

Constrained optimization with respect to stochastic dominance

Application to portfolio insurance

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Abstract

We are concerned with a classic portfolio optimization problem where the admissible strategies must dominate a floor process on every intermediate date (American guarantee). We transform the problem into a martingale one whose aim is to dominate an obstacle, or equivalently its Snell envelope. The optimization is performed w.r.t. the concave stochastic ordering on the terminal value, so that we do not impose any explicit specification of the agent's utility function. A key tool is the representation of the supermartingale obstacle in terms of a running supremum process. This is illustrated within the paper by an explicit example based on the geometrical Brownian motion.

Key words: Concave stochastic ordering, Portfolio optimization with constraints, American options, Supermartingale, Snell envelope.

1 Introduction

Our problem is motivated by Portfolio insurance which is a popular example of dynamic asset allocation. In real-world financial markets (particularly the bear ones), insurance products are observable that offer their buyers a guaranteed minimum wealth, commonly termed the floor. Sellers of insurance products are compelled to meet their guarantees whatever the market fluctuations. Therefore, they will be well-advised to pursue a portfolio strategy that is able to fulfill this requirement if they want to avoid additional cash payments.

The class of such strategies is large. They are traditionally based on an axiomatic model of risk-averse preferences, where decision makers are assumed to possess an expected utility function and the portfolio choice consists in maximizing the expected utility over the set of feasible portfolios. Hence, any rule that takes less risk at lower wealth levels and more risk at higher wealth levels is a candidate.

However, the utility function is related to individual preferences, and decision makers are often not willing or able to answer precise questions regarding their preferences. Thus the fewer restrictions that are placed on the utility function, the more general applicability the results

will have. We are led to the notion of concave stochastic order which does not impose any explicit specification of the agent's utility function. Rather, it resorts to some very general conditions of non-satiation and risk preferences.

In this paper, we aim at solving the classic decision problem “portfolio selection” in some way that has not been expressed yet in the existing literature on the subject. The basic idea is to split the problem in two steps. We first consider a new environment in which the portfolio strategies are martingales and try to optimize a general concave criterion over all martingales with American constraint. Then, we show that with an adequate change of probability, we can use our result to solve for utility-maximizing strategies under the fairly standard assumptions of complete, arbitrage free and frictionless market.

The remainder of the paper is organized as follows. We first set a martingale optimization problem motivated by portfolio insurance. Then we define the concave stochastic order and recall its basic properties. In subsection 2.3, we reformulate the martingale problem in terms of the concave order, and solve it in a particular case before giving the general solution. Following the same ideas as in [BF03], optimality results are based on a new representation of super-martingales as a conditional expectation of a running supremum process. Finally, we derive from the preceding results a closed formula for American Call options and solve the classic portfolio optimization problem when an American constraint is imposed on the liquidative value of the open fund.

2 Martingale optimization problem

2.1 Portfolio insurance framework

Throughout the paper, the uncertainty is modelled by some filtered probability space $(\Omega, \mathcal{F}, \mathbf{P})$ satisfying the usual conditions, i.e. $\{\mathcal{F}_t\}$ is an increasing, right-continuous family of σ -fields and \mathcal{F}_0 contains all the \mathbf{P} -negligible events in \mathcal{F} . We further assume $\{\mathcal{F}_t\}$ to be *quasi-left-continuous*¹. A Brownian filtration can be supposed in a first reading, so that all martingales are continuous. A more general framework is considered in the paper, in order to deal with both continuous and discontinuous cases (geometrical Brownian motion, Lévy processes, etc.). The *horizon* of the problem is a stopping time denoted by ζ and may be infinite. All processes are assumed to be defined until the horizon date, even if this latter is not finite. As usual a stopping time can take infinite values.

In finance, the problem to “find the smallest portfolio” dominating an obstacle (X_t) (assumed to be bounded or more generally dominated by a uniformly integrable martingale)² is classic, and refers in general to an American-style option problem.

In the risk-neutral environment, the self-financing portfolios are martingales. Since the inf of two martingales defines a super-martingale, the problem has always a solution in the super-martingale class, called “Snell envelope”.

In order to select an optimal “martingale” (and not an optimal “super-martingale”), we introduce

¹A filtration $\{\mathcal{F}_t\}$ is said to be quasi-left continuous if for any predictable stopping time τ , one has $\mathcal{F}_\tau = \mathcal{F}_{\tau-}$.

²In this case, X is said to be class (\mathcal{D}) .

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a utility criterion on the martingale terminal value. Note that any martingale dominating a floor process X necessarily dominates its Snell envelope Z and thus it is natural to consider the following martingale optimization problem with a super-martingale floor:

$$\max_{M_t} \mathbf{E}_{\mathbf{P}}\{u(M_\zeta), \text{ subject to } M_t \text{ martingale } \geq Z_t \ \forall t \in [0, \zeta] \text{ and } M_0 \text{ given}\},$$

where u stands for a general utility function (concave, strictly increasing, defined on \mathbb{R}^+) and the super-martingale Z is assumed to be bounded, or dominated by a uniformly integrable martingale.

The solution of the unconstrained problem does not depend on the utility function. In fact, due to the concavity of u , we have that $\mathbf{E}[u(M_\zeta)] \leq u[\mathbf{E}(M_\zeta)] = u(M_0)$, for every martingale (M_t) , and therefore the optimal strategy is to do nothing.

Similarly, when the constraints hold, we would like to find a solution which does not depend on the form of u , especially that the choice of the utility function is not easy at all. These observations lead us to the notion of concave stochastic ordering which expresses the notion of one entire probability distribution being less than or equal to another³.

2.2 Concave stochastic dominance

The stochastic order was introduced in Economics by Rothschild and Stiglitz in 1970 as a measure of risk (see [RS70]). It has been used in a wide variety of economic contexts, such as efficiency pricing, finance, insurance, risk sharing, measurement of inequality, etc. The measurement of the risk involved by random variables is made possible by using general convex functions.

Definition 2.1. *Let X_1 and X_2 be two random variables.*

1. *We say that X_2 is preferable to X_1 in the **concave stochastic order**, and we write $X_1 \leq_{cv} X_2$ if for any concave real-valued function g for which the following expectations are well defined,*

$$\mathbf{E}[g(X_1)] \leq \mathbf{E}[g(X_2)] \tag{1}$$

2. *X_2 is said to be preferable to X_1 in the **increasing concave order** (denoted by $X_1 \leq_{icv} X_2$), if Inequality (1) holds for all increasing concave functions g , for which the expectations exist.*

Roughly speaking, if $X_1 \leq_{icv} X_2$ then X_1 is both “smaller” and “more variable” than X_2 in some stochastic sense.

In addition, by considering specific concave functions ($\phi_1(x) = x$, $\phi_2(x) = -x$ and $\phi_3(x) = -(x - m)^2$), it can be easily seen that

$$X_1 \leq_{cv} X_2 \longrightarrow \mathbf{E}[X_1] = \mathbf{E}[X_2] \quad \text{and} \quad \mathbf{Var}(X_1) \geq \mathbf{Var}(X_2),$$

³We often use expected values to compare random variables, but expected values are only one-dimensional summaries of the entire probability distributions. We need something stronger than an ordering of expected values in order to compare entire probability distributions. Stochastic order serves the purpose and enables us to make the desired stronger stochastic comparisons.

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whenever the expectations exist and $\mathbf{Var}(X_1) < \infty$.

So, indeed, the concave ordering is stronger than ordering of the variances and only random variables that have the same means can be compared by the order \leq_{cv} .

As for the increasing concave order, we have that $X_1 \leq_{icv} X_2 \rightarrow \mathbf{E}[X_1] \leq \mathbf{E}[X_2]$, provided the expectations exist.

Hence, stochastic dominance provides a method of thinking about a random variable being stochastically larger than another one in a way that goes significantly beyond the overly simple procedure of comparing the expectations of these variables.

2.3 Formulation and resolution of the “new” martingale problem

Now, equipped with the previous definitions, we are able to formulate the constrained optimization problem in terms of stochastic ordering. Let us first observe that since Z is a super-martingale, the smallest initial capital that allows the investor to dominate the floor must equal Z_0 .

Introduce the following set of admissible martingales

$$\mathcal{M}^Z = \left\{ (M_t)_{t \geq 0} \text{ uniformly } \mathbf{P}\text{-integrable martingale} \mid \right. \\ \left. M_0 = Z_0 \text{ and } M_t \geq Z_t \ \forall t \in [0, \zeta] \right\}.$$

Define by M^Z the martingale of the Doob-Meyer decomposition of Z : $M_t^Z = Z_t + A_t$, where A is a predictable increasing process starting from 0. Then it is immediate to see that \mathcal{M}^Z is not empty since it already contains M^Z .

We aim at finding the greatest martingale M^\perp in \mathcal{M}^Z with respect to the concave stochastic order on the terminal value, i.e. $M_\zeta^\perp \geq_{cv} M_\zeta$ for all martingales $(M_t)_{0 \leq t \leq \zeta}$ in \mathcal{M}^Z .

As we will see in the following example, the solution is explicitly described in a Black-Scholes environment with infinite horizon.

2.3.1 Resolution in a particular case

Assume that the super-martingale Z defines a geometrical Brownian motion with negative drift:

$$\frac{dZ_t}{Z_t} = -r dt + \sigma dW_t, \quad Z_0 = x > 0. \quad (2)$$

We assume the maturity ζ to be infinite and introduce the following notation:

$$Z_{s,t}^* = \sup_{s \leq u \leq t} Z_u.$$

Since the relative increments of Z are independent,

$$\mathbf{E}(Z_{t,\infty}^* | \mathcal{F}_t) = Z_t \mathbf{E}\left(\sup_{t \leq u} \frac{Z_u}{Z_t} | \mathcal{F}_t\right) = Z_t \mathbf{E}\left(\sup_{0 \leq u} \frac{Z_{u+t}}{Z_t} | \mathcal{F}_t\right) = \frac{Z_t}{x} \mathbf{E}(Z_{0,\infty}^*).$$

It follows that

$$Z_t = b \mathbf{E}[Z_{t,\infty}^* | \mathcal{F}_t] \quad \text{and} \quad \frac{1}{b} = \frac{1}{x} \mathbf{E}\left(\sup_{0 \leq u} Z_u\right) = \mathbf{E}\left(\frac{Z_{0,\infty}^*}{x}\right). \quad (3)$$

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Theorem 2.1. *The process*

$$M_t^\perp := \mathbf{E}\left(Z_{0,\infty}^* | \mathcal{F}_t\right)$$

is the greatest martingale in \mathcal{M}^Z with respect to the concave stochastic order on the terminal value.

Proof. The optimality comes out from the general proof of Theorem 2.2. \square

Since the law of $Z_{0,\infty}^*$ is well-known, there exists a closed formula for M_t^\perp as a function of $(Z_t, Z_{0,t}^*)$.

Lemma 1. *Let γ be $\gamma = 1 + \frac{2r}{\sigma^2}$ and $b = \frac{\gamma-1}{\gamma} = \frac{1}{1+\frac{\sigma^2}{2r}}$.*

$$1. \mathbf{P}(Z_{0,\infty}^* \geq m) = \left(\frac{x}{m} \wedge 1\right)^\gamma.$$

$$2. \mathbf{E}(Z_{0,\infty}^*) = \frac{x}{b} = \frac{\gamma}{\gamma-1}x.$$

$$3. \mathbf{E}[(Z_{0,\infty}^* - m)^+] = \begin{cases} \frac{m}{\gamma-1} \left(\frac{x}{m}\right)^\gamma = \frac{\sigma^2 x}{2r} \left(\frac{x}{m}\right)^{\frac{2r}{\sigma^2}} & \text{if } m \geq x \\ \frac{\gamma}{\gamma-1}x - m & \text{if } m \leq x. \end{cases}$$

Proof. The law of $Z_{0,\infty}^*$ is well-known and can be expressed as $\mathbf{P}(Z_{0,\infty}^* \geq m) = \left(\frac{x}{m} \wedge 1\right)^\gamma$. In particular,

$$\mathbf{E}(Z_{0,\infty}^*) = \int_0^\infty \mathbf{P}(Z_{0,\infty}^* \geq \alpha) d\alpha = \int_0^x d\alpha + \int_x^\infty \left(\frac{x}{\alpha}\right)^\gamma d\alpha = \frac{\gamma}{\gamma-1}x.$$

More generally, for $m \geq x$, the price of a Call option on the supremum is given by

$$\mathbf{E}[(Z_{0,\infty}^* - m)^+] = \int_0^{+\infty} \mathbf{P}(Z_{0,\infty}^* - m \geq \alpha) d\alpha = \int_0^{+\infty} \left(\frac{x}{\alpha+m}\right)^\gamma d\alpha = \frac{m}{\gamma-1} \left(\frac{x}{m}\right)^\gamma.$$

But if $m \leq x$, $Z_{0,\infty}^* \geq m$ and

$$\mathbf{E}[(Z_{0,\infty}^* - m)^+] = \mathbf{E}[Z_{0,\infty}^* - m] = \frac{\gamma}{\gamma-1}x - m.$$

\square

We recognize the price of a perpetual American call (see [Ell82] [KS98]). The result is general as we will see in Section 2.3.1.

Now we are equipped to describe explicitly the optimal martingale M^\perp .

Proposition 2.1. *1. The floor process Z and the optimal solution M^\perp can be expressed as*

$$Z_t = \frac{\gamma-1}{\gamma} \mathbf{E}(Z_{t,\infty}^* | \mathcal{F}_t) \quad \text{and} \quad M_t^\perp = \frac{\gamma-1}{\gamma} Z_{0,t}^* \left[\frac{1}{\gamma-1} \left(\frac{Z_t}{Z_{0,t}^*}\right)^\gamma + 1 \right] := \phi(Z_t, Z_{0,t}^*).$$

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2. As a martingale, M_t^\perp can be represented as a stochastic integral

$$dM_t^\perp = \left(\frac{Z_t}{Z_{0,t}^*}\right)^{\gamma-1} \sigma Z_t dW_t = \left(\frac{Z_t}{Z_{0,t}^*}\right)^{\gamma-1} dM_t^Z,$$

where M^Z denotes the martingale of the Doob-Meyer decomposition of Z .

Proof. 1. The properties of Z have been already seen.

To study the martingale $M_t^\perp = b\mathbf{E}(Z_{0,\infty}^*|\mathcal{F}_t)$, we note that M^\perp can be also expressed as

$$M_t^\perp = b\mathbf{E}[(Z_{t,\infty}^* - Z_{0,t}^*)^+|\mathcal{F}_t] + bZ_{0,t}^* = bZ_t\mathbf{E}\left[\left(\frac{Z_{t,\infty}^*}{Z_t} - \frac{Z_{0,t}^*}{Z_t}\right)^+|\mathcal{F}_t\right] + bZ_{0,t}^*. \quad (4)$$

Since $Z_{t,\infty}^*/Z_t$ is independent of $Z_{0,t}^*/Z_t$, (4) can be rewritten as follows:

$$M_t^\perp = \frac{bZ_t}{x}\mathbf{E}\left[(Z_{0,\infty}^* - m_t)^+\right] + bZ_{0,t}^*, \quad (5)$$

with $m_t := \frac{Z_{0,t}^*}{Z_t}x$ ⁴. Substituting m_t in the last expression of Lemma 1, we hence explicitly determine M^\perp :

$$M_t^\perp = \frac{\gamma-1}{\gamma}Z_{0,t}^*\left[\frac{1}{\gamma-1}\left(\frac{Z_t}{Z_{0,t}^*}\right)^\gamma + 1\right] := \phi(Z_t, Z_{0,t}^*).$$

2. Since $M^\perp = \varphi(Z, Z^*)$ is a martingale, its decomposition as a stochastic integral needs only to know the derivative of φ w.r.t. Z : $dM_t^\perp = \varphi'_Z(Z_t, Z_{0,t}^*)Z_t\sigma dW_t$. We therefore obtain

$$dM_t^\perp = \left(\frac{Z_t}{Z_{0,t}^*}\right)^{\gamma-1} \sigma Z_t dW_t = \left(\frac{Z_t}{Z_{0,t}^*}\right)^{\gamma-1} dM_t^Z. \quad (6)$$

From (6), it can be particularly seen that M^\perp is less variable than M^Z since $\gamma \geq 1$ and $Z_{0,t}^* \geq Z_t$. □

2.3.2 Main result

Now, we address the same question in the more general case where the maturity ζ is not necessary infinite. Moreover, the floor process Z is no more a geometrical Brownian motion, but still remains a super-martingale.

Theorem 2.2. *Fix a horizon date ζ and consider a càd-làg super-martingale $Z = (Z_t)_{0 \leq t \leq \zeta}$ of class (\mathcal{D}) . Assume that exists a real progressively measurable process (L_t) with upper-right continuous paths, such that*

$$Z_t = \mathbf{E}\left(\sup_{t \leq u \leq \zeta} L_u | \mathcal{F}_t\right) \quad 0 \leq t \leq \zeta. \quad (7)$$

⁴ m_t is naturally supposed to be deterministic when computing the expectation in (5).

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Define the martingale M^\perp by

$$M_t^\perp = \mathbf{E}\left(\sup_{0 \leq u \leq \zeta} L_u \mid \mathcal{F}_t\right) \quad 0 \leq t \leq \zeta.$$

Then, M^\perp is the greatest martingale in \mathcal{M}^Z with respect to the stochastic concave order. In other terms,

$$\mathbf{E}[g(M_\zeta^\perp)] \geq \mathbf{E}[g(M_\zeta)]$$

for any martingale M in \mathcal{M}^Z ($M_t \geq Z_t \quad \forall t \in [0, \zeta]$, $M_0 = Z_0$) and any concave function g for which the above expectations are well defined.

Remark 2.1. The representation condition (7) is not really an assumption, as we have shown in [EKM04]. The proof is too technical to appear here. Hence the last theorem remains valid for all super-martingales of class (\mathcal{D}) (dominated by a uniformly integrable martingale).

Proof. We need the following lemma for the proof of Theorem 2.2.

Lemma 2. $L_{0,t}^* := \sup_{0 \leq u \leq t} L_u$ is an increasing process which satisfies the flat-off condition

$$\int_{[0, \zeta]} (M_t^\perp - Z_t) dL_{0,t}^* = 0,$$

that is, it only increases at points in time $t \leq \zeta$ when $M_t^\perp = Z_t$.

Proof of the lemma.

$(L_{0,t}^*)$ is obviously an increasing process. Let t be a right increasing point of $(L_{0,t}^*)$.

If $t < \zeta$, it necessarily satisfies $\sup_{0 \leq s \leq \zeta} L_s = \sup_{t \leq s \leq \zeta} L_s$, whence $M_t^\perp = Z_t$.

If $t = \zeta$, we have that $L_{0,\zeta}^* < L_\zeta$, otherwise the increasing process does not jump at ζ . However $L_\zeta = Z_\zeta$ by definition of Z and $M_\zeta = L_{0,\zeta}^* \vee L_\zeta$. It immediately follows that $M_\zeta^\perp = Z_\zeta$ and the flat-off condition is well satisfied.

Proof of Theorem 2.2 (sequel). Let (M_t) be an arbitrary element of \mathcal{M}^Z . We shall prove that $M_\zeta \leq_{cv} M_\zeta^\perp$.

Let g be a real concave function, for which the expectations $\mathbf{E}[g(M_\zeta)]$ and $\mathbf{E}[g(M_\zeta^\perp)]$ are well defined. The concavity of g yields

$$\mathbf{E}[g(M_\zeta)] - \mathbf{E}[g(M_\zeta^\perp)] \leq \mathbf{E}[g'(M_\zeta^\perp)(M_\zeta - M_\zeta^\perp)] = \mathbf{E}[g'(L_{0,\zeta}^*)(M_\zeta - M_\zeta^\perp)].$$

We use the classic differential rule for finite variation processes; it is convenient to introduce the discrete derivative of g' , g_d'' as

$$\begin{cases} g_d''(x, \delta) = \frac{1}{\delta}(g'(x + \delta) - g'(x)), & \text{if } \delta \neq 0 \\ g_d''(x, 0) = g_d''(x), & \text{if } \delta = 0. \end{cases}$$

$$g'(L_{0,\zeta}^*) = g'(L_{0,0}^*) + \int_0^\zeta g_d''(L_{0,s^-}^*, \Delta L_{0,s}^*) dL_{0,s}^*.$$

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Since g is a concave function, g_d'' is negative. Moreover, the current value of the martingale $M - M^\perp$ at any time s is nothing else the conditional expectation of its terminal value w.r.t the filtration \mathcal{F}_s . This yields

$$\mathbf{E}[g'(L_{0,\zeta}^*)(M_\zeta - M_\zeta^\perp)] = \mathbf{E}[g'(L_{0,0}^*)(M_\zeta - M_\zeta^\perp)] + \mathbf{E}\left[\int_0^\zeta (M_s - M_s^\perp) g_d''(L_{0,s^-}^*, \Delta L_{0,s}^*) dL_{0,s}^*\right].$$

Moreover in light of Lemma 2, $(L_{0,t}^*)$ only increases at times $t \leq \zeta$ when $M_t^\perp = Z_t$, whence

$$\int_0^\zeta (M_s - M_s^\perp) g_d''(L_{0,s^-}^*, \Delta L_{0,s}^*) dL_{0,s}^* = \int_0^\zeta (M_s - Z_s) g_d''(L_{0,s^-}^*, \Delta L_{0,s}^*) dL_{0,s}^* \leq 0.$$

Thus, these considerations lead to

$$\mathbf{E}[g'(L_{0,\zeta}^*)(M_\zeta - M_\zeta^\perp)] \geq \mathbf{E}[g'(L_{0,0}^*)(M_0 - M_0^\perp)] = 0. \quad (8)$$

If we rather consider the increasing concave order, the quantity (8) is nonpositive since $g'(L_{0,0}^*) \geq 0$ and $M_0 \leq M_0^\perp$.

Thus $\mathbf{E}[g(M_\zeta)] \leq \mathbf{E}[g(M_\zeta^\perp)]$ and the martingale M^\perp is optimal indeed.

Let us just note that the above (in)equalities remain valid for non smooth functions g . All that we need is to consider the derivatives in the sense of distributions and not in the strict sense. \square

Remark 2.2. 1. *Let us consider the specific case where the admissible martingales M dominating the floor Z have an initial value given by m . We necessarily have that $M_0 = m \geq Z_0$ and the same results hold if we replace the increasing process $(L_{0,t}^*)$ by $(L_{0,t}^* \vee m)$.*

2. *Since the initial value of any martingale is equal to its mean, the formulation of the initial condition depends on the selected stochastic order. If we consider the concave order, all admissible martingales must have the same initial value m , whereas if we consider the increasing concave order, the initial value of any admissible martingale M must not exceed the initial value of any optimal solution to the problem. It must also be equal or greater than Z_0 .*

2.3.3 Application to American Options

As a by-product of the Z_t -representation, we have an explicit characterization of American Call options as lookback ones. See [BF03] for more details on the American option pricing, based on running supremum processes.

Proposition 2.2 (Closed formula for American Call options). *Assume that $Z_t = \mathbf{E}(L_{t,\zeta}^* | \mathcal{F}_t)$ and define $C(t, Z_t, m)$ as the price of the American Call option written on the underlying Z_t , with strike m :*

$$C(t, Z_t, m) = \text{esssup}_{t \leq S \leq \zeta} \mathbf{E}((Z_S - m)^+ | \mathcal{F}_t).$$

$C(t, Z_t, m)$ has a closed formula given by

$$C(t, Z_t, m) = \mathbf{E}((L_{t,\zeta}^* - m)^+ | \mathcal{F}_t) = \mathbf{E}\left(\left(\sup_{t \leq s \leq \zeta} L_s - m\right)^+ | \mathcal{F}_t\right), \quad t \leq \zeta,$$

and the stopping time $D_t(m) := \inf\{S \in [t, \zeta]; L_{t,S}^* \geq m\}$ is optimal.

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Proof. First note that the process $\mathbf{E}((L_{t,\zeta}^* - m)^+ | \mathcal{F}_t)$ defines a supermartingale dominating $(Z_t - m)^+$ since it is the conditional expectation of a nonincreasing process and

$$\mathbf{E}((L_{t,\zeta}^* - m)^+ | \mathcal{F}_t) = \mathbf{E}(L_{t,\zeta}^* \vee m | \mathcal{F}_t) - m \geq [\mathbf{E}(L_{t,\zeta}^* | \mathcal{F}_t) \vee m] - m = (Z_t - m)^+.$$

Then, it also dominates its Snell envelope $C(t, Z_t, m)$.

Define $D_t(m)$ as the first stopping time after t , at which the process $(L_{t,u}^*)_{u \geq t}$ goes beyond m :

$$D_t(m) := \inf\{S \in [t, \zeta]; L_{t,S}^* \geq m\}.$$

We use the notation ∞^+ when the set is empty (to make the difference with $+\infty$ that could be taken by ζ). We have that

$$\begin{aligned} \mathbf{E}[(L_{t,\zeta}^* - m)^+ | \mathcal{F}_t] &= \mathbf{E}(\mathbf{E}(\mathbf{1}_{D_t(m) < \infty^+} (L_{D_t(m),\zeta}^* - m)^+ | \mathcal{F}_{D_t(m)}) | \mathcal{F}_t) \\ &= \mathbf{E}[\mathbf{1}_{D_t(m) < \infty^+} \mathbf{E}(L_{D_t(m),\zeta}^* - m | \mathcal{F}_{D_t(m)}) | \mathcal{F}_t] \\ &= \mathbf{E}[\mathbf{1}_{D_t(m) < \infty^+} (Z_{D_t(m)} - m) | \mathcal{F}_t] \\ &= \mathbf{E}[(Z_{D_t(m)} - m)^+ | \mathcal{F}_t], \end{aligned}$$

where the last equality follows from the fact that $Z_{D_t(m)} - m$ is nonnegative if $D_t(m) < \infty^+$, and nonpositive otherwise.

Hence $\mathbf{E}[(L_{t,\zeta}^* - m)^+ | \mathcal{F}_t] = \mathbf{E}[(Z_{D_t(m)} - m)^+ | \mathcal{F}_t] \leq C(t, Z_t, m)$ by definition of the Snell envelope. This completes the proof since we have already shown the reverse inequality. \square

3 Application to portfolio insurance: a “general” constrained optimization problem

Some real-world insurance products contain a minimum wealth or an income stream guarantee, both of which have to be met irrespective of capital market conditions. Therefore, the seller of such products have to choose the portfolio strategy that performs best in a reasonable worst case capital market scenario, if he wants to avoid additional cash.

As already mentioned, the traditional strategies generally try to maximize an expected utility criterion, related to individual preferences, concave and increasing.

In the present section, we are concerned with a portfolio management problem, where the goal of the manager is to exceed the performance of a given benchmark process at any time during the life of the fund.

Similar problems appear when the manager is submitted to legal constraints. In practice, the first step in the management of investment funds or pension funds is to define a Strategic allocation related to a finite horizon. According to the investor’s risk aversion, the manager decides the proportion of indexes, securities, coupon bonds to be held in a well-diversified portfolio with a current value S_t .

In a mathematical framework, (S_t) denotes the optimal portfolio for the non-constrained associated problem, with a given utility function.

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3.1 “Non-constrained” problem and change of numéraire

Let us introduce a complete financial market, where interest rate and risk premium are given through the state price density (H_t), in such a way that $H_t V_t$ is a local martingale for any self-financing portfolio (V_t).

Consider the following "non-constrained" problem of an investor who maximizes an expected utility criterion

$$\max_{V_t} \mathbf{E}\{u(V_\zeta), (V_t H_t) \text{ martingale and } V_0 = x\}, \quad (9)$$

where the expectation is performed under the subjective probability \mathbf{P} . Denote by V_ζ^{*x} the terminal value of an optimal strategy with initial wealth x . Thanks to the first order condition of the optimization problem, it can be shown that V_ζ^{*x} satisfies the marginal utility condition $u'(V_\zeta^{*x}) = \lambda H_\zeta$ ⁵, where λ denotes the Lagrange multiplier. This parameter is completely determined by the budget constraint $\mathbf{E}[H_\zeta u'^{-1}(\lambda H_\zeta)] = x$.

Furthermore, in the case of a constant relative risk aversion (CRRA) utility function u defined as: $u(x) = \frac{x^{1-\alpha}}{1-\alpha}$, for all $x \in \mathbb{R}^+$, with $\alpha \in]0, 1[$, we have that $u'^{-1}(\lambda H_\zeta) = \lambda^{-\frac{1}{\alpha}} H_\zeta^{-\frac{1}{\alpha}}$. Hence, the budget constraint $\mathbf{E}[H_\zeta \lambda^{-\frac{1}{\alpha}} H_\zeta^{-\frac{1}{\alpha}}] = x$ implies that the terminal value of the optimal strategy takes the form of $V_\zeta^{*x} = \lambda^{-\frac{1}{\alpha}} H_\zeta^{-\frac{1}{\alpha}} = x \frac{H_\zeta^{-\frac{1}{\alpha}}}{\mathbf{E}(H_\zeta^{1-\frac{1}{\alpha}})}$. It is therefore proportional to the initial wealth invested in the fund: $V_\zeta^{*x} = x S_\zeta$, where $S_\zeta = V_\zeta^{*1}$ denotes the optimal portfolio with initial capital $S_0 = 1$. In particular, we have that $S_\zeta^{-\alpha} = \lambda' H_\zeta$, with $\lambda' = [\mathbf{E}(H_\zeta^{1-\frac{1}{\alpha}})]^\alpha$.

Set $M_t^S = H_t S_t$. Since the process M^S defines a \mathbf{P} -martingale, there exists a new measure of probability \mathbf{Q}^S defined thanks to its Radon-Nikodym density with respect to \mathbf{P} : $\frac{d\mathbf{Q}^S}{d\mathbf{P}} = \frac{M_\zeta^S}{M_0^S} = H_\zeta S_\zeta$. The S -prices are \mathbf{Q}^S -local martingales and \mathbf{Q}^S defines an S -martingale measure. Thus with a change of probability and assuming that the utility function u is of power type, the criterion to maximize can be expressed as:

$$\begin{aligned} \mathbf{E}_{\mathbf{P}}(u(V_\zeta)) &= \mathbf{E}_{\mathbf{Q}^S} \left[\frac{1}{H_\zeta S_\zeta} \frac{V_\zeta^{1-\alpha}}{1-\alpha} \right] = \mathbf{E}_{\mathbf{Q}^S} \left[\frac{S_\zeta^{-\alpha}}{H_\zeta} \frac{1}{1-\alpha} \left(\frac{V_\zeta}{S_\zeta} \right)^{1-\alpha} \right] \\ &= \mathbf{E}_{\mathbf{Q}^S} \left[\frac{S_\zeta^{-\alpha}}{H_\zeta} u\left(\frac{V_\zeta}{S_\zeta}\right) \right] = \lambda' \mathbf{E}_{\mathbf{Q}^S} [u(V_\zeta^S)], \end{aligned}$$

where the last equality follows from the first order condition $u'(S_\zeta) = S_\zeta^{-\alpha} = \lambda' H_\zeta$.

Under the new probability \mathbf{Q}^S , the problem is to find an optimal portfolio in the numéraire (S), that maximizes the criterion

$$\max_{V_t^S} \mathbf{E}_{\mathbf{Q}^S} \{u(V_\zeta^S), V^S \text{ martingale and } V_0^S = x\}.$$

⁵Note that since the market is complete and the utility function u is complete, it is well known that

$$\mathbf{E}[u(V_\zeta^\lambda)] - \mathbf{E}[u(V_\zeta^{\lambda'})] \leq \mathbf{E}[u'(V_\zeta^{*\lambda})(V_\zeta^{*\lambda} - V_\zeta^{\lambda'})] = \lambda \mathbf{E}[H_\zeta (V_\zeta^{*\lambda} - V_\zeta^{\lambda'})] = \lambda [H_0 (V_0^{*\lambda} - V_0^{\lambda'})] = 0.$$

3. Application to portfolio insurance: a “general” constrained optimization problem

As noted in Section 2.1, the optimal strategy in this case is to do nothing. Here the change of probability seems to be irrelevant. However, we will see in the next section how usefully it transforms the classic portfolio selection problem into a constrained martingale one. Thus, the preceding results can be applied to derive an optimal solution to the problem.

3.2 “Constrained” problem: change of probability and characterization of the optimal solution

We characterize in this section the optimal solution to the problem of maximizing utility from terminal wealth, for an agent subject to an American constraint. We only consider the case of a CRRA utility function u . The extended result for general utility functions is given in [EJL02] by N. El Karoui, M. Jeanblanc and V. Lacoste.

Power utility functions

Let us address the following constrained decision problem, under the assumption that the utility function u is of power type:

$$\max_{V_t} \mathbf{E}\{u(V_\zeta), \text{ under the constraints } V_t \geq K_t \ \forall t \in [0, \zeta] \text{ and } V_0 = x\},$$

over all self-financing portfolios.

We make the plausible assumptions that $X_t := \frac{K_t}{S_t}$ is an optional process of class (\mathcal{D}) , and continuous in expectation w.r.t. any sequence of stopping times.

Under the new probability \mathbf{Q}^S , the problem is to find an optimal portfolio in the numéraire (S) , that maximizes the criterion

$$\max_{V_t^S} \mathbf{E}_{\mathbf{Q}^S}\{u(V_\zeta^S), \text{ subject to } V_t^S \text{ martingale} \geq \frac{K_t}{S_t} = X_t \text{ and } V_0^S = x\}. \quad (10)$$

Since the market is assumed to be complete, each martingale can be represented as a self-financing portfolio V_t^S .

In this form, the floor process X_t is not a super-martingale. However as already mentioned in Section 2.1, the problem is equivalent to an obstacle one associated with the Snell envelope of X : $Z^X = \text{esssup}_{\tau \in \mathcal{T}_{t,\zeta}} \mathbf{E}[X_\tau | \mathcal{F}_t]$. The only difference with the previous problem is that this one does not have any solution unless $x \geq Z_0^X$.

In view of Remark 2.1, Z^X admits the representation $Z_t^X = \mathbf{E}(\sup_{t \leq u \leq \zeta} L_u^X | \mathcal{F}_t)$, where L^X is a real progressively measurable process, with upper-right continuous paths. Hence thanks to the martingale property of the portfolios written in the numéraire S , Theorem 2.2 can be applied to find the optimal martingale.

Assuming that $x = Z_0^X$, the solution is given by

$$M_t^{X,\perp} = \mathbf{E}_{\mathbf{Q}^S}[L_{0,\zeta}^{X,*} | \mathcal{F}_t] = \mathbf{E}_{\mathbf{Q}^S}\left[\sup_{0 \leq u \leq t} L_u^X | \mathcal{F}_t\right] \quad 0 \leq t \leq \zeta.$$

If we come back to the initial problem, the associated optimal portfolio can be simply expressed as $V_t^* = S_t M_t^{X,\perp}$.

3. Application to portfolio insurance: a “general” constrained optimization problem

Note that even if the utility function u is given, the solution does not depend on the form of u . Hence the influence of u has been minimized here, since it has just served to find the optimal portfolio S solving the non-constrained problem.

For general utility functions, the linear property of the optimal solution of problem (9) fails to be true: it now depends on the initial wealth value. The paper [EJL02] suggests in this case a solution using similar techniques⁶.

Links with the example of Section 2.3.1

Let \mathbf{P} be the historical probability and H_t the state price process defined from the interest rate r and the risk prime λ in a complete market:

$$\frac{dH_t}{H_t} = -r dt - \lambda d\widehat{W}_t,$$

where \widehat{W}_t is a \mathbf{P} -Brownian motion.

As usual, we introduce the risk neutral probability \mathbf{Q} with density $\exp(-\lambda\widehat{W}_\zeta - \frac{\lambda^2}{2}\zeta)$ w.r.t. \mathbf{P} , so that $W_t = \widehat{W}_t - \lambda t$ defines a \mathbf{Q} -Brownian motion.

Using previous notation, the terminal value of the optimal portfolio (without constraint) is

$$S_\zeta = \lambda' H_\zeta^{-\frac{1}{\alpha}} = \exp(r\zeta + \frac{\lambda}{\alpha} W_\zeta - \frac{1}{2}(\frac{\lambda}{\alpha})^2 \zeta).$$

Within this framework, the probability measure \mathbf{Q}^S is given by

$$\frac{d\mathbf{Q}^S}{d\mathbf{P}} = H_\zeta S_\zeta = e^{-r\zeta} S_\zeta \frac{d\mathbf{Q}}{d\mathbf{P}},$$

and $W_t^S := W_t - \frac{\lambda}{\alpha} t$ is a \mathbf{Q}^S -Brownian motion.

Since $e^{-rt} S_t$ is the \mathbf{Q}^S -martingale density w.r.t. \mathbf{Q} , $(e^{-rt} S_t)^{-1}$ is a \mathbf{Q}^S -martingale of the form $\exp(-\frac{\lambda}{\alpha} W_t^S - \frac{1}{2}(\frac{\lambda}{\alpha})^2 t)$.

Let us come back to the portfolio insurance problem and assume that $\zeta = \infty$ and $K_t = K$ in (10). Then the floor process $X_t = \frac{K}{S_t}$ is a \mathbf{Q}^S -martingale denoted by Z_t :

$$\frac{dZ_t}{Z_t} = -r dt - \frac{\lambda}{\alpha} dW_t^S, \quad Z_0 = K.$$

According to Proposition 2.1 of Section 2.3.1, $Z_t = \frac{\gamma-1}{\gamma} \mathbf{E}[Z_{t,\infty}^* | \mathcal{F}_t]$ and the optimal strategy takes the following form

$$\begin{aligned} M_t^\perp &= \frac{\gamma-1}{\gamma} \mathbf{E}[Z_{0,\infty}^* | \mathcal{F}_t] = \frac{\gamma-1}{\gamma} \sup_{0,t} Z_u \left[\frac{1}{\gamma-1} \left(\frac{Z_t}{\sup_{0,t} Z_u} \right)^\gamma + 1 \right] \\ &= \frac{\gamma-1}{\gamma} \frac{K}{\inf_{0,t} S_u} \left[\frac{1}{\gamma-1} \left(\frac{\inf_{0,t} S_u}{S_t} \right)^\gamma + 1 \right]. \end{aligned}$$

⁶Representing martingales as a conditional expectation of the terminal value of an increasing process plays a key role in the resolution of the problem.

The optimal solution V_t^* of the initial problem is then given by

$$V_t^* = S_t M_t^\perp = \frac{\gamma - 1}{\gamma} K \frac{S_t}{\inf_{0,t} S_u} \left[\frac{1}{\gamma - 1} \left(\frac{S_t}{\inf_{0,t} S_u} \right)^{-\gamma} + 1 \right] = \Psi \left(\frac{S_t}{\inf_{0,t} S_u} \right). \quad (11)$$

For the sake of simplicity, we set $s_t^* = \inf_{0,t} S_u$ in the sequel. Since V_t^* is a self-financing portfolio, its finite-variation part under the risk-neutral probability \mathbf{Q} is in $rV_t^* dt$. It remains to identify its martingale part using Itô formula. The dynamics of V_t^* is then driven by the following equation:

$$\begin{aligned} dV_t^* &= rV_t^* dt + \Psi' \left(\frac{S_t}{s_t^*} \right) \frac{1}{s_t^*} \left(\frac{\lambda}{\alpha} S_t dW_t \right) \\ &= rV_t^* dt + \frac{\gamma - 1}{\gamma} K \left[1 - \left(\frac{S_t}{s_t^*} \right)^{-\gamma} \right] \frac{1}{s_t^*} \left(\frac{\lambda}{\alpha} S_t dW_t \right) \\ &= rV_t^* dt + \frac{\gamma - 1}{\gamma} K \frac{\lambda}{\alpha} \left(\frac{S_t}{\inf_{0,t} S_u} \right) \left[1 - \left(\frac{S_t}{\inf_{0,t} S_u} \right)^{-\gamma} \right] dW_t. \end{aligned} \quad (12)$$

Note that in the particular case where $S_t = \inf_{0,t} S_u$, $V_t^* = \frac{\gamma - 1}{\gamma} K \left[\frac{1}{\gamma - 1} + 1 \right] = K$ and the amount of risky assets is null. Hence the optimal strategy hits the floor K whenever S_t reaches its minimum.

4 Conclusion

The paper proposes a new approach to the classic utility maximization problem with American constraints, based on concave stochastic ordering. As we have seen, much of the appeal of stochastic dominance approach lies in its avoidance of arbitrary assumptions regarding the form of the utility function. It does not impose any explicit specification of the agent's utility function. Note that stochastic dominance criteria could be used here since we have considered a new environment in which self-financing portfolios are martingales.

Moreover, the optimization essentially lies on a key result of the working paper [EKM04], which represents any super-martingale of class $(\mathcal{D})^7$ as a conditional expectation of a running supremum process.

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⁷Recall that a super-martingale is said to be of class (\mathcal{D}) if it is bounded or more generally dominated by a uniformly integrable martingale.

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