The open-dense orbit Theorem

Abdelghani Zeghib

UMPA, ENS-Lyon
http://www.umpa.ens-lyon.fr/~zeghib/

Tlemcen, CIMPA School
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An integrability Problem

Subject, Objects

N a manifold (open subset of $\mathbb{R}^n$)

P a plane field of dimension d:
$x \to P(x) \subset T_x N$ a linear d-space,

Say, locally $P(x) = \mathrm{Span}\{\mathrm{X}_1, \ldots \mathrm{X}_\mathrm{d}\}$, $\{X_i\}$ a system of smooth linearly independent vector fields

But the X_i are not part of the data

Any vector field X such that $X(x) \in P(x)$ is called tangent, or a **section** of P

Case $d = 1$
A direction field

Leaves

Definition

A submanifold F is a **leaf** of P iff $T_xF = P(x)$ for any $x \in F$

In particular $\dim F = \dim P = d$

Remark: "Integral variety": S if $T_xS \subset P(x)$

Leaves

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Integrability Domain

Definition

The integrability domain of P is
$\mathcal{D} = \{x$ such that there exists a leaf F containing x $\}$

Example: $d = 1$, $\mathcal{D} = N$ (classical theorem of integration of ODE)

Definition

Involutivity (or Infinitesimal integrability) domain:
$\mathcal{D}^\infty \subset \mathcal{D}$ set of points x where any iterated bracket of a finite family of vector fields $\{X_i\}$ tangent to P, is tangent to P **at** x:
$[X_1, [X_2, \ldots, [X_{k-1}, X_k]](x) \in P(x)$

- P is involutive iff $\mathcal{D}^\infty = N$

Example (Frobenius Theorem, completely integrable = involutive)

$\mathcal{D} = N \iff \mathcal{D}^\infty = N$

In general:

$\mathcal{D} \subset \mathcal{D}^\infty$

Tentative

Restrict P to $\mathcal{D}^\infty$, and apply Frobenius Theorem !?

Difficulties:

- Topological structure: $\mathcal{D}^\infty$ closed but $\mathcal{D}$ is not necessarily?

- "Differentiable" structure: even closed, $\mathcal{D}^\infty$ may be "fractal"

(No fractal Frobenius Theorem is available!)

(Fractional derivative people?)

- Worse: even if $\mathcal{D}^\infty$ is smooth, P is not necessarily tangent to it (in order to apply Frobenius Theorem)

Examples

$N = \mathbb{R}^3$

$P = \ker \omega$, $\omega = dz - f(x)dy$

$d\omega = -f'(x)dx \wedge dy$

f flat at 0: $f(x) = f^{(n)}(0) = 0, \forall n$

and $f'(x) \neq 0$, for $x \neq 0$

$\mathcal{D}^\infty =$ the plane $\{x = 0\}$

But $\mathcal{D} = \emptyset$

ω is a topological contact structure...

Analytic case

Theorem (Hermann, Nagano)

If P is (real) analytic, then $\mathcal{D} = \mathcal{D}^{\infty}$

Exercise

Find intersection, dictionary, duality, ... between this section and previous talks, e.g. with F. Jean's course.

The Theorem

Partially algebraic Functions

B topological space

$f : B \times \mathbb{R}^m \to \mathbb{R}$ is **partially algebraic** if $f(x, X) = \Sigma_I a_I(x) X^I$

$I = (i_1, \ldots, i_m)$ multi-index

$X^I = X^{i_1} \ldots X^{i_m}$

$a_I : B \to \mathbb{R}$ continuous

Equivalently $f \in C^0(B)[X_1, \ldots, X_m]$

Exercise

Compare with the weaker notion: f *is such that* $f_b : u \in \mathbb{R}^m \to f(b, u)$ *is polynomial for any* $b \in B$

Reminiscent of the exercise: $f : \mathbb{R} \to \mathbb{R}$, *such that for any* x, *there exists* n *such that the derivative* $f^n(x) = 0$ *Prove that* f *is polynomial.*

Generalizations

• $N \to B$ a linear fiber bundle,
(of fiber type $\mathbb{R}^m$, basis B, and projection map $\pi : N \to B$),
They are generalization of product of spaces, but only locally:

B is covered by a family of open sets U_i over which N is a product:
$T_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^m$
On $U_i \cap U_j$: the transition $T_{ji} = T_j \circ T_i^{-1}$ has a **partially linear** form:

$$(x, u) \in (U_i \cap U_j) \times \mathbb{R}^m \to (x, A_x(u)) \in (U_i \cap U_j) \times \mathbb{R}^m$$

where $A : U_i \cap U_j \to GL(m, \mathbb{R})$

• Vector fiber bundle: family of vector spaces (linear algebra with parameter)

• Partially algebraic functions can be coherently defined on linear fiber bundle since transitions are fiberwise linear

• Smooth category: If B manifold, $f(x, X) = \Sigma a_I(x) X^I$, $a_I : B \to \mathbb{R}$ smooth,
Equivalently, $f \in C^\infty(B)[X_1, \ldots, X_m]$

Henceforth: work (often) in the smooth category

Other partially algebraic objects

Partially algebraic vector fields

Partially algebraic plane fields

Fact: The Brackets of two partially algebraic vector fields is partially algebraic.

....

More:

- Partially algebraic sets (latter on)
- Partially algebraic diffeomorphisms...

The Theorem

Theorem (Integrability and Openness)

Let $\pi : N \to B$ be a linear fiber bundle and P a partially algebraic plane field on N.
There exists an open dense subset B' such that:

- *Over B',* $\quad \mathcal{D} = \mathcal{D}^\infty$
(i.e. $\mathcal{D} \cap N' = \mathcal{D}^\infty \cap N'$, where $N' = \pi^{-1}(B')$)

- *The projection $\pi(\mathcal{D} \cap N')$ is closed in B'*

Reformulation

Corollary: *If the projection of* $\mathcal{D}$ *is dense in* B, *then it contains an open dense set*

Re-formulation: *Up to neglecting a subset* $C \subset B$ *(singularity set, catastrophe set, corrupted set...)*
which is closed and has **empty interior,**
we have $\mathcal{D}$ *equals* $\mathcal{D}^{\infty}$, *and has a closed projection in* B

Exercise

The projection of a closed set in not necessarily closed?
Construct a smooth curve $c : \mathbb{R} \to \mathbb{R}^3$*, with* c *injective ("open" Joran curve) and* $X = c(\mathbb{R})$ *a closed subset of* $\mathbb{R}^3$ *(*X *is a closed 1-dimensional of* $\mathbb{R}^3$*) such that the projection of* X *on* $\mathbb{R}^2$ *is dense.*
(e.g. construct an injective curve which contains a lattice $\mathbb{Z}^3$ *in* $\mathbb{R}^3$*...)*

Exercise

(negligible vs negligible)
A closed subset X *with empty interior in* $\mathbb{R}^n$ *may have a positive Lebesgue measure.*
- X *may be the image of a Jordan curve:* $c : [0,1] \to \mathbb{R}^n$ *continuous and INJECTIVE, such that* $X = c([0,1])$ *has positive Lebesgue measure!*
(or a "closed" Jordan curve, i.e. $c(0) = c(1)$ *and* $c_{|]0,1[}$ *injective)*

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Exercise

Prove the Theorem in the case $m = 0$, i.e. $N = B$.
(somehow empty statement!)

The example: Affine transformation groups

Affine maps in $\mathbb{R}^n$

(Exercise)

$f : \mathbb{R}^n \to \mathbb{R}^n$ affine if it has the form $f(x) = A(x) + a$, $A \in Lin(\mathbb{R}^n)$

(notation $\mathcal{L}(\mathbb{R}^n)$, $End(\mathbb{R}^n)$, $\mathrm{Mat}_n(\mathbb{R})$)

$\Longleftrightarrow$

f sends an (affine) line to an (affine) line

$\Longleftrightarrow$

$Graph(f)$ is an affine subspace in $\mathbb{R}^n \times \mathbb{R}^n$

Localization

A **local** affine map: $f : U \to V$, U, V open subsets in $\mathbb{R}^n$...

$Graph(f)$ open subset in an affine subspace...

Isometries

$(\mathbb{R}^n, \langle , \rangle)$
f isometric if $f(x) = Ax + b$, $A \in O(n)$
(f preserves the distance)

On $\mathbb{R}^n \times \mathbb{R}^n$, consider the symmetric bilinear form
$b((u_1, v_1), (u_2, v_2) = \langle u_1, u_2 \rangle - \langle v_1, v_2 \rangle$

b is non-degenerate (pseudo-scalar product) of signature (n, n)

f isometric $\iff$ *Graph*(f) is an **isotropic** affine subspace of $(\mathbb{R}^n \times \mathbb{R}^n, b)$

Exercises

1/ It suffices for Graph(f) to be an isotropic submanifold (i.e. its tangent space is everywhere isotropic) in order to be an affine subspace.

2/ Affine and isometries are solutions of a system of PDE equations. Write them?

A tautological plane field

A plane field P of dimension n on $N = (\mathbb{R}^n \times \mathbb{R}^n) \times \mathrm{Lin}(\mathbb{R}^n)$

Tangent space of N ...

$P(x, y, A) = \{0, 0\} \times Graph(A)$

Let f affine with linear part A and $f(x) = y$,

$f(u) = A(u) + y - A(x)$

$c : u \in \mathbb{R}^n \to (u, f(u), A) \in N$,

$F = \mathrm{Image}\ (c)$

F is a leaf of P through (x, y, A)

The projection of F on $\mathbb{R}^n \times \mathbb{R}^n$ is $Graph(f)$

$\{$ leaves of P $\} \sim$ (marked) local affine maps
i.e. $\{(x, y, f)/f(x) = y, f$ affine $\}$

Variants:

• f affine transformation $=$ f in addition bijective
Adapted construction: replace $Lin(\mathbb{R}^n)$ by $GL_n(\mathbb{R})$
$\mathbb{R}^n \times \mathbb{R}^n \times GL_n(\mathbb{R})$

• Similar construction on $\mathbb{R}^n \times \mathbb{R}^n \times O(n)$:
projection of leaves are graphs of isometries...

“Non-linear case”

(M, ∇) a manifold endowed with a **connection** ∇...
Affine maps are transformations preserving ∇

A connection is a “a tool” allowing one to define **geodesics**
Geodesics are generalizations of straight lines...

Riemannian manifolds

(M, g), g Riemannian metric
There is a canonical (Levi-Cevita) connection,
Its geodesics are the geodesics of g: they are solution of the Euler-Lagrange equation associated to the action $\int g(\dot{\gamma}(t), \dot{\gamma})(t))dt$

Affine map: its sends a geodesic to a geodesic

Isometry: it sends a unit speed geodesic to a unit speed geodesic (it preserves distances)

Totally geodesic submanifolds

(Role of affine subspaces)
F a submanifold of M of dimension d
F is **(totally) geodesic** (in M) if any geodesic tangent to F is contained in it (at least locally)

If $d = 1$, usual geodesics

As in the case of $\mathbb{R}^n$, *a map f is affine $\iff$ Graph(f) is a geodesic submanifold in $M \times M$, endowed with the product metric...*

f is isometric $\iff$ Graph(f) is isotropic in $M \times M$ endowed with the pseudo-Riemannian metric $g \oplus (-g)$

Geodesic up to order 1 submanifolds

Consider $\exp_x : T_xM \to M$, the exponential map:
for any $u \in T_xM$, $t \to \exp_x(tu)$ is the geodesic defined by u

For $E \subset T_xM$ a d-plane, let $\mathcal{E}_x = \exp_x E_x$

In general $\mathcal{E}_x$ is not totally geodesic.

FACT (Shur, Cartan)

Let $1 < d < n = \dim M$. If $\mathcal{E}_x$ is totally geodesic for any x, and any E, then (M, g) has constant sectional curvature...

Example

FACT

Let $f : U \subset \mathbb{S}^2 \to V \subset \mathbb{S}^2$ affine. Then f is isometric (the sphere has no non-trivial (local) affinity)

(It was known before acceptation of Non-Euclidean geometry, that in order to prove Euclid fifth axiom on parallels, it suffices to show existence of non-trivial affinities)

$M = \mathbb{S}^2 \times \mathbb{S}^2$
$x \in \mathbb{S}^2$
Let $E \subset T_x\mathbb{S}^2 \times T_x\mathbb{S}^2$,
$\mathcal{E}_{(x,x)} = \exp_{(x,x)} E$ is geodesic iff E is the graph of the derivative of an ISOMETRY $T_x \to T_x$
(The set of such $\mathcal{E}$ is a circle, $\dim(Gr^2(T_x \times T_x) = 4)$

Grassmann manifolds

For E a vector space $Gr^d(E)$ = space of d- linear subspaces of E

$A \in \mathrm{L}in(E) \to \mathrm{G}raph(A) \in Gr^n(E \times E)$, $n = \dim E$
Its image is the set of planes that are graphs on $E \times 0$

Similarly $\mathrm{G}L(E) \to Gr^n(E \times E)$, its image $Gr^*(E \times E)$, planes that are transversal to $E \times 0$ and $0 \times E$

$Gr^*(E \times E)$ is a **compactification** of $\mathrm{G}L(E)$....

Translation of the transformation problem to an integrability problem

M a manifold

$Gr^d(M) \to M$, with fiber $Gr^d(T_x M)$

FACT

There is a n-plane field P on $Gr^{n*}(M \times M)$ such that the projections of its leaves are graphs of local affine transformations $M \to M$.

Equivalently, the projection of a leaf of P is a n-totally geodesic submanifold in $M \times M$.

In addition, P is partially algebraic

- There is a similar construction on a restricted Grassmann bundle yielding graphs of isometries.

Proof

General construction:
On $Gr^d(M) \to M$,
τ^d **geodesic tautological** plane field, the projections of its leaves are geodesic d-submanifolds

For S d-submanifold of M,
$Ga^S : x \in S \to T_xS \in Gr^d(M)$ its Gauß map

Let $E \in Gr^d_x(M) \subset Gr^d(M)$, $S = \exp_x(E)$,
$D_x(Ga^S) : T_xS \to T_E(Gr^d(M))$

Define $\tau^d(E) = \text{Image of } D_x(Ga^S)$

In other words, by definition, the Gauß map of $\exp_x(E)$ is tangent up to order 1 to τ^d at E

Other definition

$T_E(Gr^d(M)) = H_E \oplus V_E$: horizontal + vertical

V_E tangent space of the fiber

H_E horizontal space given by the connection

$D_E\pi : H_E \to T_x M$ isomorphism

$\tau^d(E) =$ inverse image of E

(E plays the role of a point in $Gr^d(M)$ and a subspace in $T_x M$)

Tautologies, examples

- The Liouville form α on $T * M$, depends only of the differential structure
- $\omega = d\alpha$ the symplectic structure
- Many similar exterior differential systems on jet bundles...

By means of a Riemannian metric (or merely a connection):
- A canonical parallelism on the frame bundle...

Case $d = 1$

The geodesic flow is a vector field on TM

τ^1 is the corresponding field directions on the projectivized bundle $\mathbb{P}(TM) = Gr^1(M)$

Partial algebraicity

The notion of partial algebraicity extends naturally to projective and then Grassmann bundle...

Geodesic flow:
$\ddot{x}^k = -\Sigma\Gamma_{ij}^k(x)\dot{x}^i\dot{x}^j$

In phase space $(q, p) \in TM \sim M \times \mathbb{R}^n$
$V(q, p) = (p^1, \ldots, p^n, -\Sigma\Gamma_{ij}^1(q)p^i p^j, \ldots, -\Sigma\Gamma_{ij}^n(q)p^i p^j)$

Quadratic on p^i

Orbits of the the Isometry Pseudo-group

The pseudo-group of affine transformations is the collection of affines diffeomorphisms between open subsets of M

$\mathcal{G} = \{U, V, f : U \to V,\ U, V,$ open, and f affine $\}$
G: group of global affine transformations

Example: $M = \mathbb{R}^n$, or M an open subset of $\mathbb{R}^n$,
or M a quotient, e.g. $M = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ the torus
They have essentially the same affine pseudo-group, but different global groups

No dynamics for $\mathcal{G}$,

Delicate iteration because of collapse of domains of definition,

but one defines orbits $\mathcal{G}(x)$

Definition

M is (affinely) **locally** homogeneous if $\mathcal{G}$ has one orbit M.
(homogeneous if G has one orbit)

FACT

The orbit $\mathcal{G}(x)$ is the projection on $\{x\} \times M$ of the integrability domain $\mathcal{D}$ of the n- tautological geodesic plane field on $Gr^{n*}(M \times M)$
In particular if some $\mathcal{G}(x)$ is dense in M, then the projection of $\mathcal{D}$ is dense in $M \times M$

Corollary (Open-dense orbit Theorem, M. Gromov)

If the pseudo-group of affine transformations of M has a dense orbit, then this orbit is open. In other words, some open dense subset of $M' \subset M$ is locally homogeneous

———————

Generalization to more general situations (than connections and metrics): all "rigid" geometric structures.

M' looks like a double coset space $\Gamma \setminus G/H$: algebrico-arithmetic objet.

$M' = M - C$, C singularity set...?

Partially (real semi-) algebraic Geometry

Control Theory

The example