2002m:49045 49K45 60J60 91B16 93E20

El Karoui, N. (F-POLY-AM);
Peng, S. [Peng, Shi Ge] (PRC-SHAN-IM);
Quenez, M. C. (F-MARN-AAM)

A dynamic maximum principle for the optimization of recursive utilities under constraints. (English. English summary)

Ann. Appl. Probab. **11** (2001), *no.* 3, 664–693.

The authors present first a model of recursive utility and nonlinear wealth describing the choices of an agent who maximizes his recursive utility. A comparison theorem is then established concerning the value function of the optimization problem.

The same problem is then reformulated in a backward version and the symmetry between utility and wealth is highlighted.

An optimality characterization is then provided. Necessary and sufficient conditions are indicated. The existence and the uniqueness of the optimal solution are shown.

Finally, a characterization of the optimal wealth and utility as the solution of a forward-backward system is derived. Various examples are also provided. *Lorenzo Peccati* (I-UCOM-QM)

2001m:91066 91B24 60H10 91B28

Rouge, Richard (F-PARIS6-PB); **El Karoui, Nicole** (F-POLY-AM)

Pricing via utility maximization and entropy. (English. English summary)

INFORMS Applied Probability Conference (Ulm, 1999).
Math. Finance **10** (2000), *no.* 2, 259–276.

Summary: "In a financial market model with constraints on the portfolios, define the price for a claim C as the smallest real number p such that $\sup_\pi E[U(X_T^{x+p,\pi} - C)] \geq \sup_\pi E[U(X_T^{x,\pi})]$, where U is the negative exponential utility function and $X^{x,\pi}$ is the wealth associated with portfolio π and initial value x. We give the relations of this price with minimal entropy or fair price along the lines of I. Karatzas and S.-G. Kou [Ann. Appl. Probab. **6** (1996), no. 2, 321–369; MR 97f:90013] and superreplication. Using dynamical methods, we characterize the price equation, which is a quadratic backward SDE, and describe the optimal wealth and portfolio. Further use of backward SDE techniques allows for easy determination of the pricing function properties."

For the entire collection see 2001g:91003

J. M. Gutiérrez (Salamanca)

2000m:60069 60H10

El Karoui, N. (F-PARIS6-PB);
Huang, S.-J. [Huang, Shaojuan] (F-PARIS6-PB)

A general result of existence and uniqueness of backward stochastic differential equations.

Backward stochastic differential equations (*Paris*, 1995–1996), 27–36, *Pitman Res. Notes Math. Ser.*, 364, *Longman*, *Harlow*, 1997.

Introduction: "In this paper, we show a general result of existence and uniqueness for backward stochastic differential equations (BSDE) driven by a general càdlàg martingale. The assumption of a uniformly Lipschitz condition on the driving parameter is also relaxed. The classical results on the existence and uniqueness of BSDE suppose that the driver of the equations is uniformly Lipschitz with bounded Lipschitz constants. This last assumption is the main drawback of this result, because many problems do not satisfy it. For example, the classical pricing problem is equivalent to solving a one-dimensional linear BSDE of the form $-dY_t = (r_tY_t + Z_t^*\theta_t)dt - Z_t^*dW_t;\ Y_T = \xi$, where ξ is the contingent claim to price and to hedge. In this model, r is the short rate of the interest and θ is the risk premium vector. To suppose that the short rate r is uniformly bounded is an assumption rarely satisfied in a market. The same remark holds for the risk premium vector. Moreover, we are interested in a nondeterministic horizon T. Some recent works of Pardoux (1996) take into account this assumption.

"By introducing stronger integrability conditions on the solution which depend on the stochastic Lipschitz bounded process, we relax the two assumptions of bounded Lipschitz constant and finite horizon. Moreover, we do not refer to a Brownian filtration and consider a general BSDE driven by a general càdlàg martingale and continuous increasing process in the generalized sense."

For the entire collection see 2000k:60003

2000k:60003 60-06

★**Backward stochastic differential equations.**

Papers from the study group held at the University of Paris VI, Paris, 1995–1996.

Edited by Nicole El Karoui and Laurent Mazliak.

Pitman Research Notes in Mathematics Series, 364.

Longman, *Harlow*, 1997. *ii*+221 *pp.* *ISBN* 0-582-30733-3

Contents: N. El Karoui, Backward stochastic differential equations: a general introduction (7–26); N. El Karoui and S.-J. Huang, A general result of existence and uniqueness of backward stochastic differential equations (27–36); M. Pontier, Solutions of forward-backward stochastic differential equations (39–46); G. Barles and E. Lesigne, SDE, BSDE and PDE (47–80); M. C. Quenez, Stochastic control and BSDEs (83–99); L. Mazliak, The maximum principle in stochastic control and backward equations (101–113); S. Hamadene, J.-P. Lepeltier and S. Peng [Shi Ge Peng], BSDEs with continuous coefficients and stochastic differential games (115–128); D. Saada, Observability cone of a constrained stochastic process (129–138); S. Peng [Shi Ge Peng], Backward SDE and related g-expectation (141–159); S. Hamadene, J.-P. Lepeltier and A. Matoussi, Double barrier backward SDEs with continuous coefficient (161–175); V. Bally, Approximation scheme for solutions of BSDE (177–191); Y. Hu [Ying Hu], Stability theorems and homogenization of nonlinear PDEs with periodic structures (193–205); E. Pardoux, Generalized discontinuous backward stochastic differential equations (207–219).

{Some of the papers are being reviewed individually.}

2000f:60161 60K99 60G40 62L05 90B36

El Karoui, Nicole (F-PARIS6-PB)**;**
Karatzas, Ioannis (1-CLMB-MS)

Synchronization and optimality for multi-armed bandit problems in continuous time. (English. English, Portuguese summary)

Mat. Apl. Comput. **16** (1997), *no.* 2, 117–151.

The authors provide a complete solution to a general, continuous-time dynamic allocation (multi-armed bandit) problem with arms that are not necessarily independent or Markovian. Such models are designed to capture the essential conflict inherent in situations when one has to choose between actions that yield high reward in the short term, and actions whose reward can be reaped only later. The main difficulty in these models comes from the "interaction of different time-scales". To solve the problem, the authors use results from time-changes, optimal stopping, and multiparameter martingale theory. The so-called "synchronization identity" for dynamic allocation is shown to be necessary and sufficient for optimality in the case of decreasing rewards. This leads to the construction of optimal strategies for the general case as well. *K. M. Ramachandran* (1-SFL)

99b:60078 60H10 60G40 90A09

El Karoui, N. (F-PARIS6-PB); **Pardoux, E.** (F-PROV-LAT);
Quenez, M. C. (F-MARN-EM)

Reflected backward SDEs and American options.

Numerical methods in finance, 215–231, *Publ. Newton Inst.*,
Cambridge Univ. Press, *Cambridge*, 1997.

This is a companion paper to [N. El Karoui and M.-C. Quenez, in *Numerical methods in finance*, 181–214, Cambridge Univ. Press, Cambridge, 1997; MR 98e:90057] from which it adopts notation and results.

Omitting many details, a reflected backward stochastic differential equation can be described as a triple of data $[\xi, f, S]$ (the known information) and a triple of processes $[Y, \underline{Z}, K]$ (the derived information) which together satisfy three "constraints" $Y_t = \xi + \int_t^T f(s, Y_s, \underline{Z}_s)ds + K_T - K_t - \int_t^T \langle \underline{Z}_s, d\underline{B}_s \rangle$, $Y_t \geq S_t$, $\int_0^T (Y_t - S_t) dK_t = 0$.

The first part of the paper is devoted to the derivation of comparison theorems which insure unique progressively measurable derived data and to the proof of an existence theorem. In the second part, different control problems associated with reflected backward stochastic differential equations are solved. These include an evaluation of $K_T - K_t$ in terms of the other ingredients of the problem, an evaluation of Y as the value of a stopping time optimal problem, and classes of f's (linear, concave and convex) which yield solutions Y with optimal properties. A section is devoted to showing that reflected backward stochastic differential equations serve to produce probabilistic representations of solutions of some obstacle problems for a nonlinear parabolic partial differential equation. In the final part of the paper it is shown that the price of an American contingent claim can be expressed as the solution of a convex reflected backward stochastic differential equation.

For the entire collection see 98c:90004

A. F. Gualtierotti (Chavannes)

99a:90033 90A09 60H10 60H30 90-06

Biais, B. (F-TOUL-IEC); **Björk, T.**; **Cvitanić, J.** (1-CLMB-S);
El Karoui, N. (F-PARIS6-PB); **Jouini, E.** (F-ENSAE);
Rochet, J. C. (F-TOUL-IEC)

★**Financial mathematics.**

Lectures given at the 3rd C.I.M.E. Session held in Bressanone, July 8–13, 1996.
Edited by W. J. Runggaldier.
Lecture Notes in Mathematics, 1656.
Springer-Verlag, *Berlin*, 1997. viii+316 *pp.* $56.00.
ISBN 3-540-62642-5

Contents: Bruno Biais and Jean Charles Rochet, Risk sharing, adverse selection and market structure (1–51); Tomas Björk, Interest rate theory (53–122); Jakša Cvitanić, Optimal trading under constraints (123–190); N. El Karoui and M. C. Quenez, Non-linear pricing theory and backward stochastic differential equations (191–246); Elyès Jouini, Market imperfections, equilibrium and arbitrage (247–307).

The volume under review is composed of expanded versions of lectures delivered during the summer school on financial mathematics held in Bressanone in July 1996. Four sections are devoted to various aspects of financial modelling under market imperfections. One section deals with the problems related to the modelling of the term structure of interest rates. The whole collection provides an excellent review of the most important problems and techniques related to arbitrage pricing theory in markets with frictions.

"Risk-sharing, adverse selection and market structure", by Biais and Rochet, deals with various issues associated with financial modelling in the presence of asymmetric information. In contrast to the more standard approach to arbitrage pricing in imperfect markets, in which frictions of various kinds are exogenously specified, in the case of a competitive market with asymmetric information the market frictions are endogenous; that is, they arise as a consequence of the agents' behaviour. In particular, under the assumption that liquidity traders are irrational "noise traders", the following cases are considered: (i) the informed agents are competitive, (ii) the informed agent is strategic and trading takes place in a dealer market, (iii) the informed agent is strategic and the market structure is endogenously derived as an optimal mechanism. The case when one strategic agent, with both informational and liquidity motivations to trade, submits buy and sell orders to market makers is also studied. The problem of existence and uniqueness of the market equilibrium is examined through the calculus of variations and convex analysis techniques.

"Interest rate theory", by Björk, is an almost complete review of the present state of term structure modelling. The basic definitions of arbitrage pricing theory are first recalled. Then, the most important results and methods related to the valuation of interest rate derivatives are presented (in some instances, results are stated at an informal level, however). A large variety of probabilistic techniques related to bond price modelling and the valuation of contingent claims is examined, and existing models of the term structure are reviewed (in particular, short-term interest rate models, models of instantaneous forward rates, and models of Libor rates). Applications to the valuation and hedging of the most typical interest-rate sensitive derivatives (such as bond options, caps, swaps, or swaptions) are also presented.

"Optimal trading under constraints", by Cvitanić, is devoted to techniques and results related to the theory of optimal trading under convex constraints in a continuous-time setting (a multidimensional version of the Black-Scholes-Merton market model). Both the hedging problems and the portfolio optimization problems are considered, under the assumption that the agent's portfolio has to take values in a prespecified convex set (this covers, in particular, the case of short-sale constraints, prohibition of borrowing of cash, and market incompleteness). The approach is based on the convex duality which leads to an equivalent formulation in terms of (super-) martingale measures, and thus it is referred to as the martingale approach. The arbitrage-free property of the model is proved, under the assumption that derivative securities trade at the price which belongs to the interval determined by the lower price (i.e., the maximal price for the buyer) and the upper price (i.e., the minimal price for the seller). The case of different lending and borrowing rates and the case of proportional transaction costs are also studied via the martingale approach. Finally, models in which drift and diffusion coefficients depend on the portfolio and wealth are analysed through the associated forward-backward stochastic differential equations.

"Nonlinear pricing theory and backward stochastic differential equations", by El Karoui and Quenez, presents an alternative approach to some of the above-mentioned problems. Their main tool is the theory of backward stochastic differential equations (BSDEs), forward-backward SDEs, and reflected BSDEs. After a short introduction to the standard (i.e., linear) arbitrage pricing theory, existence, uniqueness, and comparison theorems for BSDEs with Lipschitz continuous coefficients are presented (results of this type for reflected BSDEs are also provided). Subsequently, the non-linear arbitrage pricing theory based on the link between trading with convex constrained porfolios and BSDEs is developed. The non-linear pricing of American contingent claims (e.g., American options) in an imperfect market is proved to be related to existence of solutions to reflected BSDEs. In a Markovian framework, the price of a European option is shown to correspond to the viscosity solution of a non-linear parabolic PDE. In a similar vein, the price of an American option in an incomplete market is associated with the viscosity solution of an obstacle problem for a parabolic PDE.

"Market imperfections, equilibrium and arbitrage", by Jouini, focuses on arbitrage-free properties of discrete- and continuous-time financial models in the presence of market frictions such as shortsale constraints, different lending and borrowing rates, and transaction costs. In the case of a frictionless market, the arbitrage-free property of a financial market model is known to be essentially equivalent to the martingale property of the relative price process under an equivalent probability measure (commonly referred to as the equivalent martingale measure). In a market model with shortselling restrictions, suitable generalizations are examined, namely, the super- (or sub-)martingale measure for relative price processes. The case of a market model with transaction costs (reflected here by the bid-ask spread) is also studied. In this case, the no-arbitrage feature is shown to be equivalent to the existence of a process which lies between the ask and bid prices, and admits an equivalent martingale measure. Finally, the arbitrage bounds for prices of contingent claims in imperfect markets are analysed. *Marek Rutkowski* (PL-WASWT)

98m:90016 90A09

El Karoui, Nicole (F-PARIS6-PB)**;**
Jeanblanc-Picqué, Monique (F-EVRY-EAP)**;**
Shreve, Steven E. (1-CMU)

Robustness of the Black and Scholes formula. (English. English summary)

Math. Finance **8** (1998), *no.* 2, 93–126.

Summary: "Consider an option on a stock whose volatility is unknown and stochastic. An agent assumes this volatility to be a specific function of time and the stock price, knowing that this assumption may result in a misspecification of the volatility. However, if the misspecified volatility dominates the true volatility, then the misspecified price of the option dominates its true price. Moreover, the option hedging strategy computed under the assumption of the misspecified volatility provides an almost sure one-sided hedge for the option under the true volatility. Analogous results hold if the true volatility dominates

the misspecified volatility. These comparisons can fail, however, if the misspecified volatility is not assumed to be a function of time and the stock price. The positive results, which apply to both European and American options, are used to obtain a bound and hedge for Asian options."

98k:60096 60H10 35K85 35R60 60H30

El Karoui, N. (F-PARIS6-PB); **Kapoudjian, C.** (F-ENSLY);
Pardoux, E. (F-PROV-LAT);
Peng, S. **[Peng, Shi Ge]** (PRC-SHAN-IM);
Quenez, M. C. (F-MARN)

Reflected solutions of backward SDE's, and related obstacle problems for PDE's. (English. English summary)

Ann. Probab. **25** (1997), *no.* 2, 702–737.

A backward stochastic differential equation often has the form $X_t = Y + \int_t^T f(X_s, Y_s, Z_s)ds + \int_t^T Z_s dB_s$, where B is a Brownian motion; one adds the requirement that (X, Y), and of course Z, must be predictable. The martingale representation property of B is key to proving that such equations have solutions. These equations have proved useful in control theory, stochastic finance theory, and for the indirect study of certain partial differential equations. Here the authors consider a slightly more complicated equation with a solution (X, Y, K, Z), where K is an extra singular term that keeps the solution in a certain region of the state space (e.g., reflection at a boundary). The "Skorokhod problem" provides a naive solution, but Z must again intervene to keep K predictable.

Once a theory with K is developed, the authors are able to do several pretty things, such as proving comparison theorems for solutions of backward stochastic differential equations, approximating a solution of a reflecting backwards equation using a penalization procedure, and studying optimal stopping problems for certain nonlinear parabolic partial differential equations. *Philip Protter* (1-CRNL-OR)

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98e:90057 90A60 60H10 65U05

El Karoui, N. (F-PARIS6-PB); **Quenez, M. C.** (F-MARN-EM)

Imperfect markets and backward stochastic differential equations.

Numerical methods in finance, 181–214, *Publ. Newton Inst.*, *Cambridge Univ. Press*, *Cambridge*, 1997.

The authors study the backward stochastic differential equation (BSDE) $-dY_t = f(t, Y_t, Z_t)dt - Z_t^* dW_t, \quad Y_T = \xi$.

In Section 1 they discuss situations in finance where the BSDE point of view is efficient. Hedging for different types of markets is considered. Section 2 contains a new proof of the theorem on the existence and uniqueness of the solution of a BSDE. Different properties of the solution (Y, Z) and a comparison theorem for linear, one-dimensional BSDE are given. Markov properties of solutions of forward BSDEs and relations between forward BSDEs and partial differential equations are studied in Section 3. A generalization of the Feynman-Kac formula is given. The BSDE with concave standard generator f and stochastic control problems are considered. Different properties of recursive utility Y_t are obtained. Some problems of nonlinear arbitrage pricing theory are discussed in the last section. References on the main classical results in this field are given.

For the entire collection see 98c:90004

A. Ya. Olenko (UKR-KIEVM)

98d:90030 90A09 60H10

El Karoui, N. (F-PARIS6-PB);
Peng, S. [Peng, Shi Ge] (PRC-SHAN-IM);
Quenez, M. C. (F-MARN-EM)

**Backward stochastic differential equations in finance.
(English. English summary)**

Math. Finance **7** (1997), *no.* 1, 1–71.

In this paper the authors provide an in-depth discussion of backward stochastic differential equations (BSDEs) and their applications to stochastic finance. According to É. Pardoux and Peng [Systems Control Lett. **14** (1990), no. 1, 55–61; MR 91e:60171], the solution of a BSDE consists of a pair of adapted processes (Y, Z) satisfying

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t^* dW_t, \quad Y_T = \xi,$$

where f is the generator and ξ the terminal condition. BSDEs appear in numerous problems in finance such as in the context of contingent claim valuation in complete markets, in connection with upper prices and local risk-minimizing strategies in incomplete markets, and in formulating recursive utility. The paper under review will serve as an inspiration as well as a standard reference to researchers working in this field.

After some examples of BSDEs appearing in the problem of pricing and hedging contingent claims, the authors present important technical results for BSDEs: a priori estimates of the difference between two solutions, existence and uniqueness, a comparison theorem, supersolutions, and continuity and differentiability of solutions of BSDEs with respect to parameters. Specializing to concave (convex) BSDEs the authors show that in this case the solution of the BSDE can be written as the value function of a control problem. Furthermore, solutions of BSDEs associated with a state process satisfying some classical forward SDEs are studied. These Markovian BSDEs give a Feynman-Kac representation of some nonlinear parabolic partial differential equations. Conversely, under smoothness assumptions the solution of the BSDE corresponds to the solution of a system of quasilinear parabolic PDEs. Finally the properties of differentiation on Wiener space of the solution of a BSDE are studied. As an application of these results to finance it is shown that the portfolio process of a hedging strategy corresponds to the Malliavin derivative of the price process, generalizing a result by D. L. Ocone and I. Karatzas [Stochastics Stochastics Rep. **34** (1991), no. 3-4, 187–220; MR 93b:60098]. *Rüdiger Kiesel* (4-LSE-S)

96j:90015 90A09

Geman, Hélyette; El Karoui, Nicole (F-TOUL-GR);
Rochet, Jean-Charles (F-PARIS6-PB)

Changes of numéraire, changes of probability measure and option pricing. (English. English summary)

J. Appl. Probab. **32** (1995), *no.* 2, 443–458.

A market is considered in which there is no arbitrage in the sense that there exists a numéraire (a strictly positive price process) n and an equivalent probability π such that for each basic security S, $S(t)/n(t)$ is a π-local martingale. It is shown that for any other numéraire X for which $X(t)/n(t)$ is a π-martingale there exists an equivalent probability Q_X such that each S, denominated by X, is a Q_X-local martingale; furthermore, if a contingent claim has a fair price under (n,π), then it has a fair price under (X,Q_X) and the hedging portfolios are the same. Several examples are given to demonstrate that an appropriate choice of numéraire may simplify the computation of option prices and the identification of the hedging portfolios. *Douglas P. Kennedy* (4-CAMB-S)

96h:90021 90A09 60G44 90C39

El Karoui, Nicole (F-PARIS6-PB);
Quenez, Marie-Claire (F-PARIS6-PB)

Dynamic programming and pricing of contingent claims in an incomplete market. (English. English summary)

SIAM J. Control Optim. **33** (1995), *no.* 1, 29–66.

The authors study martingale pricing methods in incomplete markets, and first characterise attainable claims as those whose expectation is constant over the set of equivalent martingale measures (EMMs), and identify EMMs via exponential densities, which also enables them to characterise the Föllmer-Schweizer minimal measures in terms of risk premia (see H. Föllmer and M. Schweizer [in *Applied stochastic analysis (London, 1989)*, 389–414, Gordon and Breach, New York, 1991; MR 92g:90029], J.-P. Ansel and C. Stricker [Ann. Inst. H. Poincaré Probab. Statist. **28** (1992), no. 3, 375–392; MR 94d:60069], N. Christopeit and M. Musiela [Stochastic Anal. Appl. **12** (1994), no. 1, 41–63; MR 95d:90015] for related results). The case of non-attainable claims is then studied in detail.

The price dynamics are familiar: stocks $(P_i)_{i\le n}$ are driven by d-dimensional Brownian motion $(W^j)_{j\le d}$ $(n\le d)$, and $(\mathcal{F}_t)$-predictable processes represent the riskless interest rate r, drift coefficients (b_i) and volatilities (σ_{ij}). The absence of arbitrage guarantees risk premia

θ_t such that $b_t - r_t\mathbf{1} = \sigma_t\theta_t$. The wealth process $X^{\pi,-C,x}$ of a portfolio strategy π with savings D, consumption $-C$, and initial value x is viewed alternatively from the perspective of a seller or buyer of a contingent claim $B \in \mathcal{F}_T$. For the seller a nonnegative optional price process Y is admissible if $Y_T = B$ and $Y_t = X^{\pi,-C,x}(t) \geq 0$ for all $t \leq T$. The minimal such price is defined to be the selling price of B. A dual notion defines the buying price. In a complete market these prices coincide and are given by the expected value of the discounted claim under the unique EMM.

While this carries over to attainable claims in incomplete markets, for a general claim B the range of possible time 0 prices is bounded by the sup and inf of these expected values, taken over the set of all EMMs. These bounds can be identified by stochastic control methods [see N. El Karoui, in *École d'Été de Probabilités de Saint-Flour IX (Saint Flour, 1979)*, 73–238, Lecture Notes in Math., 876, Springer, Berlin, 1981; MR 83c:93062] by a careful study of the minimal right-continuous supermartingale J dominating B. If $(\mathcal{F}_t)$ is generated by the Brownian motion, the predictable decomposition of J ensures that J is the selling price for B; in the general case this requires further analysis of the optional decomposition. An analogous analysis identifies the buying price as the lower bound of the range of possible prices. The resulting portfolio strategies are compared with that for optimal price proposed by Föllmer and Schweizer. Examples and general computational methods for approximating J are discussed.

P. E. Kopp (4-HULL)

95i:60035 60G40 60G60 62L10 93E20

El Karoui, Nicole (F-PARIS6-PB); **Karatzas, Ioannis** (1-CLMB-S)

Dynamic allocation problems in continuous time. (English. English summary)

Ann. Appl. Probab. **4** (1994), *no.* 2, 255–286.

This paper proposes a solution to the non-Markovian multiarmed bandit problem by formulating it as a stochastic control problem for multiparameter processes, as proposed by A. Mandelbaum [Ann. Probab. **15** (1987), no. 4, 1527–1556; MR 89b:62169]. The authors define a strategy to be of index-type provided that, at all times, only arms with maximal Gittins index are engaged (the paper contains of course a more formal definition). The main result stated in the paper (Theorem 8.1) is that if the arms in the bandit process are all independent, then all index-type allocation strategies are optimal, and one particular index-type strategy is described. Unfortunately, the proof of Theorem 8.1 is incorrect; in particular, the identity (A.15) is not valid in the generality claimed. The authors plan to publish a correction note, and claim to have a correct and more general proof that will be published in the near future.

Robert C. Dalang (CH-LSNP)

95e:93082 93E20 62K05 62N05

El Karoui, N. (F-PARIS6-PB); **Gerardi, A.** (I-ROME-IM);
Mazliak, L. (F-PARIS6-PB)

Stochastic control methods in optimal design of life testing. (English. English summary)

Stochastic Process. Appl. **52** (1994), *no.* 2, 309–328.

The problem of optimal test design of two classes of dynamic systems, based on the accelerated life testing approach, is considered. The underlying dynamic models arise from the investigation of reliability of industrial items under a stress which can involve failure of items, and from the examination of a population of seeds with a stress acting on their fertility. Both models involve some unknown parameters determining the relationship between the level of stress and the dynamics of the system.

Following the Bayesian approach, the authors consider unknown parameters as random variables with a given initial distribution, and then arrive at a stochastic optimal control problem with partial observation. By including in the state the conditional distribution of the parameters with respect to the process observed up to time t, the problem is transformed into one with complete observation (separated problem). Furthermore, it is proved that with a special form of the generator of the controlled process, the above optimal control problem is equivalent to a mixed control problem where the number of elements under the test is replaced by a stopping time. Finally, previous results on mixed control problems, obtained by L. Mazliak [C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 1, 71–74; MR 91b:49026], are extended to the problems under consideration.

Nikolai K. Krivulin (St. Petersburg)

94m:90127 90D15 62L05

El Karoui, Nicole (F-PARIS6-PB); **Karatzas, Ioannis** (1-CLMB-S)

A dynamic programming approach to the optimality of general Gittins index sequences. (English. English summary)

Stochastic processes and optimal control (*Friedrichroda*, 1992), 27–49, *Stochastics Monogr.*, 7, *Gordon and Breach*, *Montreux*, 1993.

Summary: "We extend the dynamic programming approach of P. Whittle [J. Roy. Statist. Soc. Ser. B **42** (1980), no. 2, 143–149; MR 82b:62097] and J. N. Tsitsiklis [IEEE Trans. Automat. Control **31** (1986), no. 6, 576–577; MR 87f:90132], to obtain a simple and constructive proof for the optimality of Gittins index processes in the general, non-Markovian dynamic allocation (or 'multi-armed bandit') problem. This latter is formulated as a question of stochastic control with multi-parameter time, in the manner of A. Mandelbaum [Probab. Theory Relat. Fields **71** (1986), no. 1, 129–147; MR 87f:60065]. The approach yields an explicit expression for the value of this problem, and leads to various characterizations of Gittins indices. This paper is an expanded and more detailed version of another paper by us ["General Gittins index processes in discrete time", Proc. Nat. Acad. Sci. U.S.A., to appear]."

For the entire collection see 94j:60002

93k:60108 60G40

El Karoui, N. (F-PARIS6-PB); **Lepeltier, J.-P.** (F-LMNSS); **Millet, A.** (F-PARIS10)

A probabilistic approach to the reduite in optimal stopping. (English. English summary)

Probab. Math. Statist. **13** (1992), *no.* 1, 97–121.

Let $X=(X_t)$ be a Markov process and let g be a real function on the state space of X. Let the process one wants to stop depend only on the state of X at time t, i.e., suppose it can be written as $g(X_t)\mathbf{1}_{\{t<\zeta\}}$ (ζ is the life-time of X). If its Snell envelope has the same property, then the associated function is called the reduite of g, and is denoted by Rg. The authors present a new method for studying the reduite by introducing the set $\mathcal{A}(x)$ of all stopping measures $\mu=\mu^T(g)\overset{\text{def}}{=}\mathbf{E}_x[g(X_T)\colon T<\zeta]$ (determined by stopping time T) endowed with a "good" convex compact topology induced by the Baxter-Chacon topology on processes. This unified probabilistic approach is used to study the optimal stopping problem associated with the reduite, to prove the expression of the Snell envelope in terms of the reduite under very general assumptions, and to show continuity properties of the latter. Finally, the example of diffusion processes with jumps is described. *V. Mackevičius* (Vilnius)

93c:90019 90A12 60G99

Jeanblanc-Picqué, Monique (F-ENSET-AM); **El Karoui, Nicole** (F-PARIS6-PB); **Viswanathan, Ravi**

Bounds for the price of options.

Applied stochastic analysis (*New Brunswick, NJ*, 1991), 224–237, *Lecture Notes in Control and Inform. Sci.*, 177, *Springer*, *Berlin*, 1992.

Summary: "In this paper, we present an application of stochastic calculus to show 'the robustness of the Black-Scholes formula'. The Black-Scholes formula is used extensively in order to determine the price of financial products called options. This formula is valid only when the parameters (which can, in general, be stochastic and time-dependent) are constant or deterministic. When this is not the case, this formula is still used by means of an approximation of the parameters, without theoretical justification. We prove in this paper that this methodology is correct in some sense."

{For the entire collection see MR 93a:93100}.

For the entire collection see 93a:93100

93b:60088a 60G40 60J50 60J65 93E20

El Karoui, Nicole (F-PARIS6-P); **Karatzas, Ioannis** (1-CLMB-S)

A new approach to the Skorohod problem, and its applications.

Stochastics Stochastics Rep. **34** (1991), *no.* 1-2, 57–82.

93b:60088b 60G40 60J50 60J65 93E20

El Karoui, N.; Karatzas, I.

Correction: "A new approach to the Skorohod problem, and its applications".

Stochastics Stochastics Rep. **36** (1991), *no.* 3-4, 265.

After an introduction and summary, in Section 2 of the present paper the authors consider the Skorokhod problem of reflecting a continuous process W along a moving boundary. They find a very nice and surprising representation of the solution X to this problem. Take, for example, the exponential of the process $y+W_t$ with initial value y killed at the time of first reaching the boundary and integrate this expression over $(-\infty,x]$ with respect to the Lebesgue measure. Then the logarithm of the resulting process, $X_t(x)$, is the solution to the

Skorokhod problem with initial value x. This approach is applied to establish a deep connection between a stochastic control problem, the so-called monotone follower problem, and a certain optimal stopping problem for Brownian motion. Using another methodology the monotone follower problem was investigated by Karatzas [Proc. Nat. Acad. Sci. U.S.A. **82** (1985), no. 17, 5579–5581; MR 87i:93122] and by Karatzas and S. E. Shreve [SIAM J. Control Optim. **22** (1984), no. 6, 856–877; MR 87h:93075a]. In Sections 3 and 4, the optimal stopping problem and the monotone follower problem are introduced and discussed. In Section 5 the authors show that integrating the value function of the optimal stopping problem yields the value of the monotone follower problem and the optimal process for the monotone follower problem can be constructed from the optimal stopping time explicitly. In Section 6, a further integration of the value of the optimal stopping problem leads to a new and interesting representation of the value of the monotone follower problem. This is based on a representation of the value of the optimal stopping problem which is derived in an appendix, using ideas from the theory of balayage for continuous semimartingales. Unfortunately, the paper is marred by several printing errors; see the authors' correction.

H.-J. Engelbert (D-FSU-ST)

92k:60194 60J80 60G44 60G57

El Karoui, Nicole (F-PARIS6-S); **Roelly, Sylvie** (F-PARIS6-S)

Propriétés de martingales, explosion et représentation de Lévy-Khintchine d'une classe de processus de branchement à valeurs mesures. (French. English summary) [Martingale properties, explosion and Lévy-Khinchin representation of a class of measure-valued branching processes]

Stochastic Process. Appl. **38** (1991), *no.* 2, 239–266.

This paper investigates a general class of measure branching processes. Intuitively speaking, the "infinitesimal particles" move in a space E according to a right process with generator A and branch at site $x \in E$ according to a cumulant generator $R(x,z)$ which is the sum of terms $a(x)$ and $b(x)z$, a "continuous branching" term $-\frac{1}{2}c(x)z^2$ and another nonlinear term in z describing discontinuous branching at x in terms of a Lévy measure $\nu(x,d\lambda)$ on $(0,\infty)$. In more formal terms the measure-valued Markov process X_t starting in the measure m must obey $E(\exp(-\langle X_t,f\rangle))=\exp(-\langle m,U_tf\rangle)$, where U_tf is assumed to be a (weak sense) solution of $(\partial U_tf/\partial t)(t,x)=AU_tf(x)+R(x,U_tf(x))$.

The main result of the paper is a theorem giving various characterizations of X_t in terms of martingales, and the identification of its Lévy system $\widetilde{N}(ds, dv)$ via the equality $\iint g(s)\varphi(v)\tilde{N}(ds, dv) = \int g(s)\, ds \int X_s(dx) \int \nu(x, d\lambda)\varphi(\lambda\delta_x)$.

For certain non-Lipschitz R, the total mass $X_t(E)$ may explode in finite time, even though $m(E) < \infty$. It is shown that in this case the explosion time has a totally inaccessible component of intensity $\langle X_t, a\rangle$. By means of a Girsanov-type argument, the Laplace transform of the occupation time measure of X_t is computed. Finally, the infinitely divisible random measure X_t is represented as a Poissonian superposition of clusters having survived over the time horizon t.

Anton Wakolbinger (D-FRNK)

92j:93136 93E20

El Karoui, N. (F-PARIS6-PB);
Jeanblanc-Picqué, M. (F-ENSET-AM)

Martingale measures and partially observable diffusions.

Stochastic Anal. Appl. **9** (1991), *no.* 2, 147–176.

The following partially observable problem is considered in the paper. Let (X_t) be state and (Y_t) be observation processes satisfying the stochastic differential equations: $dX_t = b(t, X_t, Y_t, U_t)dt + \sigma(t, X_t, Y_t, U_t)\, dB_t$, $dY_t = h(t, X_t, Y_t, U_t)dt + dW_t$, where $X_t, B_t \in \mathbf{R}^d$, $Y_t, W_t \in \mathbf{R}^m$, B_t, W_t are independent Brownian motions, X_0 has given probability distribution μ and $Y_t = y$. The control process (U_t) takes values in a metric compact space A and is adapted to the past of (Y_t). The system is considered over the time interval $[0, T]$. The problem is to maximize the payoff $J(U) = E\int_0^T k(s, X_s, Y_s, U_s)ds + g(X_T, Y_T)$, where $k(\cdot)$ and $d(\cdot)$ are given functions which represent the instantaneous reward and terminal reward.

The value function is defined as follows: $v = \sup\{J(U)\colon U \in \mathcal{U}^0\}$, where $\mathcal{U}^0$ is the set of strictly admissible controls. The coefficients $b(\cdot)$, $\sigma(\cdot)$, $h(\cdot)$, and functions $k(\cdot)$, $g(\cdot)$ are supposed to be bounded measurable functions on corresponding phase spaces and uniformly continuous with respect to their arguments. It is important that $h(\cdot)$ depends on the control.

The set of controls is enlarged in the paper to those which take values in a set of probability measures, so-called relaxed controls, and thus the generalized control problem is formulated. Martingale representation of the problem is introduced, the corresponding set of integro-differential equations with respect to probability measure on the space of trajectories of the process is presented, and the definition of a weak solution to this set is given. It is proved that under the assumption of the uniqueness of a weak solution to this set the generalized control problem has an optimal solution. Some properties of this solution are also obtained. *M. S. Shtil'man* (New York)

92h:90012 90A09 49L20

El Karoui, Nicole (F-PARIS6-PB); **Quenez, Marie-Claire** (F-PARIS6-PB)

Programmation dynamique et évaluation des actifs contingents en marché incomplet. (French. English summary) [Dynamic programming and pricing of contingent claims in an incomplete market]

C. R. Acad. Sci. Paris Sér. I Math. **313** (1991), *no.* 12, 851–854.

Summary: "This note studies the problem of determining the price of a contingent claim from the price dynamics of certain securities in an incomplete market. It is proved by the use of stochastic control methods that, for every contingent claim, the maximum price (selling price) is the smallest price which allows the seller to hedge completely by a controlled portfolio of the basic securities."

91k:60058 60G60 60H05

El Karoui, N. (F-PARIS6-PB); **Méléard, S.** (F-LMNS)

Martingale measures and stochastic calculus.

Probab. Theory Related Fields **84** (1990), *no.* 1, 83–101.

The notion of martingale measures was introduced by J. B. Walsh [in *École d'été de probabilités de Saint-Flour, XIV—1984*, 265–439, Lecture Notes in Math., 1180, Springer, Berlin, 1986; MR 88a:60114]. A fundamental example of a continuous and orthogonal martingale measure is the white noise which can be characterized by the property of having a deterministic intensity. In this paper the authors study some properties of martingale measures and, in particular, they obtain two representation theorems for continuous martingale measures. The first one states that a continuous martingale measure is the time-changed image measure of a white noise. The method of proof is similar to that used to represent point processes as image measures of Poisson point processes. In the second representation theorem any continuous and square-integrable martingale measure is described in terms of stochastic integrals with respect to orthogonal martingale measures. A martingale problem for martingale measures is discussed in the last section of the paper. *David Nualart* (E-BARU-MS)

91h:60046 60G40 93E20

El Karoui, Nicole (F-PARIS6-PB); **Karatzas, Ioannis** (1-CLMB-S)

Integration of the optimal risk in a stopping problem with absorption.

Séminaire de Probabilités, *XXIII*, 405–420, *Lecture Notes in Math.*, 1372, *Springer*, *Berlin*, 1989.

The following summary represents well the content of the paper: "Integration with respect to the spatial argument of the optimal risk in a stopping problem with absorption at the origin yields the value function of the so-called 'reflected follower' stochastic control problem and provides a precise description of its optimal policy."

For the entire collection see 90i:60003

J. M. Stoyanov (Ottawa, ON)

90c:49031 49A60 60G35 60H10 93E20

El Karoui, Nicole;
Du′ Hủù Nguyen [Nguyễn Hũ'u Du'] (VN-HU);
Jeanblanc-Picqué, Monique (F-ENSET)

Existence of an optimal Markovian filter for the control under partial observations.

SIAM J. Control Optim. **26** (1988), *no.* 5, 1025–1061.

Summary: "This paper concerns the control of diffusions under partial observations. Part I studies the control of the signal process $dX_t = b(t, X_t, U_t)\,dt + \sigma(t, X_t, U_t)dB_t$ when the observation is $dY_t = h(t, X_t)\,dt + dW_t$ and when the objective is to maximize a reward function $E\{\int_r^T k(s, X_s, U_s)\,ds + g(X_T)\}$. The existence of an optimal relaxed control is proved. Part II studies the separated problem and proves the existence of an optimal Markovian filter. Then, we compare the two problems and prove, under mild conditions, that the value functions for the two problems are equal."

90c:49030 49A60 90C39 93E20

El Karoui, Nicole (F-PARIS6-PB); **Karatzas, Ioannis** (1-CLMB-S)

Probabilistic aspects of finite-fuel, reflected follower problems.

Acta Appl. Math. **11** (1988), *no.* 3, 223–258.

The authors study a stochastic control problem for a process of the form $X_t = x + W_t + \eta_t + K_t,\ 0 \le t \le T$; here, $x \ge 0$ is the initial position, W a one-dimensional Brownian motion, η a control process of bounded variation $\hat{\eta}$ such that $\hat{\eta}_{T-r} \le y$ for a given $y > 0$, and K is the reflection process which guarantees the inequality $X_t \ge 0$, for all $0 \le t \le T - r$. The following expected cost is considered: $J(\eta, r, x) = E(\int_0^{T-r} h(r+t, X_t)\,dt + \int_{[0,T-r)} f(r+t)d\hat{\eta}_t + g(X_{T-r}))$. The existence of an optimal control process which minimizes the expected cost, the equation of dynamic programming, and connections with an appropriate optimal stopping problem are proved. A suitable bibliography and some results on another follower problem are also given. *Piotr Zaremba* (PL-PAN)

89m:93089 93E20 49A60 49C10

El Karoui, Nicole (F-PARIS6-PB);
Jeanblanc-Picqué, Monique (F-ENSET)

Contrôle de processus de Markov. (French) [Control of Markov processes]

Séminaire de Probabilités, XXII, 508–541, *Lecture Notes in Math.*, 1321, *Springer*, *Berlin*, 1988.

The authors study a control problem for a Markov process. The controls are processes taking their values in an action space A, the relaxed controls are processes whose values are measures on this space. The use of a control is expressed mathematically by choosing a probability on the space $\Omega \times [0, \infty]$ defined through an exponential supermartingale; this latter renders an account of the process state and of the control action (an actualization coefficient is introduced). The objective function is a compound of an instantaneous gain and a terminal one, as well as of a terminal time T. A control rule is a measure on $\Omega \times [0, T] \times A$. For a fixed initial condition, the set of control rules is convex compact relative to an "intermediate" topology. The measurability of the value function and the possibility of interchanging "sup" and "integral" enable one to state the dynamic programming principle.

The value function is characterized as the unique weak solution of a Hamilton-Jacobi-Bellman equation. The value functions associated with control processes and relaxed processes are the same. There exists an optimal Markovian control which is a process.

Unlike the style of this review, this article is written in a very technical manner.

For the entire collection see 89c:60005

Jean-Yves Ouvrard (St. Martin d'Hères)

88f:93126 93E20 49A60 60J60

El Karoui, Nicole; Hùù Nguyen, Du' [Nguyễn Hũ'u Du'] (VN-HU)**;**
Jeanblanc-Picqué, Monique (F-ENSET)

Compactification methods in the control of degenerate diffusions: existence of an optimal control.

Stochastics **20** (1987), *no.* 3, 169–219.

The authors deal with control problems involving a multidimensional diffusion process which is the weak solution of the stochastic differential equation $dX_t = b(t, X_t, U_t)\,dt + \sigma(t, X_t, U_t)\,dW_t$. A payoff functional is supposed to be of the form $J(r, z, u) = E\int_r^T h(s, X_s, U_s)\,ds + g(X_T)$ and the control process is required to be progressively measurable and A-valued, A being a given compact metric space. The infinitesimal generator associated with the stochastic equation may be degenerate. The authors derive an equation using the dynamic programming method and prove the existence of an optimal Markovian relaxed control. They extend the results to the cases of Borelian coefficients and of diffusions with jumps. The authors investigate the problems of control for the system described above in detail and include an extensive list of references on the topic.

Věra Lánská (Prague)

88f:93125 93E20 31C20 49C20

El Karoui, Nicole

Théorie du potentiel et contrôle stochastique. (French) [Potential theory and stochastic control]

Théorie du potentiel (*Orsay*, 1983), 261–279, *Lecture Notes in Math.*, 1096, *Springer*, *Berlin*, 1984.

The author presents some results concerning optimal stochastic control and potential theory. General optimal stopping control problems are studied and the connections between dynamic programming and potential theory are shown.

For the entire collection see 88d:31001

Pierre-Louis Lions (F-PARIS9-A)

87g:93078 93E11 60G40 60J60

El Karoui, Nicole (F-CENFAR);
Du' Huu Nguyen [Nguyễn Hũ'u Du'] (VN-HU);
Jeanblanc-Picqué, Monique (F-ENSET)

Existence d'un filtre markovien optimal en contrôle partiellement observable. (French. English summary) [Existence of an optimal Markov filter for control under partial observations]

C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), *no.* 1, 31–34.

Summary: "We study the control of a diffusion under partial observations. We model the problem by enlarging the filtration of the observation and by using relaxed controls. We prove the existence of an optimal Markov filter."

87a:93081 93E20 60J25

El Karoui, N.; Lepeltier, J.-P. (F-LMNS); **Marchal, B.**

Nisio semigroup associated to the control of Markov processes.

Stochastic differential systems (*Bad Honnef*, 1982), 294–301, *Lecture Notes in Control and Inform. Sci.*, 43, *Springer*, *Berlin*, 1982.

The authors recover the nonlinear Nisio semigroup under general hypotheses by means of a sequence of impulse control problems having increasing finite numbers of impulse stopping times.

For the entire collection see 86g:60006

Gérald Mazziotto (Issy-les-Moulineaux)

87a:60052 60G40 49B60 90C40

El Karoui, Nicole; Lepeltier, Jean-Pierre; Marchal, Bernard

Optimal stopping of controlled Markov processes.

Advances in filtering and optimal stochastic control (*Cocoyoc*, 1982), 106–112, *Lecture Notes in Control and Inform. Sci.*, 42, *Springer*, *Berlin*, 1982.

The paper deals with the following model of a controlled Markov process. In free evolution, the state of the system is a "good" Markov process $(\Omega, \mathcal{F}_t, X_t, P_\mu)$. An admissible control is an adapted process u_t with values in a compact set U. The control u acts on the process $X.$ as follows: Under the policy u, the law of the process $X.$ is given by $dP^u_\mu = L^u_\infty dP_\mu$, where L^u_t is a uniformly integrable martingale. With the usual reward process involving a terminal payoff, discount factor and running cost, the authors consider the problem of finding an optimal stopping time $\hat{T}$ and control $\hat{u}$ such that the expected reward is maximum. The authors show that first an optimal stopping rule $\hat{T}$ can be chosen and then the problem of choosing $\hat{u}$ can be reduced to the classical problem of instantaneous control.

For the entire collection see 86f:93005

Rajeeva L. Karandikar (New Delhi)

86k:60167 60K99 60J80

El Karoui, Nicole (F-PARIS6-S)

Nonlinear evolution equations and functionals of measure-valued branching processes.

Stochastic differential systems (*Marseille-Luminy*, 1984), 25–34, *Lecture Notes in Control and Inform. Sci.*, 69, *Springer*, *Berlin*, 1985.

In this article the author investigates the relationship between solutions of martingale problems for measure-valued branching processes which arise as limits of branching Markov processes and solutions of Riccati type evolution equations.

For the entire collection see 86g:93003 *Richard Holley* (1-CO)

84h:60084 60G40 60G07

El Karoui, Nicole

Une propriété de domination de l'enveloppe de Snell des semimartingales fortes. (French) [A domination property of the Snell envelope of strong semimartingales]

Seminar on Probability, *XVI*, *pp.* 400–408, *Lecture Notes in Math.*, 920, *Springer*, *Berlin-New York*, 1982.

The author generalizes the following result, obtained by analytical methods, in the context of the general theory of processes: If the gain function of the optimal stopping problem is the positive α-potential of a function of arbitrary sign, the smallest α-excessive function majorizing it is an α-potential. In fact, using balayage techniques she shows that the smallest strong supermartingale majorizing a strong semimartingale of class (D) is generated by an increasing process dominated by the total variation of the process generating the semimartingale. She uses the canonical decomposition of a semimartingale of class (D) and the properties of the Snell envelope. In the last section she interprets the results in a very general Markovian context.

For the entire collection see 83f:60003a *B. Marchal* (Athis-Mons)

83c:93062 93E20 60G40 60G42 60J60

El Karoui, N.

Les aspects probabilistes du contrôle stochastique. (French) [The probabilistic aspects of stochastic control]

Ninth Saint Flour Probability Summer School—1979 (*Saint Flour*, 1979), *pp.* 73–238, *Lecture Notes in Math.*, 876, *Springer*, *Berlin-New York*, 1981.

This paper is a systematic exposition of the fundamentals of stochastic control theory. Its main feature is a broad use of the ideas, methods and results of the general theory of processes and the stochastic calculus (theory of semimartingales). Because of this the author is able to decide in the framework of a single scheme questions that inevitably arise in the study of specific problems of stochastic optimal control.

In the first part he considers a general model of a controlled stochastic system. It is characterized by the fact that the hypothesis of compatibility of the set of admissible controls with a stochastic basis is formulated for Markov (rather than deterministic) stopping times. Here the Bellman principle is formulated in martingale terms.

The second part is devoted to the problem of optimal stopping. This problem is considered first in the traditional formulation, when the filtration satisfies the usual assumptions (in Dellacherie's sense), and then in a formulation in which the filtration is not right-continuous. Here the author uses in particular an approach connected with the σ-algebra of Meyer. She also studies a Markov variant of the problem of optimal stopping.

In part three, methods of the general theory of processes and the theory of semimartingales are applied to optimal control problems in which the measures P^u are equivalent to a probability measure P. The scheme considered includes both the problem of control of diffusion-type processes, and of jump processes. The author shows that the optimal control can be determined from the condition of the maximum of the Hamiltonian. Besides the general model, she considers an optimal control problem in a Markov formulation, as well as a control problem in which one must determine optimal stopping along with optimal control.

For the entire collection see 82k:60006 *Yu. M. Kabanov* (Moscow)

82k:60135 60H20 60G44 60J50

Chaleyat-Maurel, M.; El Karoui, N.; Marchal, B.

Réflexion discontinue et systèmes stochastiques. (French. English summary)

Ann. Probab. **8** (1980), *no.* 6, 1049–1067.

This paper concerns càdlàg semimartingales evolving in $G = \{x \in \mathbf{R}^d : x^1 \geq 0\}$ with a certain type of jump reflection at the boundary. First a nonprobabilistic "reflection problem" in $\mathbf{R}^1$ is studied. The results are applied to semimartingales and the relation to local time established. Maximal inequalities are obtained. Then $\mathbf{R}^d$-valued semimartingales are studied. The setup is as follows. Let (M_t) be a semimartingale in $\mathbf{R}^d$, (Y_t) and $(\overline{Y_t})$ be point processes with values in $(\mathbf{R}^d - \{0\}) \cup \{\delta\}$ and (H_t) be a càdlàg $\mathbf{R}_d^+$-valued process; q is the martingale measure associated with (Y_t); f is a real-valued function on $\mathbf{R}^+ \times \Omega \times \mathbf{R}_d^+$ and g, h are functions on $\mathbf{R}^+ \times \Omega \times \mathbf{R}_d^+ \times (\mathbf{R}^d - \{0\})$. Then a pair (X_t, K_t) ($X_t \in \mathbf{R}^d$, càdlàg; K_t càdlàg increasing, $K_0 = 0$) is a solution of the stochastic differential system $S(f, g, h)$ if

(i)

$$X_t^i = H_t^i + \int_0^t f^i(s, \omega, X_s)\, dM_s^i + \int_0^t \int_{\mathbf{R}^d - \{0\}} g^i(s, \omega, X_{s-}) q(ds, du) + \sum_{s \leq t} h^i(s, \omega, X_{s-}, \overline{Y}_s) + K_t I_{(i=1)}, \quad 1 \leq i \leq d;$$

(ii) $$X_t^1 \geq 0, \int_0^t 1_{(X_s^1 = 0)} dK_s^c = K_t^c, \Delta K_t = 2X_t^1 \quad \text{if}\, \Delta K_t \neq 0.$$

The solution is unique if any two solution pairs are indistinguishable. The main theorem in the paper gives conditions under which a unique solution exists. Roughly, these are Lipschitz conditions on f, g, h plus certain mild boundedness and integrability conditions. In the final part this result is applied to solve the Stroock-Varadhan "martingale problem" associated with an integro-differential operator of "Lévy generator" type together with Neumann boundary conditions at $X^1 = 0$.

M. H. A. Davis (London)

82f:60118a 60G48 60H05 60J45

Yor, Marc

Sur le balayage des semi-martingales continues. (French)

Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78), pp. 453–471, *Lecture Notes in Math.*, 721, *Springer, Berlin*, 1979.

82f:60118b 60G48 60H05

Stricker, C.

Semimartingales et valeur absolue. (French)

Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78), pp. 472–477, *Lecture Notes in Math.*, 721, *Springer, Berlin*, 1979.

82f:60118c 60G48 60H05 60J45

Meyer, P.-A.; Stricker, C.; Yor, M.

Sur une formule de la théorie du balayage. (French)

Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78), pp. 478–487, *Lecture Notes in Math.*, 721, *Springer, Berlin*, 1979.

82f:60118d 60G48 60H05 60J45

Stricker, C.

Encore une remarque sur la "formule de balayage". (French)

Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78), p. 610, *Lecture Notes in Math.*, 721, *Springer, Berlin*, 1979.

82f:60118e 60G48 60H05

El Karoui, Nicole

À propos de la formule d'Azéma-Yor. (French)

Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78), pp. 634–641, *Lecture Notes in Math.*, 721, *Springer, Berlin*, 1979.

82f:60118f 60G48 60J65

Meyer, P.-A.

Construction de quasimartingales s'annulant sur un ensemble donné. (French)

Séminaire de Probabilités, XIII (*Univ. Strasbourg, Strasbourg,* 1977/78), *pp.* 488–489, *Lecture Notes in Math.*, 721, *Springer, Berlin,* 1979.

Soit H un ensemble aléatoire optionnel fermé, Y une semimartingale de H^1 admettant la décomposition canonique $Y = M + B$, où M est une martingale locale et B un processus à variation finie. On pose, pour tout $t \geq 0$,

$$\tau_t(\omega) = \sup\{s < t\colon (s,\omega) \in H\}, \quad D_t(\omega) = \inf\{s > t\colon (s,\omega) \in H\}.$$

On suppose Y et H liés par la relation $Y_{D_t} = 0$ sur $\{D_t < \infty\}$. Dans beaucoup d'applications, Y sera continue et H sera l'ensemble des zéros de Y. Soit Z un processus prévisible localement borné; on a la formule (1) $Z_{\tau_t} = Z_{\tau_0} Y_0 + \int_0^t Z_{\tau_s}\, dY_s$ d'où il résulte que le processus $(Z_{\tau_t} Y_t)$ est une semimartingale. Cette formule a été établie par le rapporteur dans le cas où Y est le module d'une martingale continue. L'ensemble de ces articles est principalement consacré à diverses extensions et applications de ce résultat.

Dans le premier article, Yor, après avoir montré la formule (1) en toute généralité, montre que l'on peut calculer les balayés sur H des processus croissants classiques intervenant dans la théorie des semimartingales. Notons V^b le balayé sur H d'un processus à variation finie V et V^p sa projection duale prévisible; on a le résultat général suivant: $B^b = (Y_\infty 1_{[\tau,\infty)})^p$. On peut également calculer le processus $\langle yy \rangle$ de la manière suivante: c'est la projection duale prévisible d'un processus croissant non adapté que l'on construit à l'aide des trajectoires de B et des processus $I_t^r = \sup_{t \leq u \leq r} Y_u$.

Dans le deuxième article, Stricker, par un calcul direct de variation, montre que $(Z_{\tau_t} Y_t)$ reste une semimartingale quand Z est seulement supposé progressif; il en déduit le résultat suivant: Soit X un processus continu; X est une semimartingale si et seulement si $|X|$ a la même propriété.

Dans le troisième article on approfondit les résultats du deuxième et l'on montre la formule suivante:

$$(2) \qquad z_t Y_t = z_0 Y_0 + \int_0^t \zeta_s\, dY_s + \sum_{0 < s \leq t} (z_s - \zeta_s) Y_s + R_t^Z$$

où (z_t) désigne le processus (Z_{τ_t}), (ζ_t) sa projection prévisible, et où (R_t^Z) est un processus à variation finie continu porté par H. R^Z est nul quand Z est optionnel. On montre que dR^Z est absolument continu par rapport à une mesure aléatoire dV indépendante de Z et que la densité $r_t(\cdot, Z) = dR_t^Z/dV_t$ définit une mesure vectorielle $Z \to r(\cdot, Z)$ sur des espaces de Banach convenables. $z_t Y_t$ reste une semimartingale quand Y et H sont liés par la relation plus faible $Y_{D_t} Y_{D_{t-}} \leq 0$: c'est ce qui est remarqué dans le quatrième article.

Dans le cinquième, El Karoui donne une forme plus développée à la formule (2) et s'attache en particulier à donner une forme explicite au terme résiduel R^Z. Si l'on pose

$$K_t = \sum_{g \in H\mapsto, g \leq t} (M_{D_g} - M_g)$$

($H \mapsto$ désignant l'ensemble des extrémités gauches des intervalles contigus à H), elle montre que la projection duale optionnelle K^0 de K est continue, portée par H, et que l'on peut écrire $R_t^Z = (Z * K)^0 - ({}^0Z * K^0)_t$ (${}^0Z$ désignant la projection optionnelle de Z). On trouve ainsi, en évitant les difficiles théorèmes d'analyse fonctionnelle qui y étaient utilisés, des résultats plus précis que dans le deuxième travail.

Dans le dernier article, Meyer cherche s'il est possible, un ensemble fermé prévisible H étant donné, de construire une semimartingale continue Y vérifiant $H = \{Y = 0\}$. Il montre que c'est la cas pour la filtration naturelle d'un mouvement brownien. Dans le cas général on peut seulement construire une semimartingale n'ayant pas de sauts dans H.

For the entire collection see 80f:60003 *Jacques Azéma* (Paris)

81m:60143 60J55 60G44

El Karoui, Nicole

Temps local et balayage des semi-martingales. (French)

Séminaire de Probabilités, XIII (*Univ. Strasbourg, Strasbourg, 1977/78*), *pp.* 443–452, *Lecture Notes in Math.*, 721, *Springer, Berlin*, 1979.

In continuation of previous work by the author and others (see the collected papers of a seminar on this subject [*Temps locaux*, Astérisque, 52–53, Soc. Math. France, Paris, 1978; MR 81b:60042]) the calculus of local times for semimartingales is extended further. In particular, it is shown that the local time $L^0(X)$ of X at 0 can be obtained as the compensator of a process which grows only at the left endpoints of intervals of excursions from 0. As a concrete result in this connection we quote, for example, the following: As ε tends to 0, the processes $\sum_{0<l_i\leq t} 1_{\{l_i+\varepsilon\leq r_i\}}(X^+_{l_i+\varepsilon} - X^+_{l_i})$ converge, uniformly in t, in L^1 to $\frac{1}{2}\cdot L^0(X)$.

(Here the index i counts all excursions from 0; r_i [l_i] denotes the right [left] endpoint of the corresponding time interval.)

For the entire collection see 80f:60003

Hermann Rost (Heidelberg)

81a:60055 60G40

El Karoui, Nicole

Arrêt optimal prévisible. (French)

Measure theory applications to stochastic analysis (*Proc. Conf.*, *Res. Inst. Math.*, *Oberwolfach*, 1977), *pp.* 1–11, *Lecture Notes in Math.*, 695, *Springer*, *Berlin*, 1978.

L'enveloppe de Snell d'un processus optionnel est un outil introduit par J.-F. Mertens [Z. Wahrsch. Verw. Gebiete **22** (1972), 45–68; MR **49** #11616] qui a de multiples applications en arrêt optimal continu; c'est en effet la plus petite surmartingale optionnelle forte majorant un processus donné, et c'est en même temps le processus des espérances conditionnelles des gains maximums qu'on peut réaliser par arrêt d'un processus en un temps d'arrêt. Quand y est un processus prévisible limité à droite et à gauche tel que $\sup_{t\in[0,+\infty]}|y_t|$ est dans L_1, l'auteur construit l'enveloppe de Snell prévisible de y, qui est la plus petite surmartingale prévisible majorant y. Si y est continu à droite, c'est aussi la projection prévisible de l'enveloppe de Snell classique de y. On caractérise alors à l'aide de cette nouvelle enveloppe de Snell des temps d'arrêt généralisés optimaux, divisés en deux parties prévisibles et une partie optionnelle. Pour des problèmes liés, on pourra consulter trois articles [M. A. Maingueneau, Séminaire de Probabilités, XII (Univ. Strasbourg, Strasbourg, 1976/77), pp. 457–467, Lecture Notes in Math., Vol. 649, Springer, Berlin, 1978; the reviewer, C. R. Acad. Sci. Paris Sér. A-B **284** (1977), no. 23, A1519-A1521; MR **55** #13561; Ann. Probab. **7** (1979), no. 6, 933–964].

For the entire collection see 80c:60006

Jean-Michel Bismut (Orsay)

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58 7808 60G05

El Karoui, Nicole; Weidenfeld, Gérard

Théorie générale et changement de temps. (French)

Séminaire de Probabilités, XI (Univ. Strasbourg, Strasbourg, 1975/1976), pp. 79–108. *Lecture Notes in Math., Vol.* 581, *Springer, Berlin*, 1977.

Cet article complète le précédent [see #7807 above]. On s'intéresse maintenant au cas où le processus croissant adapté (C_t) est simplement supposé continu à droite. On donne de la même façon des expressions des différentes projections usuelles de la théorie générale des processus et l'on étudie dans la troisième partie un cas particulier important: On considère un fermé aléatoire optionnel M et l'ou pose $C_t = \sup\{s\colon s \leq t,\ (\omega, s) \in M\}$; j_t est alors le temps d'arrêt $\inf\{s > t\colon (\omega, s) \in M\}$ et l'on dispose ainsi de résultats de projection sur la filtration (F_{j_i}).

{For the entire collection see MR **55** #9186.}

Jacques Azema (Paris)

58 7807 60G05

El Karoui, Nicole; Meyer, P. A.

Les changements de temps en théorie générale des processus. (French)

Séminaire de Probabilités, XI (Univ. Strasbourg, Strasbourg, 1975/1976), pp. 65–78. *Lecture Notes in Math., Vol.* 581, *Springer, Berlin*, 1977.

Soit $(\Omega, (\mathbf{F}_t), P)$ un espace muni d'une filtration $(\mathbf{F})_{t\geq 0}$ continue à droite, ou se donne un processus croissant $(C_t)_{t\geq 0}$, continu, adapté à la famille $(\mathbf{F}_t)$. On pose $j_t = \inf\{s\colon C_s > t\}$. Il est bien connu que (j_t) est une famille croissante de temps d'arrêt. L'objet de ce travail est d'étudier la théorie générale des processus relative à la filtration continue à droite $(\mathbf{F}_{j_t})$. On exprime en particulier les opérations de projection, de projection duale, etc., en fonction des opérations correspondantes relatives à la filtration initiale.

{For the entire collection see MR **55** #9186.}

Jacques Azema (Paris)

56 6852 60G55 60H05 60G45

El Karoui, Nicole; Lepeltier, Jean-Pierre

Représentation des processus ponctuels multivariés à l'aide d'un processus de Poisson. (French)

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **39** (1977), *no.* 2, 111–133.

The authors show that, under very general conditions, every quasi-left-continuous (q.l.c.) point process (Y_t) can be obtained from some Poisson process $(\hat{Y}_t)$ with continuous Lévy measure m in the sense that $\sum_{s\le t} 1_{\{Y_s\in B\}} = \sum_{s\le t} 1_{\{n(\omega,s,\hat{Y}_s)\in B\}}$, where $n(\omega,s,u)$ is a predictable function. They also give a construction of the stochastic integral with respect to a martingale measure (for another approach see J. Jacod's paper [Séminaire de Probabilités, XI (Univ. Strasbourg, Strasbourg, 1975/1976), pp. 390–410, Lecture Notes in Math., Vol. 581, Springer, Berlin, 1977; MR **56** #16778]). A representation theorem for square integrable q.l.c. martingales as stochastic integrals with respect to some Wiener and Poisson processes is proved.

Ju. M. Kabanov (Moscow)

54 11530 60J60

El Karoui, Nicole

Processus de reflexion dans R^n. (French)

Séminaire de Probabilités, IX (Seconde Partie, Univ. Strasbourg, Strasbourg, années universitaires 1973/1974 *et* 1974/1975), *pp.* 534–554. *Lecture Notes in Math., Vol.* 465, *Springer, Berlin,* 1975.

In this exposition, diffusion processes with Ventcel′ type boundary conditions are considered. Let $\overline{G}$ be the half-space $x_1 \ge 0$ in R^n and $L = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x)\partial^2/ax_i\,\partial x_j + \sum_{j=1}^n b_j(x)\partial/\partial x_j$ be a diffusion operator on $\overline{G}$ with bounded measurable coefficients. The boundary operator Γ is given by $\Gamma = \partial/\partial x_i + \Lambda$, where Λ is the diffusion operator on $x_1 = 0$ defined by the relation $\Lambda = \frac{1}{2}\sum_{i,j=2}^n \alpha_{ij}(x)\partial^2/\partial x_i\partial x_j + \sum_{j=2}^n \nu_j(x)\partial/\partial x_j$, also with bounded measurable coefficients. The diffusion process corresponding to L in the interior G of $\overline{G}$ with instantaneous reflection along ∂G governed by Γ can be defined in one of several ways. One can use a pathwise stochastic differential formulation, or a suitable martingale (or submartingale) formulation. (See, for instance, the papers by S. Watanabe [J. Math. Kyoto Univ. **11** (1971), 169–180; MR **43** #1291] and R. F. Anderson [Indiana Univ. Math. J. **25** (1976), no. 4, 367–395; MR **54** #1402a].) The equivalence between those formulations is established in general. Existence and uniqueness of such a diffusion process is proved under Lipschitz

assumptions or b, ν and the square roots σ, $\tilde{\sigma}$ of a and α, respectively.
{For the entire collection see MR **51** #9136.} *R. S. Varadhan*

54 3864 60J50 60J55

El Karoui, Nicole; Reinhard, Hervé

★**Compactification et balayage de processus droits. (French)**

With an English summary.

Astérisque, No. 21.

Société Mathématique de France, *Paris*, 1975. iii+106 *pp.*

For many years, practitioners of Markov process theory have been concerned with some aspect of the comparison between Markov processes which evolve in the same way until they reach a boundary. In this booklet, the authors present a development of the theory which includes the most precise results obtained so far, in a general setting, about the description of the process "near the boundary". The essential techniques are the general theory of processes of Meyer and Dellacherie, and the Ray-Knight theory of compactification. The general theory allows one to make an analysis based on certain projections of random measures which refine Motoo's theory of balayage of additive functionals. In order to compare these projections, which contain essential information about the behavior of the process near the boundary, the authors introduce an entrance process which is strong Markov on a compactification of the state space for the minimal process. The various kernels used in the construction have interpretations in terms of conditional expectations given the process up to a last exit time. Their results strengthen similar ones first obtained by Pittenger and Shih, and Getoor and the reviewer. Finally, the authors examine a situation which generalizes the most classical form of the problem—the behavior of the infinitesimal generator at the boundary. An appendix contains a discussion of Mokobodski's nearly positive operators and their uses in the theory of additive functionals. *Michael Sharpe*

54 1372 60G55 60H05 60J30

El Karoui, Nicole; Lepeltier, Jean-Pierre

Processus de Poisson ponctuel associé à un processus ponctuel, représentation des martingales de carré intégrable, quasi-continue à gauche. (French. English summary)

C. R. Acad. Sci. Paris Sér. A-B **280** (1975), *A*ii, *A*1025–*A*1027.

Let $(U, \mathbf{U})$ be a measurable space and δ a "point at infinity" outside U. A process (Y_t), $t \in \mathbf{R}^+$, with values in $U \cup \{\delta\}$ is called a point process if, for each sample point ω, $Y_t(\omega) \neq \delta$ for only countably many t. A Poisson point process is one which is in a certain sense homogeneous in time with independent increments [P. A. Meyer, Séminaire de Probabilités, V (Univ. Strasbourg, année universitaire 1969–1970), pp. 177–190, Lecture Notes in Math., Vol. 191, Springer, Berlin, 1971; MR **52** #9387; erratum, Séminaire de Probabilités, VI (Univ. Strasbourg, année universitaire 1970–1971), p. 253, Lecture Notes in Math., Vol. 258, Springer, Berlin, 1972; MR **52** #9388]. The authors give conditions under which a point process with values in $\mathbf{R}^n \setminus \{0\}$ can be expressed as an "image" of a Poisson point process. Having done this they show how a left quasi-continuous, right continuous, square integrable martingale can be written as the sum of a stochastic integral with respect to Brownian motion and a stochastic integral with respect to a "Poisson martingale measure". The point is that the latter term makes up for the lack of left continuity of the martingale.

Tim Traynor

53 1751 60J55

El Karoui, Nicole

Balayage et changement de temps. (English. English summary)

C. R. Acad. Sci. Paris Sér. A-B **281** (1975), *no.* 24, *A*ii, *A*1099–*A*1101.

Author's summary: "Let A be a continuous additive functional of a right process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P^x, x \in E)$ and let τ be the right inverse of A. Let $\hat{X}_t = X_{\tau(t)}$ and $\tilde{\mathcal{F}}_t = \mathcal{F}_{\tau(t)}$, $\hat{\mathcal{F}}_t = \sigma(X_s; s \leq t)$. After studying the notions of the general theory with regard to the σ-fields $\tilde{\mathcal{F}}_t$ and $\hat{\mathcal{F}}_t$, the Lévy system of the process $\hat{X}$ is defined and the conditional law of $\tau(t)$ with respect to $\hat{\mathcal{F}}_\infty$ is calculated."

Related results were obtained previously by J.-M. Rolin [Pacific J. Math. **58** (1975), no. 2, 585–604; MR **52** #4435] under more restrictive conditions.

M. G. Sur

51 14288 60J60

Karoui, N. [El Karoui, Nicole] ; Reinhard, H.

Processus de diffusion dans R^n. (French)

Séminaire de Probabilités, VII (*Univ. Strasbourg, année universitaire* 1971–1972), *pp.* 95–117. *Lecture Notes in Math., Vol.* 321, *Springer, Berlin*, 1973.

La première partie de ce travail est centrée autour d'un théorème d'équivalence qui affirme qu'il y a existence et unicité de la diffusion (quasi-diffusion) associée à un opérateur elliptique si et seulement s'il y a existence et unicité des solutions d'une certaine équation stochastique. Dans la deuxième partie les auteurs décrivent quelques propriétés principales des diffusions dans R^n; la plus grande partie des résultats ont été exposés par les auteurs, A. Bonami et B. Roynette [Ann. Inst. H. Poincaré Sect. B (N.S.) **7** (1971), 31–80; MR **44** #7637]. Les auteurs citent également certains articles où on peut trouver d'autres propriétés importantes de ces diffusions (S. Watanabe and T. Yamada [J. Math. Kyoto Univ. **11** (1971), 155–167; MR **43** #4150], D. W. Stroock and S. R. S. Varadhan [Proc. Sixth Berkeley Sympos. Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970), Vol. III: *Probability theory*, pp. 353–359, Univ. California Press, Berkeley, Calif., 1972]). Les sujets abordés: martingales-fonctionnelles additives d'une diffusion; rang du processus; dualité des résolvantes; le support des probabilités dans l'espace des trajectoires.

{For the entire collection see MR **50** #1308.}

A. D. Wentzell (Ventcel)

49 8123 60J50

El Karoui, Nicole ; Reinhard, Hervé

Balayage d'un processus droit sur un ensemble aléatoire homogène. (French)

C. R. Acad. Sci. Paris Sér. A **278** (1974), 359–361.

Announcement of a long forthcoming paper which uses the powerful modern machinery of Markov process theory to rework and extend the results of the important, but difficult paper of M. Motoo [Proc. Fifth Berkeley Sympos. on Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: *Contributions to probability theory, Part* 2, pp. 75–110, Univ. California Press, Berkeley, Calif., 1967; MR **36** #3414], on the behavior of the "process on boundary" and the "entrance process". The techniques are variants of those of R. K. Getoor and the reviewer [Indiana Univ. Math. J. **23** (1973/74), 377–

404; MR **48** #12654] and P. A. Meyer [see B. Maisonneuve and Meyer, Séminaire de Probabilités, VIII (Univ. Strasbourg, année universitaire 1972–1973), pp. 172–175, Lecture Notes in Math., Vol. 381, Springer, Berlin, 1974; Meyer, same Séminaire, pp. 176–190; same Séminaire, pp. 191–211]. *Michael Sharpe*

44 1114 60.62

El Karoui, Nicole [Karoui, Nicole]

Diffusions avec condition frontière, associées à un opérateur elliptique dégénéré. (French)

C. R. Acad. Sci. Paris Sér. A-B **273** 1971 *A*311–*A*314

The author proves the existence and the uniqueness of a diffusion on an open region $G \subset R^d$ associated with a possibly degenerate elliptic operator

$$L = \tfrac{1}{2} \sum_{i,j \le d} a_{ij}(x)\, \partial^2/(\partial x_i \partial x_j) + \sum_{i \le d} b_i(x)\, \partial/\partial x_i.$$

The boundary of G is twice differentiable, the a_{ij}'s are bounded and continuous, the b_i's are bounded and measurable, and on ∂G one has a rather general (local) boundary condition. The technical conditions imposed are somewhat wider than those employed previously by, e.g., J.-M. Bony, P. Courrège and P. Priouret [Ann. Inst. Fourier (Grenoble) **18** (1968), fasc. 2, 369–521 (1969); MR **39** #6397] and D. W. Stroock and S. R. S. Varadhan [Comm. Pure Appl. Math. **24** (1971), 147–225; MR **43** #2774]. *H. P. McKean, Jr.*

{For errata and/or addenda to the original MR item see MR 45 Errata and Addenda in the paper version}

1 913 205 91B28

Barrieu, Pauline (F-PARIS6-PMA)**;**
El Karoui, Nicole (F-POLY-AM)

Optimal design of derivatives in illiquid markets. (English. English summary)

Quant. Finance **2** (2002), *no.* 3, 181–188.

1 791 318 91B28 60G99
El Karoui, Nicole (F-POLY-AM); **Geman, Helyette** (F-PARIS9); **Lacoste, Vincent**
On the role of state variables in interest rates models. (English. English summary)
Appl. Stoch. Models Bus. Ind. **16** (2000), *no.* 3, 197–217.

1 752 672 60H10
El Karoui, N. (F-PARIS6-PB)
Backward stochastic differential equations: a general introduction.
Backward stochastic differential equations (*Paris*, 1995–1996), 7–26, *Pitman Res. Notes Math. Ser.*, 364, *Longman*, *Harlow*, 1997.

1 478 202 90A14 60H10
El Karoui, N. (F-PARIS6-PB); **Quenez, M. C.** (F-MARN-EM)
Non-linear pricing theory and backward stochastic differential equations.
Financial mathematics (*Bressanone*, 1996), 191–246, *Lecture Notes in Math.*, 1656, *Springer*, *Berlin*, 1997.