaped network.
- PDEs on networks:
  [S. Nicaise, 1987],
  [G. Lumer, 1980],
  [R. Dáger, E. Zuazua, 2006],
  [J. Valein, E. Zuazua, 2009]...
- Stability with intermittent signals:
  [M. Gugat, M. Sigalotti, 2010].



Transport equation on a network of circles
Hypotheses

If $\frac{L_i}{L_j} \in \mathbb{Q}$ for every i, j and the damping intervals are small enough, one may find periodic solutions (depending on M).

Hypothesis

There exist $i, j \in [\![1, N]\!]$ such that $\frac{L_i}{L_j} \notin \mathbb{Q}$.

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The total mass $\sum_{i=1}^{N} \int_0^{L_i} u_i(t, x) dx$ is preserved and non-negative initial conditions imply non-negative solutions $\iff$ M is left stochastic.

Hypothesis

1. $\left|M\right|_{\ell^1} \leq 1$.
2. For every $i, j \in [\![1, N]\!]$, we have $m_{ij} \neq 0$.

Transport equation on a network of circles

Main result

Theorem (Y. Chitour, G. M., M. Sigalotti)

For every $T \geq \mu > 0$, there exist $C \geq 1$ and $\gamma > 0$ such that, for every $p \in [1, +\infty]$, every initial condition $u_{i,0} \in L^p(0, L_i)$, $i \in [\![1, N]\!]$, and every choice of signals $\alpha_i \in \mathcal{G}(T, \mu)$, $i \in [\![1, N_d]\!]$, the corresponding solution satisfies

$$\sum_{i=1}^{N} \left\|u_i(t)\right\|_{L^p(0,L_i)} \leq Ce^{-\gamma t} \sum_{i=1}^{N} \left\|u_{i,0}\right\|_{L^p(0,L_i)}, \qquad \forall t \geq 0.$$

Transport equation on a network of circles

Main result

- Our proof relies on the explicit formula for the solutions.
- The main difficulty comes from the fact that the α_i are PE and may be zero on several time intervals, switching off the damping.
- It is also important to take into account the fact that $\frac{L_i}{L_j} \notin \mathbb{Q}$ for certain i, j and combine it with the persistence of excitation of the α_i.

Transport equation on a network of circles
Explicit solution

We can give an explicit formula for the solutions of this system.
To simplify: $N = 2$, no damping, $L_1 < L_2$.

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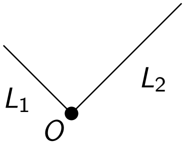

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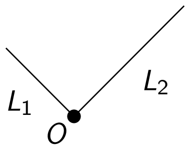

$$u_1(t, 0) = m_{11} u_1(t - s, L_1 - s) + m_{12} u_2(t - s, L_2 - s)$$

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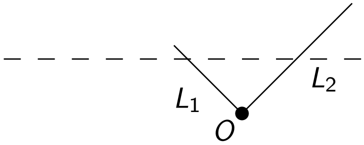

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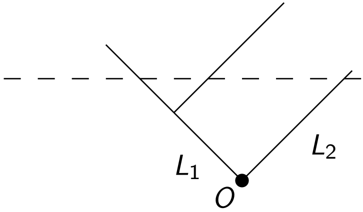

$$u_1(t,0) = m_{11}\left[m_{11}u_1(t-s, L_1-(s-L_1)) + m_{12}u_2(t-s, L_2-(s-L_1))\right]$$
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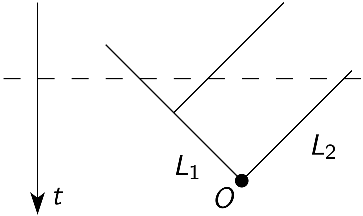

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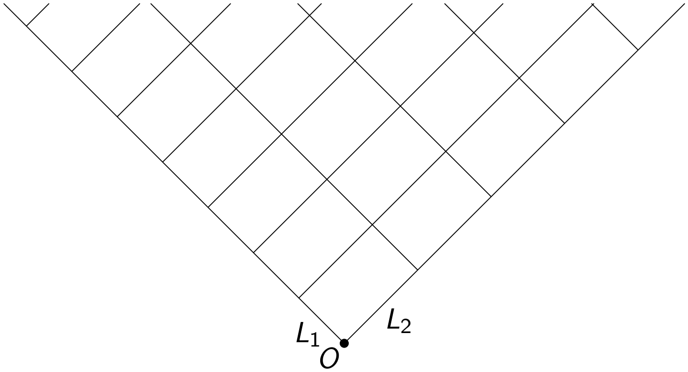

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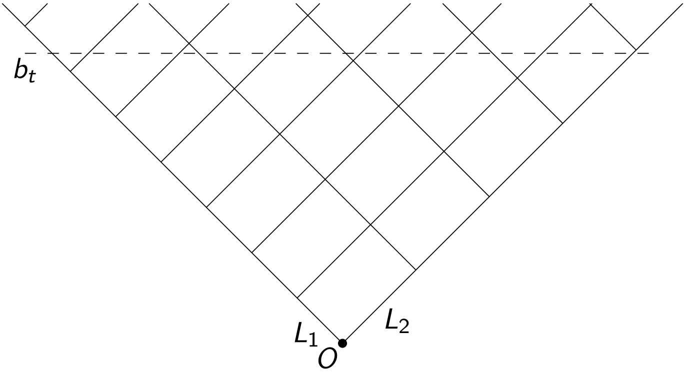

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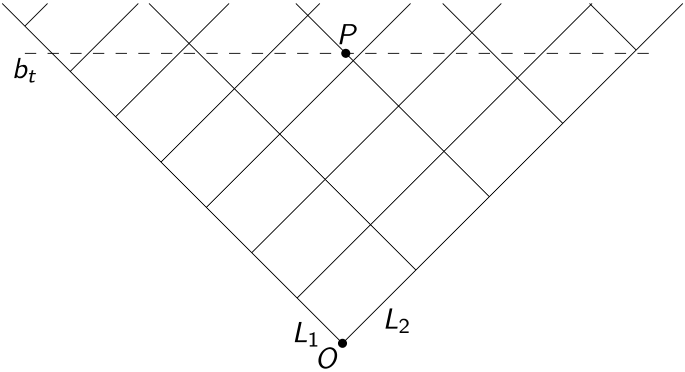


Transport equation on a network of circles

Explicit solution

$$u_1(t,0) = \sum_{n=0}^{\left\lfloor \frac{t}{L_2} \right\rfloor} \beta_{1,n,t} u_{1,0}(L_1 - \{t - nL_2\}_{L_1}) + \sum_{m=0}^{\left\lfloor \frac{t}{L_1} \right\rfloor} \beta_{2,m,t} u_{2,0}(L_2 - \{t - mL_1\}_{L_2})$$

- Notation: $\{x\}_y = x - \left\lfloor \frac{x}{y} \right\rfloor y$.
- $\beta_{j,n,t}$ can be computed from M.
- This formula can be generalized to larger N and to take the damping into account.

Transport equation on a network of circles

Explicit solution

Theorem

The solution satisfies

$$u_i(t,0) = \sum_{j=1}^{N} \sum_{\substack{\mathfrak{n} \in \mathfrak{N}_j \\ L(\mathfrak{n}) \leq t}} \vartheta^{(i)}_{j, \mathfrak{n} + \left\lfloor \frac{t - L(\mathfrak{n})}{L_j} \right\rfloor \mathbf{1}_j, L_j - \{t - L(\mathfrak{n})\}_{L_j}, t} u_{j,0}\left(L_j - \{t - L(\mathfrak{n})\}_{L_j}\right)$$

- Notations: $\mathfrak{N}_j = \mathbb{N}^{j-1} \times \{0\} \times \mathbb{N}^{N-j}$, $L(\mathfrak{n}) = n_1 L_1 + \cdots + n_N L_N$.
- $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ is defined for $i, j \in [\![1, N]\!]$, $\mathfrak{n} \in \mathbb{N}^N$, $x \in [0, L_j]$ and $t \geq L(\mathfrak{n})$.

Transport equation on a network of circles

Exponential convergence of the coefficients

Lemma

Let $T \geq \mu > 0$. If $\exists C_0 \geq 1,\ \gamma_0 > 0$ s.t.

$$\left|\vartheta_{j,\mathfrak{n},x,t}^{(i)}\right| \leq C_0 e^{-\gamma_0 \left|\mathfrak{n}\right|_{\ell^1}}$$

for every choice of PE signals $\alpha_k \in \mathcal{G}(T, \mu)$, then $\exists C \geq 1,\ \gamma > 0$ s.t., for every $p \in [1, +\infty]$, every solution satisfies

$$\sum_{i=1}^{N} \left\|u_i(t)\right\|_{L^p(0, L_i)} \leq C e^{-\gamma t} \sum_{i=1}^{N} \left\|u_{i,0}\right\|_{L^p(0, L_i)}, \qquad \forall t \geq 0.$$

It suffices to study the coefficients $\vartheta_{j,\mathfrak{n},x,t}^{(i)}$.

Transport equation on a network of circles

Exponential convergence of the coefficients

Theorem

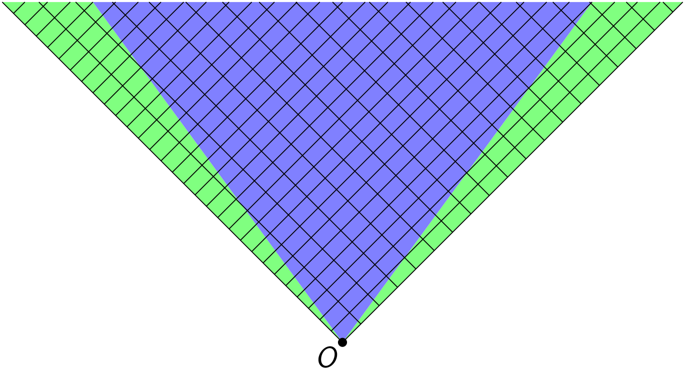
The coefficients $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ *satisfy*

$$\vartheta^{(i)}_{j,\mathbf{0},L_j,t} = m_{ij},$$

$$\vartheta^{(i)}_{j,\mathfrak{n},L_j,t} = \sum_{\substack{k=1\\ n_k \geq 1}}^{N} m_{kj} \; \vartheta^{(i)}_{k,\mathfrak{n}-\mathbf{1}_k,L_k,t} \; e^{-\int_{t-L(\mathfrak{n})+a_k}^{t-L(\mathfrak{n})+b_k} \alpha_k(s) ds}.$$

Transport equation on a network of circles
Exponential convergence of the coefficients

Decomposition of the set $\mathbb{N}^N$.



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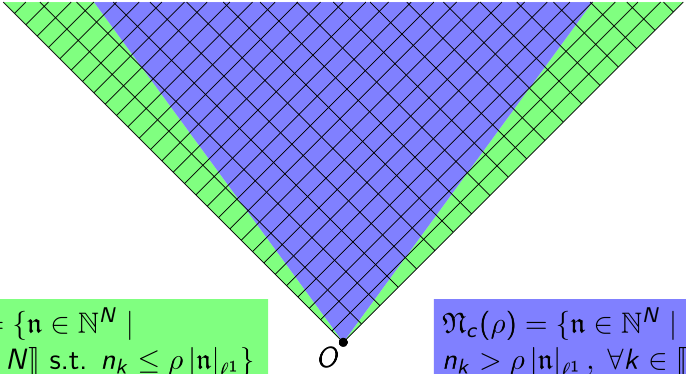


$\mathfrak{N}_b(\rho) = \{\mathfrak{n} \in \mathbb{N}^N \mid$
$\exists k \in [\![1, N]\!] \text{ s.t. } n_k \leq \rho \left|\mathfrak{n}\right|_{\ell^1}\}$

$\mathfrak{N}_c(\rho) = \{\mathfrak{n} \in \mathbb{N}^N \mid$
$n_k > \rho \left|\mathfrak{n}\right|_{\ell^1}, \ \forall k \in [\![1, N]\!]\}$

Transport equation on a network of circles

Exponential convergence of the coefficients

In $\mathfrak{N}_b(\rho)$:

Lemma

$\exists \mu \in (0, 1)$ s.t., $\forall i, j, \mathfrak{n}, x, t$ and $\forall k \in [\![1, N]\!]$, we have

$$\left|\vartheta_{j,\mathfrak{n},x,t}^{(i)}\right| \leq \binom{\left|\mathfrak{n}\right|_{\ell^1}}{n_k} \mu^{\left|\mathfrak{n}\right|_{\ell^1}}.$$

Transport equation on a network of circles

Exponential convergence of the coefficients

In $\mathfrak{N}_b(\rho)$:

Lemma

$\exists \mu \in (0, 1)$ s.t., $\forall i, j, \mathfrak{n}, x, t$ and $\forall k \in [\![1, N]\!]$, we have

$$\left|\vartheta_{j,\mathfrak{n},x,t}^{(i)}\right| \leq \binom{\left|\mathfrak{n}\right|_{\ell^1}}{n_k} \mu^{\left|\mathfrak{n}\right|_{\ell^1}}.$$

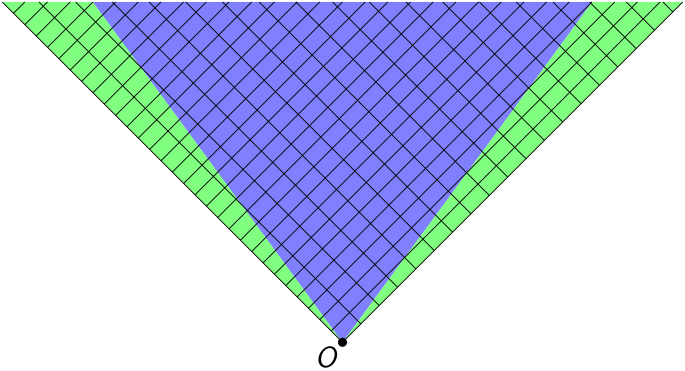
Corollary

$\exists \rho > 0,\ C \geq 1,\ \gamma > 0$ s.t., $\forall i, j, x, t$ and $\forall \mathfrak{n} \in \mathfrak{N}_b(\rho)$, we have

$$\left|\vartheta_{j,\mathfrak{n},x,t}^{(i)}\right| \leq C e^{-\gamma\left|\mathfrak{n}\right|_{\ell^1}}.$$

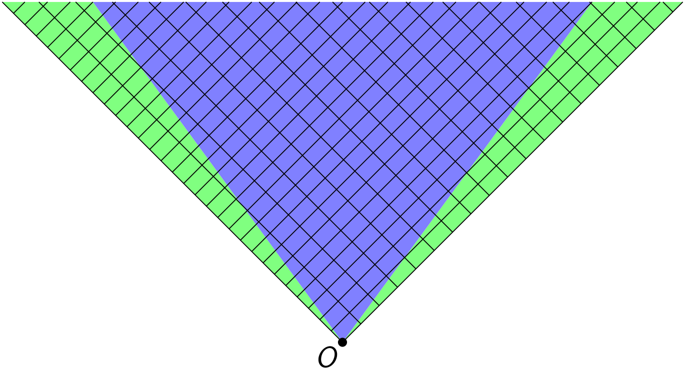
Transport equation on a network of circles

Exponential convergence of the coefficients



Transport equation on a network of circles

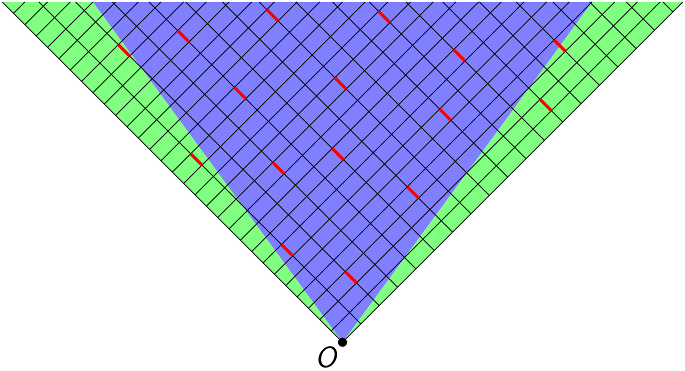
Exponential convergence of the coefficients



$$e^{-\int_{t-L(\mathfrak{n})+a_k}^{t-L(\mathfrak{n})+b_k} \alpha_k(s) ds}$$

Transport equation on a network of circles

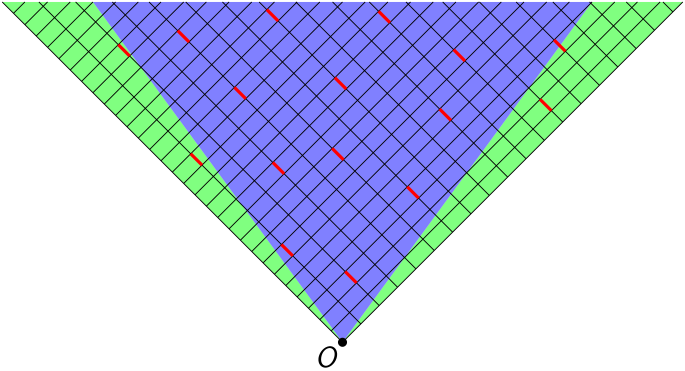
Exponential convergence of the coefficients



$$e^{-\int_{t-L(\mathfrak{n})+a_k}^{t-L(\mathfrak{n})+b_k} \alpha_k(s) ds} \leq \eta$$

Transport equation on a network of circles

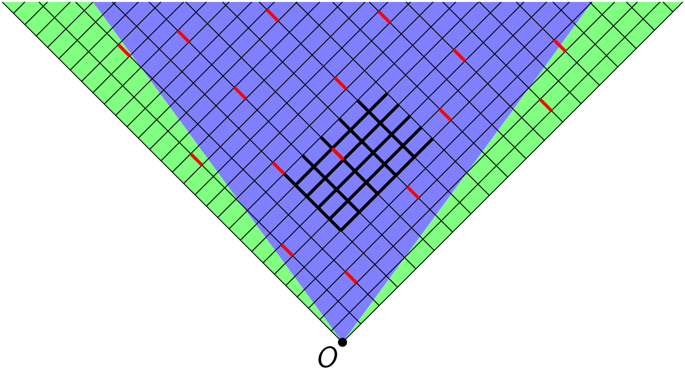
Exponential convergence of the coefficients



Find η such that $e^{-\int_{t-L(\mathfrak{n})+a_k}^{t-L(\mathfrak{n})+b_k} \alpha_k(s) ds} \leq \eta$ "often enough"

Transport equation on a network of circles

Exponential convergence of the coefficients



Find η such that $e^{-\int_{t-L(\mathfrak{n})+a_k}^{t-L(\mathfrak{n})+b_k} \alpha_k(s) ds} \leq \eta$ "often enough"

Transport equation on a network of circles

Exponential convergence of the coefficients

Lemma

Let $T \geq \mu > 0$. $\exists \eta \in (0, 1)$ *and* $K \in \mathbb{N}$ *s.t., for every pair of circles* $k_1 \in [\![1, N]\!]$ *and* $k_2 \in [\![1, N_d]\!] \setminus \{k_1\}$ *with* $\frac{L_{k_1}}{L_{k_2}} \notin \mathbb{Q}$, *every* $\alpha_{k_2} \in \mathcal{G}(T, \mu)$ *and every suitable* $\mathfrak{n}$, t, *there exists* $\mathfrak{r} \in \mathfrak{N}$ *with* $n_j \leq r_j \leq K + n_j$, $j \in \{k_1, k_2\}$, *and* $r_j = n_j$ *for* $j \in [\![1, N]\!] \setminus \{k_1, k_2\}$, *such that*

$$e^{-\int_{t-L(\mathfrak{r})+a_{k_2}}^{t-L(\mathfrak{r})+b_{k_2}} \alpha_{k_2}(s) ds} \leq \eta.$$

Key hypotheses: $\frac{L_{k_1}}{L_{k_2}} \notin \mathbb{Q}$ and $\alpha_{k_2} \in \mathcal{G}(T, \mu)$.

Transport equation on a network of circles

Exponential convergence of the coefficients

Since the decay $e^{-\int_{t-L(\mathfrak{n})+a_k}^{t-L(\mathfrak{n})+b_k} \alpha_{k_2}(s) ds}$ is "active enough" "often enough", we can obtain the following result.

Lemma

Let $T \geq \mu > 0$ and $\sigma \in (0,1)$. $\exists \gamma > 0$, $\Lambda_0 \in \mathbb{N}^*$ s.t., $\forall i, j, x, t$ and $\forall \mathfrak{n} \in \mathfrak{N}_c(\sigma)$ with $\left|\mathfrak{n}\right|_{\ell^1} \geq \Lambda_0$, we have

$$\left|\vartheta_{j,\mathfrak{n},x,t}^{(i)}\right| \leq e^{-\gamma \left|\mathfrak{n}\right|_{\ell^1}}.$$

Transport equation on a network of circles

Exponential convergence of the coefficients

Since the decay $e^{-\int_{t-L(\mathfrak{n})+a_k}^{t-L(\mathfrak{n})+b_k} \alpha_{k_2}(s)ds}$ is "active enough" "often enough", we can obtain the following result.

Lemma

Let $T \geq \mu > 0$ *and* $\sigma \in (0,1)$. $\exists \gamma > 0$, $\Lambda_0 \in \mathbb{N}^*$ *s.t.,* $\forall i, j, x, t$ *and* $\forall \mathfrak{n} \in \mathfrak{N}_c(\sigma)$ *with* $\left|\mathfrak{n}\right|_{\ell^1} \geq \Lambda_0$, *we have*

$$\left|\vartheta_{j,\mathfrak{n},x,t}^{(i)}\right| \leq e^{-\gamma\left|\mathfrak{n}\right|_{\ell^1}}.$$

This proves that the coefficients decrease exponentially with $\left|\mathfrak{n}\right|_{\ell^1}$, and so our theorem is proved.

Discussion on the result

Remarks

- If the damping terms are always active ($\alpha_i(t) \equiv 1$ for every i), one can also show the exponential convergence of the solutions to zero without the hypothesis $\frac{L_i}{L_j} \notin \mathbb{Q}$ for certain i, j.
- With the PE damping, exponential convergence cannot be true in general without this hypothesis.
- If the damping terms are always active, one can replace the hypothesis $|M|_{\ell^1} \leq 1$ by $|M|_{\ell^p} \leq 1$ for a certain $p \in [1, +\infty]$ when $N_d \geq N - 1$.
- We do not know if this still holds true for the PE damping.

Discussion on the result
Open problems

- To which classes of matrices can we generalize this result? $|M|_{\ell^p} \leq 1$ for a certain $p \in [1, +\infty]$? Orthogonal matrices?
- What about coefficients $m_{ij} = 0$? Can we have some of them? Under which hypotheses?
- Can these ideas be used to study waves on a star-shaped network of strings with a persistently excited damping?