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# BSDEs and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations

N. El-Karoui[a], S. Hamadène[b,*]

[a] Ecole Polytechnique, 91128 Palaiseau, France
[b] Laboratoire de Statistique et Processus, Université du Maine, 72085 Le Mans Cedex 9, France

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**Abstract**

We deal with the risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations. Using backward stochastic differential equations we show the existence of an optimal control and, a saddle-point and an equilibrium point for respectively the zero-sum and nonzero-sum games.

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*Keywords:* Backward SDEs; Risk-sensitive control; Zero-sum game; Nonzero-sum game; Optimal control; Saddle point; Equilibrium point

## 0. Introduction

Let us consider a controlled system whose state evolution is described by a process $x = (x_t)_{t \leqslant 1}$ which is the solution of the following stochastic functional differential equation (or functional diffusion),

$$\begin{cases} \mathrm{d}x_t = f(t, x., u_t)\,\mathrm{d}t + \sigma(t, x.)\,\mathrm{d}B_t, \quad t \leqslant 1, \\ x_0 = 0, \end{cases} \tag{1}$$

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* Corresponding author.

*E-mail addresses:* elkaroui@cmapx.polytechnique.fr (N. El-Karoui), said.hamadene@univ-lemans.fr (S. Hamadène).

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where $B = (B_t)_{t \leqslant 1}$ is a standard Brownian motion and $u = (u_t)_{t \leqslant 1}$ is an admissible control process.

In *risk-sensitive control problems*, it is taken into account the attitude with respect to risk of the controller in choosing an appropriate criterion to minimize, and which is of exponential type. Namely it is given by

$$J(u) = E\left[\exp\left\{\theta\left(\int_0^1 h(s, x., u_s)\,\mathrm{d}s + \xi\right)\right\}\right]. \tag{2}$$

Indeed, if we set $G(u) := \int_0^1 h(s, x., u_s)\,\mathrm{d}s + \xi$ and $\gamma_\theta(u) := \theta^{-1} \ln E[\exp(\theta G)]$ then $\gamma_\theta(u) \sim E[G(u)] + \frac{\theta}{2}\operatorname{var}(G)$, provided that $\theta \operatorname{var}(G)$ is small ($\operatorname{var}(G)$ is the variance of $G$). On the other hand, roughly speaking, minimizing $J(u)$ is equivalent to minimize $\gamma_\theta(u)$. Therefore if $\theta < 0$ (resp. $\theta > 0$) then variations of $G$, which create the situation of risk, improve (resp. worsen) the criterion. So, an economist would term a *risk seeking* (resp. *risk averse*) attitude on the part of the optimizer if $\theta < 0$ (resp. $\theta > 0$). Now, the *risk neutral* attitude of the optimizer corresponds to $J(u) = E[G(u)]$ since $\gamma_\theta(u) \to E[G(u)]$ as $\theta \to 0$ (we can see e.g. Bertsekas, 1976 for more details on this subject).

Since the early work of Jacobson (1973), introducing the risk-sensitive control problems, followed by Whittle (1974) and many others, among them Bensoussan, Elliott, Fleming, Nagai, etc. (see Bensoussan et al., 1998 and the references therein), there has been many lines of research on this subject. This concerns at least the following topics:

(i) Bellman equation and, existence and characterization of an optimal control.
(ii) Existence of the value function and the study of cases where no breaking down occurs.
(iii) Asymptotic behavior of the value function when the white noise intensity tends to 0.
(iv) Risk-sensitive optimal control with partial observation.

However, in the risk-sensitive control problems with full observation, theses works concern only the Markovian frame (Bensoussan and Nagai, 1997; Bensoussan et al., 1998; Dupuis and McEneaney, 1996; Fleming and McEneaney, 1995; Nagai, 1996), that is, at any time $t$ the functions $f$ and $h$ of (1) and (2) depend on $x$ only by $x_t$ and not on the path of $x$ up to $t$, which is called the functional frame. Basically this is due to the fact that these authors, to handle the considered problems, use approaches which are more or less linked to partial differential equations (PDEs in short). However it is well known that PDEs are not an appropriate tool for dealing with control problems of stochastic functional differential equations. As we will see it later, the suitable tool is the notion of backward stochastic differential equations (BSDEs in short).

Nonlinear BSDEs have been first introduced by Pardoux and Peng (1990), who proved the existence and uniqueness of a solution under suitable assumptions on the coefficient and the terminal value of the BSDE. Their aim was to give a probabilistic interpretation of a solution of second order quasilinear PDE. Since, these equations have gradually become an important mathematical tool which is encountered in many fields such as mathematical finance (El-Karoui et al., 1997; Buckdahn and Hu, 1998; Cvitanic and Karatzas, 1996), stochastic games and optimal control (Hamadène and Lepeltier, 1995a, b; Hamadène et al., 1997), partial differential equations (Pardoux and Peng, 1992; Pardoux, 1999; Peng, 1991).

In this paper, mainly, we deal with the risk-sensitive control and zero-sum game (which we describe below) problems of functional diffusions. The main tool is the notion of BSDEs which has been already used in *risk neutral* optimal stochastic control and game problems (see e.g. Hamadène and Lepeltier, 1995a, b; Hamadène et al., 1997). According to our knowledge risk-sensitive control and game problems of stochastic functional DEs have not been studied yet.

The problems which we consider here rise naturally in mathematical finance. Indeed, the optimal investment model, when the factors which determine the performance of the market are functional diffusions and the utility function is of power type, turns into a risk-sensitive control problem (see e.g. Fleming and Sheu, 2002; Kuroda and Nagai, 2002). Another domain where our models could have an application is linked to the risk measure theory and insurance of derivatives in illiquid markets (Barrieu and El-Karoui, 2002a, b). Finally when $\sigma = \epsilon I$, $h = 0$, $\theta < 0$ and $\xi$ is the first exiting time of the functional diffusion $(x_t)_{t \leqslant 1}$ from a set $A$, we obtain the "escape criterion" introduced by Dupuis and McEneaney (1996) (see therein the examples which motivate the consideration of such a criterion).

The paper is organized as follows:

In Section 1, we give the precise statement of the risk-sensitive control problem of stochastic functional differential equations.

Section 2 is devoted to the study of a special BSDE which is encountered both in risk-sensitive control and zero-sum game problems of functional diffusions. We show existence of a solution for such an equation.

In Section 3, the link between the costs $J(u)$ of (2) and the BSDE studied in Section 2 is established and the *value function* of the control problem is characterized. Moreover we prove the existence of an optimal control and we give its expression.

In Section 4 we deal with the zero-sum game problem of stochastic functional DEs. Let us describe it briefly.

Assume, instead of having one acting controller, we have two of them, say $c_1$ and $c_2$, and their advantages are antagonistic. The dynamic of the system in this case is

$$\begin{cases} \mathrm{d}x_t = f(t, x., u_t, v_t)\,\mathrm{d}t + \sigma(t, x.)\,\mathrm{d}B_t, \qquad t \leqslant 1, \\ x_0 = 0. \end{cases} \tag{3}$$

The control process for $c_1$ (resp. $c_2$) is $u = (u_t)_{t \leqslant 1}$ (resp. $v = (v_t)_{t \leqslant 1}$). The cost functional $J(u, v)$, which is a cost (resp. reward) for $c_1$ (resp. $c_2$) and which $c_1$ (resp. $c_2$) looks for to minimize (resp. maximize), is given by:

$$J(u, v) = E\left[\exp\left\{\theta\left(\int_0^1 h(s, x., u_s, v_s)\,\mathrm{d}s + \xi\right)\right\}\right]. \tag{4}$$

The problem is to find a saddle point (a fair strategy) for the controllers $c_1$ and $c_2$, that is an admissible control $(u^*, v^*)$ such that $J(u^*, v) \leqslant J(u^*, v^*) \leqslant J(u, v^*)$ for any $(u, v)$.

We solve completely this problem under the natural Isaacs'condition and we give the expression of the saddle point, furthermore we characterize the *upper* and *lower values* of this zero-sum game.

In the last section we consider the nonzero-sum risk-sensitive game problem which, in few words, can be described as follows:

Assume we have only two players $c_1$ and $c_2$ who intervene on a system whose state evolution is $x = (x_t)_{t \leqslant 1}$ solution of (3). The player $c_1$ (resp. $c_2$) acts with the control $u$ (resp. $v$). Since the control is not free then it corresponds to $c_1$ (resp. $c_2$) a cost $J_1(u, v)$ (resp. $J_2(u, v)$) whose expression is of type (4). The problem is to find an equilibrium point for the game, that is, an admissible pair of controls $(u^*, v^*)$ for the players such that $J_1(u^*, v^*) \leqslant J_1(u, v^*)$ and $J_2(u^*, v^*) \leqslant J_2(u^*, v)$ for any $(u, v)$.

We show that the resolution of this problem turns into the resolution of its associated multidimensional BSDE. In the Markovian frame, and under some regularity assumptions on the data of the game, this latter BSDE is solved and the nonzero-sum game has an equilibrium point for which we give the expression.

## 1. Statement of the risk-sensitive control problem

Throughout this paper $(\Omega, \mathscr{F}, P)$ is a fixed probability space on which is defined a standard $d$-dimensional Brownian motion $B = (B_t)_{t \leqslant 1}$ whose natural filtration is $(F_t)_{t \leqslant 1}$ i.e. $\forall t \leqslant 1$, $F_t = \sigma\{B_s, s \leqslant t\}$. On the other hand, let $\mathscr{P}$ be the $\sigma$-algebra on $[0,1] \times \Omega$ of $F_t$-progressively measurable processes, let $H^{2,k}$ be the set of $\mathscr{P}$-measurable processes $a = (a_t)_{t \leqslant 1}$ with values in $R^k$ such that $E[\int_0^1 |a_s|^2 \, \mathrm{d}s] < \infty$. Finally, let $\mathscr{B}$ be the set of $\mathscr{P}$-measurable uniformly bounded processes $(Y_t)_{t \leqslant 1}$ i.e. there exists a constant $C \geqslant 0$ such that $P$-a.s., $|Y_t| \leqslant C, \forall t \leqslant 1$.

Let $\mathscr{C}$ be the set of continuous functions $\bar{w}$ from $[0,1]$ into $R^d$ endowed with the uniform norm $\|.\|$ and let $\sigma$ be a function from $[0,1] \times \mathscr{C}$ into $R^{d \times d}$ such that:

(A1.1) $\sigma$ is $\mathscr{P}$-measurable i.e. for any continuous $\mathscr{P}$-measurable process $p = (p_t)_{t \leqslant 1}$ the process $(\sigma(t, p))_{t \leqslant 1}$ is $\mathscr{P}$-measurable.

(A1.2) there exists a constant $k$ such that:

- (a) $\forall t \in [0,1]$, $\forall \bar{w}, \bar{w}' \in \mathscr{C}$, $|\sigma(t, \bar{w}) - \sigma(t, \bar{w}')| \leqslant k \|\bar{w} - \bar{w}'\|_t$ where $\|\bar{w}\|_t = \sup_{s \leqslant t} |\bar{w}(s)|$, $t \leqslant 1$.
- (b) $\sigma$ is invertible and its inverse $\sigma^{-1}$ satisfies $|\sigma^{-1}(t, \bar{w})| \leqslant k(1 + \|\bar{w}\|_t^{\delta})$ for some constant $\delta \geqslant 0$.
- (c) $\forall t \in [0,1]$ and $\bar{w} \in \mathscr{C}$, $|\sigma(t, \bar{w})| \leqslant k(1 + \|\bar{w}\|_t)$.

Let $x = (x_t)_{t \leqslant 1}$ be the unique process solution of the following stochastic functional differential equation (which exists since $\sigma$ satisfies conditions (A1.1)–(A1.2) above (cf. Karatzas and Shreve, 1991; Revuz and Yor, 1991)):

$$\begin{cases} \mathrm{d}x_t = \sigma(t, x) \, \mathrm{d}B_t, & t \leqslant 1, \\ x_0 = x \in R^d. \end{cases} \tag{5}$$

It is well known that $x$ has moments of any order i.e. $\forall n \geqslant 1$, $E[\|x\|_1^n] \leqslant C_n$.

Let us now consider $U$ a compact metric space and $\mathscr{U}$ the set of $\mathscr{P}$-measurable processes $u = (u_t)_{t\leqslant 1}$ with values in $U$. Hereafter $\mathscr{U}$ is called the set of admissible controls.

Let $f$ and $h$ be two measurable functions from $[0,1] \times \mathscr{C} \times U$ into $R^d$ and $R^+$ respectively such that:

(A1.3) $f$ and $h$ are $\mathscr{P}$-measurable i.e. for any $u \in \mathscr{U}$ the processes $(f(t,x,u_t))_{t\leqslant 1}$ and $(h(t,x,u_t))_{t\leqslant 1}$ are $\mathscr{P}$-measurable.

(A1.4) $\forall t \in [0,1]$, $\bar{w} \in \mathscr{C}$, the mappings $f(t,\bar{w},.)(u) := f(t,\bar{w},u)$ and $h(t,\bar{w},.)(u) := h(t,\bar{w},u)$ are continuous on $U$. Moreover $h$ is bounded and $f$ is of linear growth i.e. there exists a constant $k$ such that $|f(t,\bar{w},u)| \leqslant k(1 + \|\bar{w}\|_t)$, $\forall (t,\bar{w},u) \in [0,1] \times \mathscr{C} \times U$.

For $u \in \mathscr{U}$, let $P^u$ be the measure on $(\Omega, \mathscr{F})$ defined as follows:

$$\mathrm{d}P^u = \mathscr{E}_1\left(\int_0^{\cdot} \sigma^{-1}(s,x) f(s,x,u_s)\,\mathrm{d}B_s\right)\mathrm{d}P$$

where for any $(F_t, P)$-continuous local martingale $M = (M_t)_{t\leqslant 1}$, $\mathscr{E}(M) := (\exp\{M_t - \frac{1}{2}\langle M\rangle_t\})_{t\leqslant 1}$.

In taking into account the assumptions (A1.2) and (A1.4) on $\sigma$ and $f$ we can infer that $P^u$ is a probability on $(\Omega, \mathscr{F})$ (see Appendix A for the proof).

Now for $t \leqslant 1$, let $B_t^u = B_t - \int_0^t \sigma^{-1}(s,x) f(s,x,u_s)\,\mathrm{d}s$. As $P^u$ is a probability, it is well known, from Girsanov's theorem (Girsanov, 1960), that the process $(B_t^u)_{t\leqslant 1}$ is a $(F_t, P^u)$-Brownian motion and $(x_t)_{t\leqslant 1}$ satisfies:

$$\begin{cases} \mathrm{d}x_t = f(t,x,u_t)\,\mathrm{d}t + \sigma(t,x)\,\mathrm{d}B_t^u, \quad t \leqslant 1, \\ x_0 = x \in R^d. \end{cases} \tag{6}$$

In general the process $x = (x_t)_{t\leqslant 1}$ is not adapted to the filtration generated by the Brownian motion $(B_t^u)_{t\leqslant 1}$. Thereby $x = (x_t)_{t\leqslant 1}$ is called a weak solution for the standard SFDE (6).

Now let $\xi$ be a bounded, $F_1$-measurable random variable and $J(u)$, $u \in \mathscr{U}$, be the cost functional defined as follows:

$$J(u) = E^u\left[\exp\theta\left\{\int_0^1 h(s,x,u_s)\,\mathrm{d}s + \xi\right\}\right] \tag{7}$$

where $E^u$ is the expectation under $P^u$. The quantity $J(u)$ is the cost that a controller has to pay for his action on a system whose state evolution has the same law as the one of $(x_t)_{t\leqslant 1}$ under $P^u$. Since the control is not free then the problem is to find an admissible control $u^*$ such that $J(u^*) \leqslant J(u)$, $\forall u \in \mathscr{U}$.

In the expression (7) of $J(u)$, $h$ (resp. $\xi$) stands for the instantaneous (resp. terminal) cost. The parameter $\theta$ expresses the attitude of the controller with respect to risk. He/she is *risk-averse* (resp. *seeking*) if $\theta > 0$ (resp. $\theta < 0$). However the resolution of the problem is the same in both cases ($\theta > 0$ or $\theta < 0$), hence for the sake of simplicity, from now on, we assume $\theta = 1$ in (7).

Now it is well-known that the stochastic Pontryagin's principle leads to necessary conditions which insure that a control is optimal (see e.g. Bensoussan, 1988). On the other hand, the application of the stochastic dynamic programming principle for functional diffusions with risk-neutral costs established by Davis and Varaiya (1973) gives necessary and sufficient conditions for a control to be optimal. Those conditions are expressed via BSDEs even it is not explicitly said. Roughly speaking, it has been stated that if the BSDE associated with the optimal payoff has a solution then an optimal control exists which, moreover, is constructed by means of the solution of the BSDE. The coefficient of that BSDE is the infimum, with respect to $u \in U$, of the Hamiltonian function associated with the control problem.

This idea still valid when the costs are of risk-sensitive types, just the coefficient changes.

So to begin with we are going to deal with a specific types of BSDEs which are encountered in risk-sensitive control problems.

## 2. BSDEs related to risk-sensitive problems

First, let us recall the notion of a solution of a BSDE. Let $\Psi$ be a function from $[0,1] \times \Omega \times R^{p+p\times d}$ into $R^d$ such that for any $(y,z) \in R^{p+p\times d}$, the process $(\Psi(t,\omega,y,z))_{t\leqslant 1}$ is $\mathscr{P}$-measurable. A solution of the backward stochastic differential equation whose *coefficient* is $\Psi$ and *terminal value* $\xi$ is a pair of $\mathscr{P}$-measurable processes $(Y,Z)$ such that:

$$\begin{cases} E\left[\sup_{t\leqslant 1} |Y_t|^2 + \int_0^1 |Z_s|^2\,\mathrm{d}s\right] < \infty, \\ \forall t \leqslant 1,\ Y_t = \xi + \int_t^1 \Psi(s,Y_s,Z_s)\,\mathrm{d}s - \int_t^1 Z_s\,\mathrm{d}B_s. \end{cases} \tag{8}$$

In the case when $\Psi$ is Lipschitz in $(y,z)$ uniformly with respect to $(t,\omega)$ and $(\Psi(t,0,0))_{t\leqslant 1}$ belongs to $H^{2,p}$, Pardoux and Peng (1990) have proved the existence and uniqueness of $(Y,Z)$ which satisfies (8).

We are now going to introduce a specific type of BSDE which intervenes in risk-sensitive optimal control and zero-sum game problems. For this, let $\Phi$ be a map from $[0,1] \times \Omega \times R^d$ into $R$ which satisfies:

(A2.1) $\forall z \in R^d$, the process $(\Phi(t,\omega,z))_{t\leqslant 1}$ is $\mathscr{P}$-measurable; furthermore $(\Phi(t,\omega,0))_{t\leqslant 1}$ belongs to $\mathscr{B}$.

(A2.2) there exists a $\mathscr{P}$-measurable and nonnegative process $(k_t)_{t\leqslant 1}$ which belongs to $L^2([0,1] \times \Omega, \mathrm{d}t \otimes \mathrm{d}P)$ such that:

$$\forall t \in [0,1],\ \forall z,z' \in R^d,\ |\Phi(t,z) - \Phi(t,z')| \leqslant k_t|z - z'|, \quad P\text{-a.s.}$$

Let us now consider the following specific BSDE associated with $(\Phi + \frac{1}{2}|z|^2, \xi)$ that is,

$$\begin{cases} Y \in \mathcal{B}, \quad Z \in H^{2,d}, \\ -\mathrm{d}Y_t = \{\Phi(t, Z_t) + \frac{1}{2}|Z_t|^2\}\,\mathrm{d}t - Z_t\,\mathrm{d}B_t, \quad t \leqslant 1, \\ Y_1 = \xi. \end{cases} \tag{9}$$

As we will show it later, the cost functional $J(u)$, $u \in \mathcal{U}$, and the optimal control can be expressed by means of processes $(Y, Z)$ solutions of BSDEs which are of type (9). So first let us prove that (9) has a solution with some specific properties.

To begin with, we recall the following result of Kobylanski (2000) on the one hand and Lepeltier and San Martin (1998) on the other hand, which is related to one dimensional BSDEs whose coefficients are quadratic in $z$.

Assume, in addition, the function $\Psi$ of (8) takes its values in $R$ ($p=1$) and satisfies:

(A2.3) for any $(t, \omega)$, $\Psi$ is continuous with respect to $(y, z)$ and for any $(y, z)$, the process $(\Psi(t, y, z))_{t \leqslant 1}$ is $\mathcal{P}$-measurable.

(A2.4) $P$-a.s., $|\Psi(t, y, z)| \leqslant C(1 + |y| + |z|^2)$, $\forall (t, y, z) \in [0, 1] \times R^{1+d}$. We say that $\Psi$ is of linear (resp. quadratic) growth with respect to $y$ (resp. $z$).

Then we have the following theorem which provides a solution for the BSDE associated with $(\Psi, \xi)$.

**Theorem 2.1** (Kobylanski, 2000; Lepeltier and San Martin, 1998). *There exists a $\mathcal{P}$-measurable process $(Y, Z)$ with values in $R^{1+d}$ such that*:

$$\begin{cases} (Y, Z) \in \mathcal{B} \times H^{2,d}, \\ -\mathrm{d}Y_t = \Psi(t, Y_t, Z_t)\,\mathrm{d}t - Z_t\,\mathrm{d}B_t, \quad t \leqslant 1, \\ Y_1 = \xi. \end{cases} \tag{10}$$

*Moreover (10) has a maximal bounded solution, i.e., if $(\bar{Y}, \bar{Z})$ is another solution which belongs to $\mathcal{B} \times H^{2,d}$, then we have, $P$-a.s. $Y \geqslant \bar{Y}$.*

The following result is of crucial role in the proof of the existence of a solution for (9) since it allows the comparison of solutions of BSDEs of types (10).

**Theorem 2.2** (Lepeltier and San Martin, 1998). *Let $\Lambda$ be a $\mathcal{P} \otimes B(R^{1+d})$-measurable function with values in $R$. Let $(Y', Z')$ (resp. $(Y, Z)) \in \mathcal{B} \times H^{2,d}$ be a solution (resp. the maximal solution) of the BSDE associated with $(\Lambda, \xi)$ (resp. $(\Psi, \xi)$). Assume that $\Lambda \leqslant \Psi$. Then $Y' \leqslant Y$ $P$-a.s.*

We are now ready to give the main result of this part which states the existence of a solution $(Y, Z) \in \mathcal{B} \times H^{2,d}$ for the BSDE (9). Let us stress that this equation is not of type (10) since the process $(k_t)_{t \leqslant 1}$ of the condition (A2.2) is not bounded.

**Theorem 2.3.** *There exists a* $\mathscr{P}$*-measurable process* $(Y,Z)$ *with values in* $R^{1+d}$ *such that*:

$$\begin{cases} (Y,Z)\in\mathscr{B}\times H^{2,d}, \\ -\mathrm{d}Y_t=\{\Phi(t,Z_t)+\frac{1}{2}|Z_t|^2\}\,\mathrm{d}t-Z_t\,\mathrm{d}B_t, \quad t\leqslant 1, \\ Y_1=\xi. \end{cases} \tag{11}$$

**Proof.** It will be divided into several steps.

*Step* 1: Construction of the approximating sequence. Some properties.

For any $n,m\in N$, let $\Phi^{n,m}(t,\omega,z)$ be the function defined as follows:

$$\Phi^{n,m}(t,\omega,z)=\Phi^{+}(t,\omega,z)1_{[k_t\leqslant n]}-\Phi^{-}(t,\omega,z)1_{[k_t\leqslant m]}, \quad \forall(t,\omega,z),$$

where $\Phi^{+}:=\max(\Phi,0)$ and $\Phi^{-}=\max(-\Phi,0)$. So it is easily seen that $\Phi^{n,m}$ is Lipschitz with respect to $z$ and satisfies $\Phi^{n,m}\leqslant\Phi^{n+1,m}$, $\Phi^{n,m}\geqslant\Phi^{n,m+1}$ for any $n,m\geqslant 0$. Moreover we have $\lim_{m\mapsto\infty}\lim_{n\mapsto\infty}\Phi^{n,m}(t,\omega,z)=\Phi(t,\omega,z)$, for any $(t,\omega,z)\in[0,1]\times\Omega\times R^d$.

Now for $n,m\geqslant 0$, let $(Y^{n,m},Z^{n,m})$ be the maximal bounded solution of the BSDE associated with $(\Phi^{n,m}(t,z)+\frac{1}{2}|z|^2,\xi)$ i.e.

$$\begin{cases} (Y^{n,m},Z^{n,m})\in\mathscr{B}\times H^{2,d}, \\ -\mathrm{d}Y_t^{n,m}=\left\{\Phi^{n,m}(t,Z_t^{n,m})+\dfrac{1}{2}\,|Z_t^{n,m}|^2\right\}\mathrm{d}t-Z_t^{n,m}\,\mathrm{d}B_t, \quad t\leqslant 1, \\ Y_1^{n,m}=\xi \end{cases}$$

and is a maximal solution in the sense of Theorem 2.1.

The comparison of the coefficients $\Phi^{n,m}(t,z)+\frac{1}{2}|z|^2$ implies that, through Theorem 2.2, for any $n,m\geqslant 0$, $Y^{n,m}\leqslant Y^{n+1,m}$ and $Y^{n,m+1}\leqslant Y^{n,m}$.

(a) There exists a positive constant $C$ which does not depend on $n,m$ such that $P$-a.s., $|Y^{n,m}|\leqslant C$.

Indeed, let $\bar{Y}_t^{n,m}=\exp(Y_t^{n,m})$, $n,m\geqslant 0$ and $t\leqslant 1$. Using Itô's formula we obtain

$$\begin{aligned} \mathrm{d}\bar{Y}_t^{n,m}&=-\bar{Y}_t^{n,m}\Phi^{n,m}(t,Z_t^{n,m})\,\mathrm{d}t+\bar{Y}_t^{n,m}Z_t^{n,m}\,\mathrm{d}B_t \\ &=\{-\bar{Y}_t^{n,m}\Phi^{n,m}(t,0)-\bar{Y}_t^{n,m}(\Phi^{n,m}(t,Z_t^{n,m})-\Phi^{n,m}(t,0))\}\,\mathrm{d}t+\bar{Y}_t^{n,m}Z_t^{n,m}\,\mathrm{d}B_t \\ &=\{-\bar{Y}_t^{n,m}\Phi^{n,m}(t,0)-\bar{Y}_t^{n,m}Z_t^{n,m}\delta_t^{n,m}\}\,\mathrm{d}t+\bar{Y}_t^{n,m}Z_t^{n,m}\,\mathrm{d}B_t \end{aligned}$$

where $(\delta_t^{n,m})_{t\leqslant 1}$ is the bounded process defined as follows:

$$\delta_t^{n,m}=\begin{cases} [\Phi^{n,m}(t,Z_t^{n,m})-\Phi^{n,m}(t,0)]\dfrac{Z_t^{n,m}}{|Z_t^{n,m}|^2} & \text{if } Z_t^{n,m}\neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that for any $t \leqslant 1$ we have

$$\bar{Y}_t^{n,m} = e^{\xi} + \int_t^1 \bar{Y}_s^{n,m} \Phi^{n,m}(s,0)\,\mathrm{d}s - \int_t^1 \bar{Y}_s^{n,m} Z_s^{n,m}\,\mathrm{d}B_s^{n,m} \tag{12}$$

where $B_t^{n,m} = B_t - \int_0^t \delta_s^{n,m}\,\mathrm{d}s$, $t \leqslant 1$, is a Brownian motion under the probability $P^{n,m}$ on $(\Omega, \mathscr{F})$ whose density with respect to $P$ is $\mathscr{E}_1(\int_0^{\cdot} \delta_s^{n,m}\,\mathrm{d}B_s)$. Now since $Z^{n,m}$ belongs to $H^{2,d}$ and $Y^{n,m}$, $\delta^{n,m}$ are bounded processes then using the Burkholder–Davis–Gundy inequality (Karatzas and Shreve, 1991; Revuz and Yor, 1991) we can easily deduce that $(\int_0^t \bar{Y}_s^{n,m} Z_s^{n,m}\,\mathrm{d}B_s^{n,m})_{t \leqslant 1}$ is a $(F_t, P^{n,m})$-martingale. Taking the conditional expectation with respect to $E^{n,m}$, the expectation under $P^{n,m}$, in both sides of (12) and since $\xi$ and $(\Phi(t,0))_{t \leqslant 1}$ are bounded we obtain,

$$|\bar{Y}_t^{n,m}| \leqslant C \left\{ 1 + \int_t^1 E^{n,m}[|\bar{Y}_r^{n,m}| \,|\, F_t]\,\mathrm{d}r \right\}, \quad t \leqslant 1,$$

where $C$ is a constant which does not depend on $n, m$ and, from now on, may change from a line to another. Henceforth for any $s \geqslant t$ we have

$$E^{n,m}[|\bar{Y}_s^{n,m}| \,|\, F_t] \leqslant C \left\{ 1 + \int_s^1 E^{n,m}[|\bar{Y}_r^{n,m}| \,|\, F_t]\,\mathrm{d}r \right\}.$$

Using Gronwall's inequality and then taking $s = t$ we obtain $|\bar{Y}_t^{n,m}| \leqslant C$, which yields $P$-a.s., $Y_t^{n,m} \leqslant C$, $\forall t \leqslant 1$.

Now for $m \geqslant 0$, let $(\tilde{Y}^m, \tilde{Z}^m)$ be the solution of the following BSDE:

$$\begin{cases} E\left[ \sup_{t \leqslant 1} |\tilde{Y}_t^m|^2 + \int_0^1 |\tilde{Z}_s^m|^2\,\mathrm{d}s \right] < \infty, \\ -\mathrm{d}\tilde{Y}_t^m = -\Phi^{m-}(t, \tilde{Z}_t^m)\,\mathrm{d}t - \tilde{Z}_t^m\,\mathrm{d}B_t, \quad t \leqslant 1; \ \ \tilde{Y}_1^m = \xi, \end{cases}$$

where $\Phi^{m-}(t,z) = \Phi(t,z)^{-} 1_{[k_t \leqslant m]}$. This solution exists and is unique, through Pardoux and Peng's theorem (Pardoux and Peng, 1990), since $\Phi^{m-}(t,z)$ is Lipschitz with respect to $z$. An easy calculation shows that the component $\tilde{Y}^m$ is a uniformly bounded process since $\xi$ and $\Phi(t,0)^{m-}$ are so. Indeed,

$$\begin{aligned} -\mathrm{d}\tilde{Y}_t^m &= -\Phi^{m-}(t,0) - (\Phi^{m-}(t,\tilde{Z}_t^m) - \Phi^{m-}(t,0))\,\mathrm{d}t - \tilde{Z}_t^m\,\mathrm{d}B_t \\ &= -\Phi^{m-}(t,0)\,\mathrm{d}t - \tilde{Z}_t^m\,\mathrm{d}\tilde{B}_t, \end{aligned}$$

where $\tilde{B}_t = B_t + \int_0^t (\Phi^{m-}(s,\tilde{Z}_s^m) - \Phi^{m-}(s,0))(\tilde{Z}_s^m / |\tilde{Z}_s^m|^2) 1_{[\tilde{Z}_s^m \neq 0]}\,\mathrm{d}s$, $t \leqslant 1$, is a Brownian motion under a new probability equivalent to $P$. Now taking into account the fact that $\Phi^{m-}(t,0)$ and $\xi$ are uniformly bounded then $\tilde{Y}^m$ is so, i.e., $|\tilde{Y}^m| \leqslant \tilde{C}$ where $\tilde{C}$ is independent of $m$.

On the other hand $\Phi^{n,m} \geqslant \Phi^{m-}$ then, according to Theorem 2.2, we have, $\forall n, m \geqslant 0$, $P$-a.s., $Y_t^{n,m} \geqslant \tilde{Y}_t^m \geqslant -\tilde{C}$, $\forall t \leqslant 1$. It follows that $P$-a.s., $\forall t \leqslant 1$, $|Y_t^{n,m}| \leqslant C$ whence the desired result.

(b) There exists a constant $C$ such that $\forall n, m \geqslant 0$, $E[\int_0^1 |Z_s^{n,m}|^2\,\mathrm{d}s] \leqslant C$.

Using Itô's formula with $\bar{Y}^{n,m}$ we have

$$d(\bar{Y}_t^{n,m})^2 = 2\bar{Y}_t^{n,m}\{-\bar{Y}_t^{n,m}\Phi^{n,m}(t, Z_t^{n,m})\,\mathrm{d}t + \bar{Y}_t^{n,m} Z_t^{n,m}\,\mathrm{d}B_t\} + (\bar{Y}_t^{n,m})^2 |Z_t^{n,m}|^2\,\mathrm{d}t.$$

It implies that

$$\xi^2-(\bar{Y}_t^{n,m})^2=-2\int_t^1(\bar{Y}_s^{n,m})^2\Phi^{n,m}(s,Z_s^{n,m})\,\mathrm{d}s+2\int_t^1(\bar{Y}_s^{n,m})^2Z_s^{n,m}\,\mathrm{d}B_s$$
$$+\int_t^1(\bar{Y}_s^{n,m})^2|Z_s^{n,m}|^2\,\mathrm{d}s.$$

Hence,

$$\begin{aligned}(\bar{Y}_t^{n,m})^2&+\int_t^1(\bar{Y}_s^{n,m})^2|Z_s^{n,m}|^2\,\mathrm{d}s\\ &=\xi^2+2\int_t^1(\bar{Y}_s^{n,m})^2\Phi^{n,m}(s,Z_s^{n,m})\,\mathrm{d}s-2\int_t^1(\bar{Y}_s^{n,m})^2Z_s^{n,m}\,\mathrm{d}B_s\\ &\leqslant C\left\{1+\int_t^1k_s|Z_s^{n,m}|\,\mathrm{d}s\right\}-2\int_t^1(\bar{Y}_s^{n,m})^2Z_s^{n,m}\,\mathrm{d}B_s.\end{aligned}$$

But since $\bar{Y}^{n,m}\geqslant C>0$, $k.\in L^2(\mathrm{d}t\otimes\mathrm{d}P)$, $|ab|\leqslant\epsilon a^2+\epsilon^{-1}b^2$, $\forall a,b\in R$ and $\epsilon>0$, and $\int_0^{\cdot}(\bar{Y}_s^{n,m})^2Z_s^{n,m}\,\mathrm{d}B_s$ is an $(F_t,P)$-martingale then $E[\int_0^1|Z_s^{n,m}|^2\,\mathrm{d}s]\leqslant C$.

*Step* 2: Convergence of the sequences $(Y^{n,m})_{n,m\geqslant0}$ and $(Z^{n,m})_{n,m\geqslant0}$.

For any fixed $m$, the sequence $(Y^{n,m})_{n\geqslant0}$ is non-decreasing and is uniformly bounded then there exists a process $Y^m$ such that $P$-a.s. $\forall t\leqslant1$, $Y_t^{n,m}\to Y_t^m$ and $E[\int_0^1|Y_t^{n,m}-Y_t^m|^2\,\mathrm{d}t]\to0$ as $n\to\infty$. On the other hand, since the sequence $(Y^{n,m})_{n,m\geqslant0}$ is decreasing with respect to $m$ then the sequence $(Y^m)_{m\geqslant0}$ is decreasing and uniformly bounded hence there exists a process $Y$ such that $P$-a.s. $\forall t\leqslant1$, $Y_t^m\to Y_t$ and $E[\int_0^1|Y_t^m-Y_t|^2\,\mathrm{d}t]\to0$ as $m\to\infty$. Let us set $Y_t:=\lim_{m\to\infty}\lim_{n\to\infty}Y_t^{n,m}$, $\forall t\leqslant1$.

Now let us show the convergence of the sequence $(Z^{n,m})_{n,m\geqslant0}$. Using Itô's formula with $\bar{Y}^{n,m}$ we have,

$$d(\bar{Y}_t^{n,m}-\bar{Y}_t^{p,q})^2=2(\bar{Y}_t^{n,m}-\bar{Y}_t^{p,q})d(\bar{Y}_t^{n,m}-\bar{Y}_t^{p,q})+|\bar{Z}_t^{n,m}-\bar{Z}_t^{p,q}|^2\,\mathrm{d}t$$

where $\forall n,m\geqslant0$, $\bar{Z}_t^{n,m}:=\bar{Y}_t^{n,m}Z_t^{n,m}$ , $t\leqslant1$. It follows that,

$$\begin{aligned}E&\left[|\bar{Y}_t^{n,m}-\bar{Y}_t^{p,q}|^2+\int_t^1|\bar{Z}_s^{n,m}-\bar{Z}_s^{p,q}|^2\,\mathrm{d}s\right]\\ &=-2E\left[\int_t^1(\bar{Y}_s^{n,m}-\bar{Y}_s^{p,q})(\bar{Y}_s^{n,m}\Phi^{n,m}(s,Z_s^{n,m})-\bar{Y}_s^{p,q}\Phi^{p,q}(s,Z_s^{p,q}))\,\mathrm{d}s\right]\\ &\leqslant2E\left[\int_t^1|\bar{Y}_s^{n,m}-\bar{Y}_s^{p,q}|C\{1+k_s(|Z_s^{n,m}|+|Z_s^{p,q}|)\}\,\mathrm{d}s\right]\end{aligned}$$

$$\leqslant 2CE\left[\int_t^1 |\bar{Y}_s^{n,m} - \bar{Y}_s^{p,q}|\,\mathrm{d}s\right] + 2C\sqrt{E\left[\int_t^1 |\bar{Y}_s^{n,m} - \bar{Y}_s^{p,q}|^2 k_s^2\,\mathrm{d}s\right]}$$

$$\times\left\{\sqrt{E\left[\int_t^1 |Z_s^{n,m}|^2\,\mathrm{d}s\right]} + \sqrt{E\left[\int_t^1 |Z_s^{p,q}|^2\,\mathrm{d}s\right]}\right\}. \qquad (13)$$

But the sequence $(Z^{n,m})_{n,m\geqslant 0}$ is bounded in $H^{2,d}$, $|\bar{Y}^{n,m}| \leqslant C$ and finally the process $(k_t)_{t\leqslant 1}$ belongs to $L^2(\mathrm{d}t\otimes\mathrm{d}P)$. Therefore if we fix $m$ and take $q = m$ we deduce that, in using the dominated convergence theorem, the sequence $(\bar{Z}^{n,m})_{n\geqslant 0}$ converges in $H^{2,d}$ to a process $\bar{Z}^m \in H^{2,d}$. Moreover we have (in taking the limit as $n, p \to \infty$ in (13)),

$$E\left[\int_0^1 |\bar{Z}_s^m - \bar{Z}_s^q|^2\,\mathrm{d}s\right]$$

$$\leqslant 2C\left\{E\left[\int_0^1 |\bar{Y}_s^m - \bar{Y}_s^q|\,\mathrm{d}s\right] + \sqrt{E\left[\int_0^1 |\bar{Y}_s^m - \bar{Y}_s^q|^2 k_s^2\,\mathrm{d}s\right]}\right\}.$$

It implies that the sequence $(\bar{Z}^m)_{m\geqslant 0}$ converges also in $H^{2,d}$ to a process which we denote $\bar{Z}$. Let us set $Z = \bar{Z}/e^Y$, therefore the process $Z$ belongs to $H^{2,d}$ since $\bar{Z}$ is so and $e^Y \geqslant C > 0$.

*Step* 3: The pair of processes $(Y, Z)$ is a solution for the BSDE (11).

Let $m$ be fixed. Since the sequence $(\bar{Z}^{n,m})_{n\geqslant 0}$ converges in $H^{2,d}$ to $\bar{Z}^m$ then there is a subsequence $(\bar{Z}^{n_k^m,m})_{k\geqslant 0}$ which converges to $\bar{Z}^m$ in $H^{2,d}$ and $\bar{Z}_t^{n_k^m,m}(\omega) \to \bar{Z}_t^m(\omega)$, $\mathrm{d}t\otimes\mathrm{d}P$. In addition $\sup_{k\geqslant 0}|\bar{Z}_t^{n_k^m,m}(\omega)|$ belongs to $H^{2,d}$. It follows that, in using the dominated convergence theorem and taking into account the uniform boundedness of $Y^{n,m}$, the sequence $(Z^{n_k^m,m} = \bar{Z}^{n_k^m,m}/\bar{Y}^{n_k^m,m})_{k\geqslant 0}$ converges in $H^{2,d}$ to $Z^m := \bar{Z}^m/\bar{Y}^m$.

Now for any $k \geqslant 0$ and $t \leqslant 1$ we have

$$Y_t^{n_k^m,m} = \xi + \int_t^1 \left\{\Phi^{n_k^m,m}(s, Z_s^{n_k^m,m}) + \frac{1}{2}|Z_s^{n_k^m,m}|^2\right\}\mathrm{d}s - \int_t^1 Z_s^{n_k^m,m}\,\mathrm{d}B_s$$

$$= \xi + \int_t^1 \left\{\Phi^+(s, Z_s^{n_k^m,m})1_{[k_s\leqslant n_k^m]} - \Phi^-(s, Z_s^{n_k^m,m})1_{[k_s\leqslant m]} + \frac{1}{2}|Z_s^{n_k^m,m}|^2\right\}\mathrm{d}s$$

$$- \int_t^1 Z_s^{n_k^m,m}\,\mathrm{d}B_s.$$

But for any $t \leqslant 1$,

$$E\left[|Y_t^{n_k^m,m} - Y_t^m| + \sup_{t\leqslant 1}\left|\int_t^1 (Z_s^{n_k^m,m} - Z_s^m)\,\mathrm{d}B_s\right|\right.$$

$$\left. + \int_t^1 \| Z_s^{n_k^m,m}|^2 - |Z_s^m|^2|\mathrm{d}s\right] \to 0 \quad \text{as } k \to \infty.$$

On the other hand

$$E\left[\int_t^1 |\Phi^-(s,Z_s^{n_k^m,m})1_{[k_s\leqslant m]} - \Phi^-(s,Z_s^m)1_{[k_s\leqslant m]}|\,\mathrm{d}s\right]$$

$$\leqslant mE\left[\int_t^1 |Z_s^{n_k^m,m}-Z_s^m|\,\mathrm{d}s\right]\to 0 \quad \text{as } k\to\infty.$$

Finally $|\Phi^+(s,Z_s^{n_k^m,m})1_{[k_s\leqslant n_k^m]}|\leqslant C+k_s\sup_{k\geqslant 0}|Z_s^{n_k^m,m}|$ and then through the dominated convergence theorem we have

$$E\left[\int_t^1 |\Phi^+(s,Z_s^{n_k^m,m})1_{[k_s\leqslant n_k^m]} - \Phi^+(s,Z_s^m)|\,\mathrm{d}s\right]\to 0 \quad \text{as } k\to\infty.$$

Therefore for any $m\geqslant 0$ the pair of processes $(Y^m,Z^m)$ satisfies,

$$Y_t^m=\xi+\int_t^1\left\{\Phi^+(s,Z_s^m)-\Phi^-(s,Z_s^m)1_{[k_s\leqslant m]}+\frac{1}{2}|Z_s^m|^2\right\}\mathrm{d}s$$

$$-\int_t^1 Z_s^m\,\mathrm{d}B_s,\qquad t\leqslant 1.$$

Next, once again, since the sequence $(\bar{Z}^m)_{m\geqslant 0}$ converges in $H^{2,d}$ to $\bar{Z}$, there is a subsequence $(\bar{Z}^{m_l})_{l\geqslant 0}$ such that $\bar{Z}_t^{m_l}(\omega)\to\bar{Z}_t(\omega)$, $\mathrm{d}t\otimes\mathrm{d}P$ and $\sup_{l\geqslant 0}|\bar{Z}_t^{m_l}(\omega)|$ belongs to $H^{2,d}$. Therefore the subsequence $(Z^{m_l}=\bar{Z}^{m_l}/\bar{Y}^{m_l})_{l\geqslant 0}$ converges in $H^{2,d}$ to the process $Z$.

Now for $l\geqslant 0$ and $t\leqslant 1$ we have

$$Y_t^{m_l}=\xi+\int_t^1\left\{\Phi^+(s,Z_s^{m_l})-\Phi^-(s,Z_s^{m_l})1_{[k_s\leqslant m]}+\frac{1}{2}|Z_s^{m_l}|^2\right\}\mathrm{d}s-\int_t^1 Z_s^{m_l}\,\mathrm{d}B_s.$$

We can argue as previously in order to obtain that the process $(Y_t,Z_t)_{t\leqslant 1}$ satisfies

$$Y_t=\xi+\int_t^1\left\{\Phi^+(s,Z_s)-\Phi^-(s,Z_s)+\frac{1}{2}|Z_s|^2\right\}\mathrm{d}s-\int_t^1 Z_s\,\mathrm{d}B_s,\qquad t\leqslant 1.$$

Therefore the pair of processes $(Y,Z)$ is solution for the BSDE (11). In addition $(Y,Z)\in\mathscr{B}\times H^{2,d}$. The proof is now over. □

**Remark 2.4.** A comparison result.

Let $\xi'$ be a bounded $F_1$-measurable random variable and $\Phi'(t,\omega,z)$ be another $\mathscr{P}\otimes B(R^d)$-measurable map from $[0,1]\times\Omega\times R^d$ into $R$ such that:

(i) the process $(\Phi'(t,0))_{t\leqslant 1}$ is bounded.
(ii) $|\Phi'(t,z)-\Phi'(t,z')|\leqslant k_t|z-z'|$ for the same process $k$ as the one of $\Phi$ in (A2.2).
(iii) $\Phi'(t,z)\leqslant\Phi(t,z)$.
(iv) $\xi'\leqslant\xi$.

Let $(Y',Z')$ be the solution of the BSDE associated with $(\Phi'(t,z)+\frac{1}{2}|z|^2,\xi')$ constructed as in Theorem 2.3, then $Y'\leqslant Y$.

## 3. The risk-sensitive control problem

We are now going to use the results of the previous section to solve the risk-sensitive control problem.

First, let us introduce the Hamiltonian function $H$ associated with this control problem i.e. the function which to $(t,\bar{w},p,u)\in[0,1]\times\mathscr{C}\times R^d\times U$ associates $H(t,\bar{w},p,u):=p\sigma^{-1}(t,\bar{w})f(t,\bar{w},u)+h(t,\bar{w},u)$.

The costs $J(u), u\in\mathscr{U}$, can be expressed by means of BSDEs of type (11) in the following way:

**Proposition 3.1.** *For any $u\in\mathscr{U}$, there exists a unique process $(Y^u,Z^u)$ such that*:

$$\begin{cases} (Y^u,Z^u)\in\mathscr{B}\times H^{2,d}, \\ -\mathrm{d}Y_t^u=\{H(t,x,Z_t^u,u_t)+\frac{1}{2}|Z_t^u|^2\}\,\mathrm{d}t-Z_t^u\,\mathrm{d}B_t, \quad t\leqslant 1;\ \ Y_1^u=\xi. \end{cases} \tag{14}$$

*Moreover $J(u)=\exp\{Y_0^u\}$.*

**Proof.** The existence of the process $(Y^u,Z^u)$ is a direct consequence of Theorem 2.3 since

$$|H(t,x,p,u)-H(t,x,p',u)|\leqslant|\sigma^{-1}(t,x)f(t,x,u)|\,|p-p'|\leqslant C(1+\|x\|_t^{1+\delta})|p-p'|$$

and $(\|x\|_t^{1+\delta})_{t\leqslant 1}$ belongs to $L^2(\Omega\times[0,1],\mathscr{P},\mathrm{d}P\otimes dt)$. Let us focus on the uniqueness.

Let $(Y,Z)$ be another process with the same properties as $(Y^u,Z^u)$ and let $\bar{Y}_t=e^{Y_t}$, $t\leqslant 1$. Using Itô's formula we obtain

$$\begin{aligned}\mathrm{d}\bar{Y}_t&=-(\bar{Y}_tZ_t\sigma^{-1}(t,x)f(t,x,u_t)+\bar{Y}_th(t,x,u_t))\,\mathrm{d}t+\bar{Y}_tZ_t\,\mathrm{d}B_t\\&=-\bar{Y}_th(t,x,u_t)\,\mathrm{d}t+\bar{Y}_tZ_t\,\mathrm{d}B_t^u,\end{aligned}$$

where $B^u$ is a Brownian motion with respect to $P^u$. Now let us set $\tilde{Y}_t=\bar{Y}_t\exp\{\int_0^t h(s,x,u_s)\,\mathrm{d}s\}$, $t\leqslant 1$, then $\tilde{Y}$ is bounded since $\bar{Y}$ and $h$ (see (A1.4)) are so. On the other hand

$$\mathrm{d}\tilde{Y}_t=\tilde{Y}_tZ_t\,\mathrm{d}B_t^u,\ \ t\leqslant 1,\quad\text{and}\quad\tilde{Y}_1=\exp\left\{\xi+\int_0^1 h(s,x,u_s)\,\mathrm{d}s\right\}.$$

Therefore the local martingale $(\int_0^t\tilde{Y}_tZ_t\,\mathrm{d}B_t^u)_{t\leqslant 1}$, which is equal to $(\tilde{Y}_t-\tilde{Y}_0)_{t\leqslant 1}$, is bounded and then it is a $(F_t,P^u)$-martingale. It follows that

$$\tilde{Y}_t=E^u\left[\exp\left\{\xi+\int_0^1 h(s,x,u_s)\,\mathrm{d}s\right\}|F_t\right]\quad\text{and}$$

$$Y_t=\mathrm{Ln}\left\{E^u\left[\exp\left\{\xi+\int_t^1 h(s,x,u_s)\,\mathrm{d}s\right\}|F_t\right]\right\},\quad t\leqslant 1. \tag{15}$$

Henceforth we have, first $Y=Y^u$ and, second, the continuous local martingale $(\int_0^t(Z_s^u-Z_s)\,\mathrm{d}B_s)_{t\leqslant 1}$ is of bounded variation. Then we have also $Z^u=Z$, whence the uniqueness.

In addition since $F_0$ is the trivial tribe then from (15), we have $\exp(Y_0^u) = J(u)$, for any $u \in \mathscr{U}$. □

Now let $H^*(t,x,p) := \inf_{u\in U} H(t,x,p,u)$ and $u^*(t,x,p)$ a $\mathscr{P}\otimes B(U)$-measurable function such that $H^*(t,x,p) = H(t,x,p,u^*(t,x,p))$. The function $u^*(t,x,p)$ exists according to the Benes' selection theorem (Benes, 1970). We give now the main result of this section.

**Theorem 3.2.** *There exists a unique $\mathscr{P}$-measurable process $(Y^*,Z^*)$ such that*

$$\begin{cases} (Y^*,Z^*) \in \mathscr{B} \times H^{2,d}, \\ -\mathrm{d}Y_t^* = \{H^*(t,x,Z_t^*) + \frac{1}{2}|Z_t^*|^2\}\,\mathrm{d}t - Z_t^*\,\mathrm{d}B_t, \quad t \leqslant 1;\ Y_1^* = \xi. \end{cases} \tag{16}$$

*Moreover, $\forall t \leqslant 1$, $Y_t^* = \operatorname{essinf}_{u\in\mathscr{U}} Y_t^u$, and $(u^*(t,x,Z_t^*))_{t\leqslant 1}$ is an optimal control for the risk-sensitive control problem.*

**Proof.** First let us point out that we have,

$$\begin{aligned} |H^*(t,x,p) - H^*(t,x,p')| &= \left|\inf_{u\in U} H(t,x,p,u) - \inf_{u\in U} H(t,x,p',u)\right| \\ &\leqslant \sup_{u\in U} |H(t,x,p,u) - H(t,x,p',u)| \\ &= \sup_{u\in U} |\sigma^{-1}(t,x)f(t,x,u).(p-p')| \\ &\leqslant C(1 + \|x\|_t^{1+\delta})|p-p'|. \end{aligned}$$

Therefore the existence of $(Y^*,Z^*)$ stems from Theorem 2.3. Let us show that $Y_t^* = \operatorname{essinf}_{u\in\mathscr{U}} Y_t^u$. This characterization implies, in particular, that if $(\bar{Y}^*,\bar{Z}^*)$ is another solution of (16) then $Y^* = \bar{Y}^*$ and $Z^* = \bar{Z}^*$. Henceforth the solution of (16) is unique.

Let $u \in \mathscr{U}$ and $(Y^u,Z^u)$ be the solution of (14). If we set $\bar{Y}_t^* = e^{Y_t^*}$ and $\bar{Y}_t^u = e^{Y_t^u}$ and using Itô's formula we obtain, for any $t \leqslant 1$,

$$\begin{aligned} -d(\bar{Y}_t^* - \bar{Y}_t^u) &= \{\bar{Y}_t^* H^*(t,x,Z_t^*) - \bar{Y}_t^u H(t,x,Z_t^u,u_t)\}\,\mathrm{d}t - (\bar{Y}_t^* Z_t^* - \bar{Y}_t^u Z_t^u)\,\mathrm{d}B_t \\ &= \{\bar{Y}_t^* H^*(t,x,Z_t^*) - \bar{Y}_t^* H(t,x,Z_t^*,u_t) + \bar{Y}_t^* H(t,x,Z_t^*,u_t) \\ &\quad - \bar{Y}_t^u H(t,x,Z_t^u,u_t)\}\,\mathrm{d}t - (\bar{Y}_t^* Z_t^* - \bar{Y}_t^u Z_t^u)\,\mathrm{d}B_t \\ &= \{\bar{Y}_t^*(H^*(t,x,Z_t^*) - H(t,x,Z_t^*,u_t)) + (\bar{Y}_t^* - \bar{Y}_t^u)h(t,x,u_t)\}\,\mathrm{d}t \\ &\quad - (\bar{Y}_t^* Z_t^* - \bar{Y}_t^u Z_t^u)\,\mathrm{d}B_t^u. \end{aligned}$$

Next let us set $D_t = \exp\{\int_0^t h(s,x,u_s)\,\mathrm{d}s\}(\bar{Y}_t^* - \bar{Y}_t^u)$, $t \leqslant 1$, then

$$\begin{aligned} -\mathrm{d}D_t &= \left[\exp\left\{\int_0^t h(s,x,u_s)\,\mathrm{d}s\right\}\bar{Y}_t^*(H^*(t,x,Z_t^*) - H(t,x,Z_t^*,u_t))\right]\mathrm{d}t \\ &\quad - \exp\left\{\int_0^t h(s,x,u_s)\,\mathrm{d}s\right\}(\bar{Y}_t^* Z_t^* - \bar{Y}_t^u Z_t^u)\,\mathrm{d}B_t^u, \quad t \leqslant 1;\ D_1 = 0. \end{aligned} \tag{17}$$

Now for $n \geqslant 0$ let $\tau_n = \inf\{t \geqslant 0,\ \int_0^t (|Z_s^*|^2 + |Z_s^u|^2)\,\mathrm{d}s \geqslant n\} \wedge 1$, therefore $\tau_n$ is a stopping time such that $\tau_n \nearrow 1$ as $n \to \infty$. In addition we have,

$$D_{\tau_n} - D_{t\wedge\tau_n} = -\int_{t\wedge\tau_n}^{\tau_n} \exp\left\{\int_0^s h(r,x,u_r)\,\mathrm{d}r\right\} \bar{Y}_s^*(H^*(s,x,Z_s^*) - H(s,x,Z_s^*,u_s))\,\mathrm{d}s$$

$$+ \int_{t\wedge\tau_n}^{\tau_n} \exp\left\{\int_0^s h(r,x,u_r)\,\mathrm{d}r\right\} (\bar{Y}_s^* Z_s^* - \bar{Y}_s^u Z_s^u)\,\mathrm{d}B_s^u.$$

As $\bar{Y}^* \geqslant 0$ and $H^*(t,x,Z_t^*) - H(t,x,Z_t^*,u_t) \leqslant 0$, then taking the conditional expectation in the previous inequality yields $D_{t\wedge\tau_n} \leqslant E^u[D_{\tau_n}|F_{t\wedge\tau_n}]$. But $(D_t)_{t\leqslant 1}$ is bounded then the sequence $(E^u[D_{\tau_n}|F_{t\wedge\tau_n}])_{n\geqslant 0}$ converges to 0 in $L^1(\mathrm{d}P^u)$, therefore we have $D_t \leqslant 0$ for any $t \leqslant 0$. It implies that $\forall t \leqslant 1$, $\bar{Y}_t^* \leqslant \bar{Y}_t^u$ and then $Y_t^* \leqslant Y_t^u$. Henceforth we have $Y_t^* \leqslant \operatorname{essinf}_{u\in\mathscr{U}} Y_t^u$. On the other hand, let $u^* := (u^*(t,x,Z_t^*))_{t\leqslant 1}$ then $\forall t \leqslant 1$, $Y_t^* = Y_t^{u^*}$ which yields $Y_t^* = \operatorname{essinf}_{u\in\mathscr{U}} Y_t^u$, $t \leqslant 1$.

The optimality of $u^*$ is obvious since it is $\mathscr{P}$-measurable and $J(u^*) = e^{Y_0^*} \leqslant e^{Y_0^u} = J(u), \forall u \in \mathscr{U}$. □

**Remark.** The process $(e^{Y_t^*})_{t\leqslant 1}$ is the *value function* of the risk-sensitive control problem.

## 4. The zero-sum risk-sensitive game problem

We now deal with the risk-sensitive zero-sum game, which is basically a control problem where, instead of one acting controller, we have two of them, say $c_1$ and $c_2$, and whose interests are antagonistic. Each one acts such as to improve its own advantage. The problem is to find a fair strategy for both players $c_1$ and $c_2$. To be more precise let us describe briefly the position of the problem that we consider.

Let $x$, $U$, $\mathscr{U}$ and $\xi$ are as in the previous section. Let $V$ be another compact metric space and $\mathscr{V}$ the space of $\mathscr{P}$-measurable processes with values in $V$. Hereafter $\mathscr{U}$ (resp. $\mathscr{V}$) is the set of admissible controls for $c_1$ (resp. $c_2$).

Let $f$ be a function from $[0,1] \times \mathscr{C} \times U \times V$ into $R^d$ such that:

(A4.1) For any $(u,v) \in \mathscr{U} \times \mathscr{V}$, the process $(f(t,x,u_t,v_t))_{t\leqslant 1}$ is $\mathscr{P}$-measurable and $\forall t \leqslant 1$, $|f(t,x,u_t,v_t)| \leqslant C(1 + \|x\|_t)$ $P$-a.s.

(A4.2) The mapping $f(t,x,.,.)$ which to $(u,v) \in U \times V$ associates $f(t,x,u,v)$ is continuous.

Now let $(u,v) \in \mathscr{U} \times \mathscr{V}$ and $P^{u,v}$ the probability on $(\Omega, \mathscr{F})$ defined as

$$\mathrm{d}P^{u,v} := \mathscr{E}_1\left(\int_0^\cdot \sigma^{-1}(s,x) f(s,x,u_s,v_s)\,\mathrm{d}B_s\right)\mathrm{d}P.$$

As in Section 1, the process $x$ of (5) is a weak solution of the following functional diffusion:

$$\begin{cases} \mathrm{d}x_t = f(t,x,u_t,v_t)\,\mathrm{d}t + \sigma(t,x)\,\mathrm{d}B_t^{u,v} t \leqslant 1, \\ x_0 = x \in R^d. \end{cases}$$

where $B^{u,v}$ is a Brownian motion under $P^{u,v}$.

The control is not free then there is a payment between the two controllers, which is a cost (resp. a reward) for $c_1$ (resp. $c_2$) and whose expression is:

$$J(u,v) = E^{u,v}\left[\exp\left\{\int_0^1 h(s,x,u_s,v_s)\,\mathrm{d}s + \xi\right\}\right];$$

$E^{u,v}$ is the expectation under $P^{u,v}$. The function $h : [0,1]\times\mathscr{C}\times U\times V \to R^+$ which stands for the instantaneous payment is Borel-measurable and satisfies:

(A4.3) $h$ is bounded and for any $(u,v)\in\mathscr{U}\times\mathscr{V}$, the process $(h(t,x,u_t,v_t))_{t\leqslant 1}$ is $\mathscr{P}$-measurable. Moreover for any $t\leqslant 1$, the map $h(t,x,.,.) : (u,v)\mapsto h(t,x,u,v)$ is continuous.

The problem is to find $(u^*,v^*)\in\mathscr{U}\times\mathscr{V}$ such that $J(u^*,v)\leqslant J(u^*,v^*)\leqslant J(u,v^*)$, $\forall(u,v)\in\mathscr{U}\times\mathscr{V}$. The strategy $(u^*,v^*)$ is called a *saddle point* for the game and can be considered as the fair strategy for the two controllers $c_1$ and $c_2$.

Let $H$ be the Hamiltonian function associated with this game problem i.e. the function which to $(t,x,p,u,v)$ associates $H(t,x,p,u,v) := p\sigma^{-1}(t,x)f(t,x,u,v)+h(t,x,u,v)$. On the other hand, let us consider the functions $\bar{H}$ and $\underline{H}$ such that $\bar{H}(t,x,p) := \inf_{u\in U}\sup_{v\in V} H(t,x,p,u,v)$ and $\underline{H}(t,x,p) := \sup_{v\in V}\inf_{u\in U} H(t,x,p,u,v)$. In taking into account the properties which satisfy $f$ and $h$ and using the Benes' selection theorem (Benes, 1970), there exist two $\mathscr{P}\otimes B(R^d)$-measurable functions $u^*(t,x,p)$ and $v^*(t,x,p)$ with values, respectively, in $U$ and $V$ such that $\bar{H}(t,x,p)=\sup_{v\in V} H(t,x,p,u^*(t,x,p),v)$ and $\underline{H}(t,x,p) := \inf_{u\in U} H(t,x,p,u,v^*(t,x,p))$.

Now it is obvious that $H$, $\bar{H}$ and $\underline{H}$ verify, with respect to $p$, the same assumption (A2.2) as $\Phi$ of (11) with $k_t = C(1+\|x\|_t^{1+\delta})$. So according to Theorem 2.3 there exist processes $(\bar{Y},\bar{Z})$, $(\underline{Y},\underline{Z})$ and $(Y^{u,v},Z^{u,v})$, for $(u,v)\in\mathscr{U}\times\mathscr{V}$, such that

$$\begin{cases}(\bar{Y},\bar{Z})\in\mathscr{B}\times H^{2,d},\\ -\mathrm{d}\bar{Y}_t = \{\bar{H}(t,x,\bar{Z}_t)+\frac{1}{2}|\bar{Z}_t|^2\}\,\mathrm{d}t - \bar{Z}_t\,\mathrm{d}B_t, \quad t\leqslant 1,\\ \bar{Y}_1=\xi,\end{cases}\tag{18}$$

$$\begin{cases}(\underline{Y},\underline{Z})\in\mathscr{B}\times H^{2,d},\\ -\mathrm{d}\underline{Y}_t = \{\underline{H}(t,x,\underline{Z}_t)+\frac{1}{2}|\underline{Z}_t|^2\}\,\mathrm{d}t - \underline{Z}_t\,\mathrm{d}B_t, \quad t\leqslant 1,\\ \underline{Y}_1=\xi\end{cases}$$

and

$$\begin{cases}(Y^{u,v},Z^{u,v})\in\mathscr{B}\times H^{2,d},\\ -\mathrm{d}Y_t^{u,v} = \{H(t,x,Z_t^{u,v},u_t,v_t)+\frac{1}{2}|Z_t^{u,v}|^2\}\,\mathrm{d}t - Z_t^{u,v}\,\mathrm{d}B_t, \quad t\leqslant 1,\\ Y_1^{u,v}=\xi.\end{cases}$$

As in the previous section $J(u,v)$ can be expressed by means of $Y^{u,v}$. In other respects $\bar{Y}$ and $\underline{Y}$ are linked to the *upper* and *lower* values of the zero-sum game.

**Proposition 4.1.** (i) $\forall (u,v)\in\mathscr{U}\times\mathscr{V}$, $J(u,v)=e^{Y_0^{u,v}}$.
(ii) $\forall t\leqslant 1$, $\bar{Y}_t=\operatorname{essinf}_{u\in\mathscr{U}}\operatorname{esssup}_{v\in\mathscr{V}} Y_t^{u,v}$.
(iii) $\forall t\leqslant 1$, $\underline{Y}_t=\operatorname{esssup}_{v\in\mathscr{V}}\operatorname{essinf}_{u\in\mathscr{U}} Y_t^{u,v}$.

**Proof.** For (i), it is the same as in Proposition 3.1. Let us show (ii). Let $u\in\mathscr{U}$, $\tilde{H}(t,x,p,u):=\sup_{v\in V}H(t,x,p,u,v)$ and $(\tilde{Y}^u,\tilde{Z}^u)$ a $\mathscr{P}$-measurable process such that:

$$\begin{cases}(\tilde{Y}^u,\tilde{Z}^u)\in\mathscr{B}\times H^{2,d},\\ -\mathrm{d}\tilde{Y}_t^u=\{\tilde{H}(t,x,\tilde{Z}^u,u_t)+\frac{1}{2}|\tilde{Z}_t^u|^2\}\,\mathrm{d}t-\tilde{Z}_t^u\,\mathrm{d}B_t,\qquad t\leqslant 1,\\ \tilde{Y}_1^u=\xi.\end{cases}\tag{19}$$

As in the proof of Theorem 3.2, we have $\forall t\leqslant 1$, $\tilde{Y}_t^u=\operatorname{esssup}_{v\in\mathscr{V}}Y_t^{u,v}$ which implies that the solution of (19) is unique. On the other hand, since $\bar{H}(t,x,p)\leqslant\tilde{H}(t,x,p,u)$ for any $u\in\mathscr{U}$, then the comparison result of Remark 2.4 implies that $\forall u\in\mathscr{U}$, $\forall t\leqslant 1$, $\bar{Y}_t\leqslant\tilde{Y}_t^u=\operatorname{esssup}_{v\in\mathscr{V}}Y_t^{u,v}$. Now $u^*(t,x,p)$ is a $\mathscr{P}\otimes B(R^d)$-measurable function such that $\bar{H}(t,x,p)=\sup_{v\in V}H(t,x,p,u^*(t,x,p),v)$, hence $u^*:=(u^*(t,x,\bar{Z}_t))_{t\leqslant 1}$ belongs to $\mathscr{U}$ and $\forall t\leqslant 1$, $\bar{Y}_t=\tilde{Y}_t^{u^*}$ whence $\bar{Y}_t=\operatorname{essinf}_{u\in\mathscr{U}}\operatorname{esssup}_{v\in\mathscr{V}}Y_t^{u,v}$.

Finally the proof of (iii) can be done in the same way as the one of (ii). □

**Remark.** The processes $(e^{\bar{Y}_t})_{t\leqslant 1}$ and $(e^{\underline{Y}_t})_{t\leqslant 1}$ are, respectively, the *upper* and *lower values* of the game.

Now as usual (e.g. Hamadène and Lepeltier, 1995a, b), the zero-sum risk-sensitive game problem will be solved under the following basic assumption called the *Isaacs' condition*.

**Assumption H1.** For any $(t,p)\in[0,1]\times R^d$ we have,

$$\inf_{u\in U}\sup_{v\in V}H(t,x,p,u,v)=\sup_{v\in V}\inf_{u\in U}H(t,x,p,u,v),\qquad P\text{-a.s.}$$

**Proposition 4.2.** *Under Assumption* H1, *the* $\mathscr{P}\otimes B(R^d)$*-measurable functions* $u^*(t,x,p)$ *and* $v^*(t,x,p)$ *satisfy*: *for any* $(t,p,u,v)\in[0,1]\times R^d\times U\times V$,

(i) $H(t,x,p,u,v^*(t,x,p))\leqslant H(t,x,p,u^*(t,x,p),v^*(t,x,p))\leqslant H(t,x,p,u^*(t,x,p,),v)$, $P$-a.s.
(ii) $\bar{H}(t,x,p)=\underline{H}(t,x,p)=H(t,x,p,u^*(t,x,p),v^*(t,x,p))$ $P$-a.s.

**Proof.** Let us recall that $u^*(t,x,p)$ and $v^*(t,x,p)$ satisfy $\bar{H}(t,x,p)=\sup_{v\in V}H(t,x,p,u^*(t,x,p),v)$ and $\underline{H}(t,x,p)=\inf_{u\in U}H(t,x,p,u,v^*(t,x,p))$. It follows that,

$$\begin{aligned}\sup_{v\in V}\inf_{u\in U}H(t,x,p,u,v)&=\inf_{u\in U}H(t,x,p,u,v^*(t,x,p))\\&\leqslant H(t,x,p,u^*(t,x,p),v^*(t,x,p))\\&\leqslant\sup_{v\in V}H(t,x,p,u^*(t,x,p),v)=\inf_{u\in U}\sup_{v\in V}H(t,x,p,u,v).\end{aligned}$$

Now taking into account Assumption H1 we deduce,

$$\begin{aligned}\bar{H}(t,x,p) &= \inf_{u\in U} H(t,x,p,u,v^*(t,x,p)) = H(t,x,p,u^*(t,x,p),v^*(t,x,p))\\ &= \sup_{v\in V} H(t,x,p,u^*(t,x,p),v) = \underline{H}(t,x,p)\end{aligned}$$

and the proof is over. □

We are now ready to give the main result of this part.

**Theorem 4.3.** *Under Assumption* H1, *the pair of controls* $(u^*,v^*) := (u^*(t,x,\bar{Z}_t), v^*(t,x,\bar{Z}_t))_{t\leqslant 1}$, *where* $(\bar{Y},\bar{Z})$ *is the solution of* (18), *belongs to* $\mathscr{U}\times\mathscr{V}$ *and satisfies* $J(u^*,v)\leqslant J(u^*,v^*)\leqslant J(u,v^*)$ *for any* $(u,v)\in\mathscr{U}\times\mathscr{V}$, *i.e.* $(u^*,v^*)$ *is a saddle point for the zero-sum risk-sensitive game problem.*

**Proof.** It is obvious that $(u^*,v^*)\in\mathscr{U}\times\mathscr{V}$. On the other hand we have, $\bar{H}(t,x,p)=H(t,x,p,u^*(t,x,p),v^*(t,x,p))=\underline{H}(t,x,p)$ and then $\bar{Y}_t=\underline{Y}_t=Y_t^{u^*,v^*}$. But as it is shown in the proof of Proposition 4.1 we have also $\bar{Y}_t=\mathrm{esssup}_{v\in\mathscr{V}}Y_t^{u^*,v}$ and $\underline{Y}_t=\mathrm{essinf}_{u\in\mathscr{U}}Y_t^{u,v^*}$ which implies $Y_t^{u^*,v}\leqslant Y_t^{u^*,v^*}\leqslant Y_t^{u,v^*}$, $t\leqslant 1$. Now taking $t=0$ we get $J(u^*,v)=e^{Y_0^{u^*,v}}\leqslant J(u^*,v^*)=e^{Y_0^{u^*,v^*}}\leqslant e^{Y_0^{u,v^*}}$, whence $(u^*,v^*)$ is a saddle-point for the game. □

**Remark.** A direct proof without using the results of Proposition 4.1 can be given (see Proposition 5.1 below).

## 5. The risk-sensitive nonzero-sum game

We now consider the case when many controllers intervene on the dynamic of the system and their advantages are not necessarily antagonistic but each one acts such as to save its own interest. We are in a situation of a nonzero-sum game. To begin with let us precise in few words the model which we deal with.

For the sake of simplicity we suppose we have only two controllers $c_1$ and $c_2$. The generalization to the case where we have $n\geqslant 3$ controllers $c_1,\ldots,c_n$ does not rise any difficulty and can be done in the same way as for $n=2$.

Let $x$, $U$, $V$, $\mathscr{U}$, $\mathscr{V}$, $f$ and $P^{u,v}$ are as in the previous sections. On the other hand we assume that $\mathscr{U}$ (resp. $\mathscr{V}$) is the set of admissible strategies for $c_1$ (resp. $c_2$) when he intervenes on the state dynamic of the system.

We are going to define the cost functionals associated with the controllers or *players*. For $i=1,2$, let $h_i$ be a bounded measurable function from $[0,1]\times\mathscr{C}\times U\times V$ into $R$ which to $(t,\bar{w},u,v)$ associates $h_i(t,\bar{w},u,v)$.

Suppose the players act on the system with a strategy $(u,v)\in\mathscr{U}\times\mathscr{V}$, then the costs associated with $c_1$ and $c_2$ are, respectively, $J_1(u,v)$ and $J_2(u,v)$ whose expressions are:

$$J_i(u,v)=E^{u,v}\left[\exp\left\{\int_0^1 h_i(s,x,u_s,v_s)\,\mathrm{d}s+\xi_i\right\}\right],\qquad i=1,2,$$

where $\xi_1$ (resp. $\xi_2$) is a bounded $F_1$-measurable r.v. with values in $R$. The functions $h_1$ and the r.v. $\xi_1$ (resp. $h_2$ and $\xi_2$) are, respectively, the instantaneous and terminal costs for $c_1$ (resp. $c_2$).

The problem is to find a control $(u^*, v^*) \in \mathcal{U} \times \mathcal{V}$ such that:

$$J_1(u^*, v^*) \leqslant J_1(u, v^*) \quad \text{and} \quad J_2(u^*, v^*) \leqslant J_2(u^*, v), \quad \forall (u, v) \in \mathcal{U} \times \mathcal{V}.$$

The pair of controls $(u^*, v^*)$ is called an *equilibrium point* for the game because when $c_1$ (resp. $c_2$) acts with the strategy $u^*$ (resp. $v^*$), the best that has to do $c_2$ (resp. $c_1$) is to act with $v^*$ (resp. $u^*$).

As we will show it later the resolution of this game problem is also obtained via its associated BSDE. This latter equation is of multidimensional type since we have several players and each one with its own associated cost functional.

Let us introduce now the Hamiltonian functions associated with this game, namely two measurable functions $H_1$ (resp. $H_2$) from $[0,1] \times \mathscr{C} \times R^d \times U \times V$ into $R$ which to $(t, \bar{w}, p, u, v)$ (resp. $(t, \bar{w}, q, u, v)$) associates $p\sigma^{-1}(t, \bar{w}) f(t, \bar{w}, u, v) + h_1(t, \bar{w}, u, v)$ (resp. $q\sigma^{-1}(t, \bar{w}) f(t, \bar{w}, u, v) + h_2(t, \bar{w}, u, v)$).

For the resolution of this game problem we are led to assume the following assumption called *the generalized Isaacs' condition* which is the analogue of Assumption H1 in the zero-sum game frame.

**Assumption H2.** There exist two measurable functions $u^*(t, \bar{w}, p, q)$ and $v^*(t, \bar{w}, p, q)$ which satisfy: for any $(t, \bar{w}, p, q, u, v) \in [0,1] \times \mathscr{C} \times R^{d+d} \times U \times V$,

$$H_1(t, \bar{w}, p, u^*(t, \bar{w}, p, q), v^*(t, \bar{w}, p, q)) \leqslant H_1(t, \bar{w}, p, u, v^*(t, \bar{w}, p, q))$$

and

$$H_2(t, \bar{w}, q, u^*(t, \bar{w}, p, q), v^*(t, \bar{w}, p, q)) \leqslant H_2(t, \bar{w}, q, u^*(t, \bar{w}, p, q), v).$$

The link between the nonzero-sum risk-sensitive game problem and its associated multidimensional BSDE is:

**Proposition 5.1.** *Assume there exists a $\mathscr{P}$-measurable process $(Y^1, Y^2, Z^1, Z^2)$ with values in $R^{1+1+d+d}$ such that:*

$$\begin{cases} Y^1, Y^2 \in \mathscr{B} \quad \text{and} \quad Z^1, Z^2 \in H^{2,d}, \\ -\mathrm{d}Y_t^1 = \{H_1(t, x, Z_t^1, (u^*, v^*)(t, x, Z_t^1, Z_t^2)) + \frac{1}{2}|Z_t^1|^2\}\,\mathrm{d}t - Z_t^1\,\mathrm{d}B_t, \\ \qquad t \leqslant 1; \quad Y_1^1 = \xi_1, \\ -\mathrm{d}Y_t^2 = \{H_2(t, x, Z_t^2, (u^*, v^*)(t, x, Z_t^1, Z_t^2)) + \frac{1}{2}|Z_t^2|^2\}\,\mathrm{d}t - Z_t^2\,\mathrm{d}B_t, \\ \qquad t \leqslant 1; \quad Y_1^2 = \xi_2. \end{cases} \tag{20}$$

*Then $(u^*, v^*) = (u^*(t, x, Z_t^1, Z_t^2), v^*(t, x, Z_t^1, Z_t^2))_{t \leqslant 1}$ is an equilibrium point for the game. In addition $J_1(u^*, v^*) = e^{Y_0^1}$ and $J_2(u^*, v^*) = e^{Y_0^2}$.*

**Proof.** Let $(u,v)\in\mathscr{U}\times\mathscr{V}$ and $(Y^{u,v},Y'^{u,v},Z^{u,v},Z'^{u,v})$ the process (it exists according to Theorem 2.3) which satisfies:

$$\begin{cases} Y^{u,v},Y'^{u,v}\in\mathscr{B} \quad\text{and}\quad Z^{u,v},Z'^{u,v}\in H^{2,d},\\ -\mathrm{d}Y_t^{u,v}=\{H_1(t,x,Z_t^{u,v},u_t,v_t)+\frac{1}{2}\,|Z_t^{u,v}|^2\}\,\mathrm{d}t-Z_t^{u,v}\,\mathrm{d}B_t, \quad t\leqslant 1;\ Y_1^{u,v}=\xi_1,\\ -\mathrm{d}Y_{t'}^{u,v}=\{H_2(t,x,Z_{t'}^{u,v},u_t,v_t)+\frac{1}{2}\,|Z_{t'}^{u,v}|^2\}\,\mathrm{d}t-Z_{t'}^{u,v}\,\mathrm{d}B_t, \quad t\leqslant 1;\ Y_{1'}^{u,v}=\xi_2, \end{cases}$$

as in the proof of Proposition 3.1 we have $J_1(u,v)=e^{Y_0^{u,v}}$ and $J_2(u,v)=e^{Y_{0'}^{u,v}}$ and then $J_1(u^*,v^*)=e^{Y_0^1}$ and $J_2(u^*,v^*)=e^{Y_0^2}$. Now let us show that $J_1(u^*,v^*)\leqslant J_1(u,v^*)$. First we have $J_1(u,v^*)=e^{Y_0^{u,v^*}}$ where $(Y^{u,v^*},Z^{u,v^*})$ satisfies:

$$\begin{cases} Y^{u,v^*}\in\mathscr{B} \quad\text{and}\quad Z^{u,v^*}\in H^{2,d},\\ -\mathrm{d}Y_t^{u,v^*}=\{H_1(t,x,Z_t^{u,v^*},u_t,v^*(t,x,Z_t^1,Z_t^2))+\frac{1}{2}\,|Z_t^{u,v^*}|^2\}\,\mathrm{d}t-Z_t^{u,v^*}\,\mathrm{d}B_t,\\ \qquad\qquad t\leqslant 1;\ Y_1^{u,v^*}=\xi_1. \end{cases}$$

On the other hand, if we set $\bar{X}:=\exp(Y^1)$, $X':=\exp(Y^{u,v^*})$ and $\Gamma_t^1:=(\bar{X}_t-X_t')\exp\{\int_0^t h_1(s,x,u_s,v_s^*)\,\mathrm{d}s\}$, $t\leqslant 1$, then using Itô's formula yields:

$$\mathrm{d}\Gamma_t^1=-\exp\left\{\int_0^t h_1(s,x,u_s,v_s^*)\,\mathrm{d}s\right\}\left[\bar{X}_t\Delta_t^1\,\mathrm{d}t-(\bar{X}_t\bar{Z}_t-X_t'Z_t^{u,v^*})\,\mathrm{d}B_t^{u,v^*}\right],$$

$$t\leqslant 1;\ \Gamma_1^1=0, \tag{21}$$

where $\Delta_t^1:=H_1(t,x,Z_t^1,(u^*,v^*)(t,x,Z_t^1,Z_t^2))-H_1(t,x,Z_t^1,u_t,v^*(t,x,Z_t^1,Z_t^2))$. But since $\Gamma^1$ is bounded and $\{\int_0^t h_1(s,x,u_s,v_s^*)\,\mathrm{d}s\}\bar{X}_t\Delta_t^1\leqslant 0$ then using a localization argument (as in the proof of Theorem 3.2) we deduce that $\Gamma_t^1\leqslant 0$. It implies that $\bar{X}\leqslant X'$ and, taking $t=0$, we obtain $J_1(u^*,v^*)=\bar{X}_0=e^{Y_0^1}\leqslant e^{Y_0^{u,v^*}}=X_0'=J_1(u,v^*)$. In the same way we can show that $J_2(u^*,v^*)\leqslant J(u^*,v)$, $\forall v\in\mathscr{V}$, whence the desired result. □

The existence of a solution for multidimensional BSDEs whose coefficients are not uniformly Lipschitz w.r.t. to $Z$ still an open problem in the general case. So we are not able to affirm the existence of a process $(Y^1,Y^2,Z^1,Z^2)$ which satisfies (20) since its coefficient is not uniformly Lipschitz. Consequently we do not know whether or not the game has an equilibrium point in the general case. This situation is not new since we face the same problem in the nonzero-sum games with risk-neutral costs. Nevertheless in the Markovian frame we can show that (20) has a solution if, in addition, we assume some regularity hypotheses on the data of the game $\sigma$, $f$, $h_i$ and $\xi_i$. In the rest of this paper we will focus on this latter frame.

The Markovian frame means that the functions which intervene in the model depend on $x$ at time $t$ only by $x_t$ and not the path of $x$ up to $t$ as it may be in the previous sections.

First, let us precise the assumptions on $\sigma$.

(A5.1)
(a) $\forall(t,\bar{w})\in[0,1]\times\mathscr{C}$, $\sigma(t,\bar{w})=\sigma(t,\bar{w}_t)$,
(b) the function $\sigma(t,\bar{x})$ is uniformly Lipschitz with respect to $\bar{x}$ and there exists a constant $\bar{c}>0$ such that $\bar{c}I\leqslant\sigma\sigma^*(t,\bar{x})\leqslant\bar{c}^{-1}I$ ($^*$ stands for the transpose). This latter condition implies that $\sigma$ and $\sigma^{-1}$ are bounded.

Then the process $x=(x_t)_{t\leqslant 1}$ of (5) satisfies

$$dx_t=\sigma(t,x_t)\,dB_t,\qquad t\leqslant 1;\ \ x_0=x. \tag{22}$$

and is a Markov process on $(\Omega,\mathscr{F},P)$ (cf. Karatzas and Shreve, 1991; Revuz and Yor, 1991).

The following result of Hamadène et al. (1997) provides solutions for multidimensional BSDEs with continuous coefficients when the randomness stems from a Markov process as in (22).

**Theorem 5.2** (Hamadène et al., 1997). *Let $(x_t)_{t\leqslant 1}$ be the process of (22) with $\sigma$ satisfying (A5.1). Let $\varphi$ (resp. $g'$) be a measurable function from $[0,1]\times R^{d+p+p\times d}$ (resp. $R^p$) with values in $R^p$ such that:*

(i) $|g'(\bar{x})|+|\varphi(t,\bar{x},y,z)|\leqslant C(1+|\bar{x}|^{\rho}+|y|+|z|)$ *for some constants $\rho$ and $C$.*
(ii) $\forall(t,\bar{x})\in[0,1]\times R^d$, *the mapping which to $(y,z)$ associates $\varphi(t,\bar{x},y,z)$ is continuous.*

*Then there exists a $\mathscr{P}$-measurable process $(Y,Z)$ with values in $R^{p+p\times d}$ solution of the BSDE associated with $(\varphi(t,x_t,.,.),g'(x_1))$ that is,*

$$\begin{cases} E\left[\sup_{t\leqslant 1}|Y_t|^2\right]<\infty \quad \text{and} \quad Z\in H^{2,p\times d}, \\ -dY_t=\varphi(t,x_t,Y_t,Z_t)\,dt-Z_t\,dB_t, \quad t\leqslant 1,\ \ Y_1=g'(x_1). \end{cases}$$

Now, in addition to (A5.1), suppose $\xi_i$ and the functions $f$, $h_i$ satisfy:

(A5.2) $f(t,\bar{w},u,v)=f(t,\bar{w}_t,u,v)$ and $h_i(t,\bar{w},u,v)=h_i(t,\bar{w}_t,u,v)$, $i=1,2$, $\forall(t,z,\bar{w},v)\in[0,1]\times\mathscr{C}\times U\times V$,
(A5.3) the functions $f$, $h_1$ and $h_2$ are bounded,
(A5.4) there exist two bounded Borel measurable functions $g_1$ and $g_2$ from $R^d$ into $R$ such that $\xi_i=g_i(x_1)$, $i=1,2$,
(A5.5) $\forall(t,\bar{x})\in[0,1]\times R^d$, the function $(p,q)\mapsto(H_1(t,\bar{x},p,(u^*,v^*)(t,\bar{x},p,q)),H_2(t,\bar{x},q,(u^*,v^*)(t,\bar{x},p,q)))$ is continuous.

An example where the Assumption H2 and (A5.5) are satisfied is: assume $m=1$, $U=[-1,1]$, $V=[0,1]$, $f(t,\bar{x},u,v):=\bar{f}(t,\bar{x})+bu+cv$ and $h_i(t,\bar{x},u,v):=\bar{h}_i(t,\bar{x})+a_{i1}u^2+a_{i2}v^2$, $i=1,2$, where $a_{11}>0$ and $a_{22}>0$. Then Assumption H2 and (A5.5) are satisfied with $u^*(t,\bar{x},p,q)=\delta(-pb/2a_{11})$ and $v^*(t,\bar{x},p,q)=\bar{\delta}(-qc/2a_{22})$ with $\delta(x)=x1_{[|x|\leqslant 1]}+1_{[x>1]}-1_{[x<1]}$ and $\bar{\delta}(x)=x1_{[0,1]}(x)+1_{[x>1]}$.

Assuming (A5.1)–(A5.5), then the BSDE (20) reads,

$$\begin{cases} Y^1,Y^2\in\mathscr{B} \quad \text{and} \quad Z^1,Z^2\in H^{2,d}, \\ -\mathrm{d}Y_t^1=\{H_1(t,x_t,Z_t^1,(u^*,v^*)(t,x_t,Z_t^1,Z_t^2))+\frac{1}{2}\,|Z_t^1|^2\}\,\mathrm{d}t-Z_t^1\,\mathrm{d}B_t, \\ \qquad\qquad t\leqslant 1;\ \ Y_1^1=g_1(x_1), \\ -\mathrm{d}Y_t^2=\{H_2(t,x_t,Z_t^2,(u^*,v^*)(t,x_t,Z_t^1,Z_t^2))+\frac{1}{2}\,|Z_t^2|^2\}\,\mathrm{d}t-Z_t^2\,\mathrm{d}B_t, \\ \qquad\qquad t\leqslant 1;\ \ Y_1^2=g_2(x_1), \end{cases} \tag{23}$$

and we have:

**Theorem 5.3.** *Under the assumptions* (A5.1)–(A5.5), *there exists a process* $(Y^1,Y^2,Z^1,Z^2)$ *solution of* (23).

**Proof.** Let $c>0$ and $\gamma$ be a function from $R$ into $R^+$ such that $\gamma(x)=\max\{x,c\}$. According to Theorem 5.2 and taking into account (A5.1)–(A5.5), there exists a $\mathscr{P}$-measurable process $(\bar{Y}^1,\bar{Y}^2,\bar{Z}^1,\bar{Z}^2)$ with values in $R^{1+1+m+m}$ such that:

$$\begin{cases} E\left[\sup_{t\leqslant 1}(|\bar{Y}_t^1|^2+|\bar{Y}_t^2|^2)\right]<\infty \quad \text{and} \quad \bar{Z}^1, \quad \bar{Z}^2\in H^{2,d}, \\ -\mathrm{d}\bar{Y}_t^i=\bar{Y}_t^i\left\{\dfrac{\bar{Z}_t^i}{\gamma(\bar{Y}_t^i)}\,\sigma^{-1}(t,x_t)f\left(t,x_t,u^*\left(t,x_t,\dfrac{\bar{Z}_t^1}{\gamma(\bar{Y}_t^1)},\dfrac{\bar{Z}_t^2}{\gamma(\bar{Y}_t^2)}\right),\right.\right. \\ \qquad v^*\left(t,x_t,\dfrac{\bar{Z}_t^1}{\gamma(\bar{Y}_t^1)},\dfrac{\bar{Z}_t^2}{\gamma(\bar{Y}_t^2)}\right)\right)+h_i\left(t,x_t,u^*\left(t,x_t,\dfrac{\bar{Z}_t^1}{\gamma(\bar{Y}_t^1)},\dfrac{\bar{Z}_t^2}{\gamma(\bar{Y}_t^2)}\right),\right. \\ \qquad \left.\left. v^*\left(t,x_t,\dfrac{\bar{Z}_t^1}{\gamma(\bar{Y}_t^1)},\dfrac{\bar{Z}_t^2}{\gamma(\bar{Y}_t^1)}\right)\right)\right\}\mathrm{d}t-\bar{Z}_t^i\,\mathrm{d}B_t, \quad \bar{Y}_1^i=e^{g_i(x_1)};\ \ i=1,2. \end{cases} \tag{24}$$

There exists a constant $\tilde{c}>0$ which does not depend on $c$ (of $\gamma$ above) such that for $i=1,2$, $\tilde{c}^{-1}\leqslant\bar{Y}_t^i\leqslant\tilde{c}$, $\forall t\leqslant 1$. Indeed, via Girsanov's Theorem (Girsanov, 1960), there exists a probability $\tilde{P}$ equivalent to $P$, and an $(F_t,\tilde{P})$-Brownian motion $\tilde{B}$ such that

$$-\mathrm{d}\bar{Y}_t^1=\bar{Y}_t^1h_1(t)\,\mathrm{d}t-\bar{Z}_t^1\,\mathrm{d}\tilde{B}_t, \quad \bar{Y}_1=e^{g_1(x_1)},$$

where $h_1(t):=h_1(t,x_t,u^*(t,x_t,\frac{\bar{Z}_t^1}{\gamma(\bar{Y}_t^1)},\frac{\bar{Z}_t^2}{\gamma(\bar{Y}_t^2)}),v^*(t,x_t,\frac{\bar{Z}_t^1}{\gamma(\bar{Y}_t^1)},\frac{\bar{Z}_t^2}{\gamma(\bar{Y}_t^1)}))$. It follows that

$$\bar{Y}_t^1=\tilde{E}\left[\exp\left\{\int_t^1 h_1(s)\,\mathrm{d}s+g_1(x_1)\right\}|F_t\right], \quad t\leqslant 1,$$

where $\tilde{E}$ is the expectation under the probability $\tilde{P}$. Now since $h_1$ and $g_1$ are bounded then there exists a constant $c' > 0$ such that $\forall t \leqslant 1$, $c'^{-1} \leqslant \bar{Y}_t^1 \leqslant c'$. In the same way there exists another constant $c'' > 0$ such that $c''^{-1} \leqslant \bar{Y}_t^2 \leqslant c''$, $\forall t \leqslant 1$, whence the desired result.

Assume now that $c < \tilde{c}$, then the process $(\bar{Y}^1, \bar{Y}^2, \bar{Z}^1, \bar{Z}^2)$ of (24) satisfies:

$$\begin{cases} \bar{Y}^1, \bar{Y}^2 \in \mathscr{B}, \quad \text{and} \quad Z^1, Z^2 \in H^{2,d}, \\ -\mathrm{d}\bar{Y}_t^i = \bar{Y}_t^i \left\{ \dfrac{\bar{Z}_t^i}{\bar{Y}_t^i} \sigma^{-1}(t,x_t) f\left(t, x_t, u^*\left(t, x_t, \dfrac{\bar{Z}_t^1}{\bar{Y}_t^1}, \dfrac{\bar{Z}_t^2}{\bar{Y}_t^2}\right), v^*\left(t, x_t, \dfrac{\bar{Z}_t^1}{\bar{Y}_t^1}, \dfrac{\bar{Z}_t^2}{\bar{Y}_t^2}\right)\right)\right. \\ \qquad \left. + h_i\left(t, x_t, u^*\left(t, x_t, \dfrac{\bar{Z}_t^1}{\bar{Y}_t^1}, \dfrac{\bar{Z}_t^2}{\bar{Y}_t^2}\right), v^*\left(t, x_t, \dfrac{\bar{Z}_t^1}{\bar{Y}_t^1}, \dfrac{\bar{Z}_t^2}{\bar{Y}_t^1}\right)\right)\right\} \mathrm{d}t \\ \qquad - \bar{Z}_t^1 \, \mathrm{d}B_t, \quad \bar{Y}_i^1 = e^{g_i(x_1)}. \end{cases}$$

Now let us set $Y_t^1 := \ln(\bar{Y}_t^1)$, $Y_t^2 := \ln(\bar{Y}_t^2)$, $Z_t^1 := \bar{Z}_t^1/\bar{Y}_t^1$ and $Z_t^2 := \bar{Z}_t^2/\bar{Y}_t^2$ $\forall t \leqslant 1$, then using Itô's formula, it is easily seen that $(Y^1, Y^2, Z^1, Z^2)$ satisfies the BSDE (23). □

As a consequence of Proposition 5.1 and Theorem 5.3 we have,

**Theorem 5.4.** *Under Assumptions* H2 *and* (A5.1)–(A5.5), *the admissible control* $(u^*, v^*) = (u^*(t, x_t, Z_t^1, Z_t^2), v^*(t, x_t, Z_t^1, Z_t^2))_{t \leqslant 1}$ *is an equilibrium point for the Markovian nonzero-sum risk-sensitive game.*

## Appendix A

**Proposition A.1.** *Assume that* $\sigma$ *and* $f$ *satisfy* (A1.2) *and* (A1.3)–(A1.4) *respectively, then for any* $u \in \mathscr{U}$ *the measure* $P^u$ *is a probability on* $(\Omega, \mathscr{F})$.

**Proof.** We give it for the reader's convenience since, according to our knowledge, it has not been given elsewhere with $\sigma^{-1}$ satisfying (A1.2.b). However it is inspired by the work of Lepeltier and Marchal (1977).

In order to prove that $P^u$ is probability it is sufficient to show that

$$E\left[\mathscr{E}_1\left(\int_0^{\cdot} \sigma^{-1}(s,x) f(s,x,u_s)\,\mathrm{d}B_s\right)\right] = 1.$$

For $n \geqslant 0$, let $f_n(t,x,u_t) = f(t,x,u_t) 1_{[\|x\|_t \leqslant n]}$. Then the process $(\sigma^{-1}(t,x) f_n(t,x,u_t))_{t \leqslant 1}$ is $\mathscr{P}$-measurable and bounded, hence $(R_t^n := \exp\{\int_0^t \sigma^{-1}(s,x) f_n(s,x,u_s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t |\sigma^{-1}(s,x) f_n(s,x,u_s)|^2\,\mathrm{d}s\})_{t \leqslant 1}$ is an $(F_t, P)$-martingale. Therefore we have $E[R_1^n] = 1$.

On the other hand, $\sup_{t \leqslant 1} |x_t| < \infty$ $P$-a.s. and on the set $\{\omega, \sup_{s \leqslant 1} |x_s| \leqslant n_0\}$ we have $\forall n \geqslant n_0$, $R_1^n = \mathscr{E}_1(.)$ hence $R_1^n \to \mathscr{E}_1(.)$, $P$-a.s. as $n \to \infty$.

Now let us show that $(R_1^n)_{n \geqslant 1}$ is uniformly integrable. Let $\varepsilon$ and $c > 0$, and for $m \geqslant 1$, $\tau_m = \inf\{t \leqslant 1, \ \|x\|_t \geqslant m\}$ if the set is nonempty, and $\tau_m = 2$ if empty.

We have,

$$\int_{(R_1^n>c)} R_1^n\,\mathrm{d}P = \int_{(R_1^n>c)(\tau_m\leqslant 1)} R_1^n\,\mathrm{d}P + \int_{(R_1^n>c)(\tau_m>1)} R_{1\wedge\tau_m}^n\,\mathrm{d}P$$

$$\leqslant \int_{(\tau_m\leqslant 1)} R_1^n\,\mathrm{d}P + \int_{(R_{1\wedge\tau_m}^n>c)(\tau_m>1)} R_{1\wedge\tau_m}^n\,\mathrm{d}P.$$

Now if $E^n$ is the expectation under the probability $\mathrm{d}P^n = R_1^n\,\mathrm{d}P$, then $\int_{(\tau_m\leqslant 1)} R_1^n\,\mathrm{d}P = P^n[\tau_m \leqslant 1] = P^n[\sup_{t\leqslant 1}|x_t| \geqslant m] \leqslant (E^n[\sup_{t\leqslant 1}|x_t|]/m) \leqslant \bar{C}/m$ where $\bar{C}$ is a constant which does not depend on $n$ since $\forall n \geqslant 1$, $\forall t \leqslant 1$, $|f_n(t,x,u_t)| \leqslant C(1+\|x\|_t)$. So let $m_0$ such that $\bar{C}/m_0 < \varepsilon$; we have $\forall n \geqslant m_0$, $R_{1\wedge\tau_{m_0}}^n = R_{1\wedge\tau_{m_0}}^{m_0}$ and then

$$\sup_{n\geqslant 1} \int_{(R_{1\wedge\tau_{m_0}}^n>c)} R_{1\wedge\tau_{m_0}}^n\,\mathrm{d}P = \max_{n\leqslant m_0} \int_{(R_{1\wedge\tau_{m_0}}^n>c)} R_{1\wedge\tau_{m_0}}^n\,\mathrm{d}P.$$

Hence there exists a constant $c_0$ such that $\forall c \geqslant c_0$ we have

$$\sup_{n\geqslant 1} \int_{(R_1^n>c)} R_1^n\,\mathrm{d}P \leqslant 2\varepsilon.$$

It implies that $(R_1^n)_{n\geqslant 1}$ is uniformly integrable and since it converges $P$-a.s. to $\mathscr{E}_1(.)$ then it converges also in $L^1$, whence $E[\mathscr{E}(.)_1] = 1$. $\square$

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