When discontinuity matters: portfolio insurance in presence of jumps

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REFERENCES


Constant Proportion Portfolio Insurance


Ingredients:

Risky asset $S$: fund, index

Guarante: value of fund at maturity $V_T$ must be $V_T > N$

Reserve asset: bond $B_t$ (here: zero coupon with maturity $T$)

Self-financing strategy whose goal is to leverage the returns of a risky asset (traded fund or index) through dynamic trading while guaranteeing a fixed amount $N$ of capital at maturity $T$. 
Rule-based trading strategy

A fraction of the wealth is invested into the risky asset $S_t$ and the remainder is invested in bonds (ex: zero-coupon bond with maturity $T$ and nominal $N$) $B_t$. Denoting the value of the fund by $V_t$,

- if $V_t > B_t$, the exposure to the risky asset (wealth invested into the risky asset) is given by $mC_t \equiv m(V_t - B_t)$, where $C_t$ is the ’cushion’ and $m > 1$ is a constant multiplier.

- if $V_t \leq B_t$, the entire portfolio is invested into the zero-coupon.
CPPI: Black Scholes model

\[ \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad \frac{dB_t}{B_t} = r dt \]

Cushion \( C_t = V_t - B_t \) satisfies

\[ \frac{dC_t}{C_t} = (m\mu + (1 - m)r) dt + m\sigma dW_t, \]

\[ C_T = C_0 \exp \left( rT + m(\mu - r)T + m\sigma W_T - \frac{m^2 \sigma^2 T}{2} \right). \]

\[ V_T = N + (V_0 - Ne^{-rT}) \exp \left( rT + m(\mu - r)T + m\sigma W_T - \frac{m^2 \sigma^2 T}{2} \right) \]
\[ V_T = N + (V_0 - Ne^{-rT}) \exp \left( rT + m(\mu - r)T + m\sigma W_T - \frac{m^2 \sigma^2 T}{2} \right) \]

Interpretation: in the Black-Scholes model with continuous trading, a CPPI strategy is equivalent to a long position in a zero-coupon bond with nominal \( N \) to guarantee the capital at maturity and investing the remaining sum into a (fictitious) risky asset which has \( m \) times the excess return and \( m \) times the volatility of \( S \) and is perfectly correlated with \( S \).

No risk, expected returns increasing with leverage \( m \):

\[ E[V_T] = N + (V_0 - Ne^{-rT}) \exp(rT + m(\mu - r)T). \]
CPPI: stochastic volatility case

\[ \frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \quad \frac{dB_t}{B_t} = r dt \]

Cushion \( C_t = V_t - B_t \) still satisfies

\[ \frac{dC_t}{C_t} = (m\mu_t + (1 - m)r) dt + m\sigma_t dW_t = dX_t, \]

so \( C_t = C_0 \mathcal{E}(X) \) where \( X \) is a continuous semimartingale.

In particular \( C_t > 0 \): no risk of going below the floor.
Stochastic exponentials

$X$ semimartingale, $Y_0 > 0$

$$Y_t = Y_0 \mathcal{E}(X)_t \iff dY_t = Y_t dX_t$$

Solution:

$$Y_t = Y_0 \exp(X_t - \frac{1}{2}[X]_t) \prod_{0<s\leq t} (1 + \Delta X_s) e^{-\Delta X_s}$$

If $X$ continuous $\Rightarrow Y$ continuous and $Y_t > 0$

$X$ discontinuous: $Y > 0 a.s. \iff \forall s > 0, \Delta X_s > -1] a.s.$
Model setup

\[ \frac{dS_t}{S_{t-}} = dZ_t \quad \text{and} \quad \frac{dB_t}{B_{t-}} = dR_t, \]

where \( Z \) Lévy process with Lévy measure \( \nu \), \( R \) continuous semimartingale.

Ex. \( R_t = rT \)

Ex. 2 Vasicek model \( dr_t = (\alpha - \beta r_t)dt + \sigma dW_t \)

\( B_t = B(t, T) = E[e^{-\int_t^T r_s ds}] \) follows

\[ \frac{dB_t}{B_t} = r_t dt - \sigma \frac{1 - e^{-\beta(T-t)}}{\beta} dW_t. \]

\( S > 0 \) so \( \Delta Z_t > -1 \) almost surely.
CPPI strategy: discontinuous case

Time at which floor is reached $\tau = \inf\{t : V_t \leq B_t\}$.

CPPI is self-financing: for $t \leq \tau$

$$dV_t = m(V_t - B_t)\frac{dS_t}{S_t} + \{V_t - m(V_t - B_t)\} \frac{dB_t}{B_t},$$

Cushion $C_t = V_t - B_t$

$$\frac{dC_t}{C_t} = mdZ_t + (1 - m)dR_t,$$
Dynamics of the cushion

Solution by change of numeraire: \( C'_t = \frac{C_t}{B_t} \)

Ito formula:

\[
\frac{dC'_t}{C'_{t-}} = m(dZ_t - d[Z, R]_t - dR_t + d[R]_t),
\]

(1)

Define \( L_t \equiv Z_t - [Z, R]_t - R_t + [R]_t \)

\( C'_t = C'_0 \mathcal{E}(mL)_t, t \leq \tau \)

For \( t > \tau \), \( C'_* \) remains constant so

\[
C'_t = C'_0 \mathcal{E}(mL)_{t \wedge \tau},
\]

\[
\frac{V_t}{B_t} = 1 + \left( \frac{V_0}{B_0} - 1 \right) \mathcal{E}(mL)_{t \wedge \tau}.
\]

(2)
Gap risk

\[ C^*_t = C^*_0 \mathcal{E}(mL)_t, \quad t \leq \tau \quad \text{where} \quad L = Z - [Z, R] - R + [R] \]

Since \( L \) is discontinuous, \( C \) can go negative as soon as \( \Delta L_t < -1/m \)

**Proposition 1** The probability of going below the floor is given by

\[ P[\exists t \in [0, T] : V_t \leq B_t] = 1 - \exp \left( -T \int_{-\infty}^{-1/m} \nu(dx) \right). \]

Proof: since \( \mathcal{E}(mL)_t \) does negative as soon as \( m\Delta L_t < -1 \),

\[ \tau \leq T \iff \exists t \leq T, \Delta L_t < -\frac{1}{m} \]

The number of jumps of the Lévy process \( L^j \) in the interval \([0, T]\), whose sizes fall in \((-\infty, -1/m]\) is a Poisson random variable with intensity \( T\nu((-\infty, -1/m]) \).
Example: Kou model

\[ \nu(x) = \frac{\lambda (1- p)}{\eta_+} e^{-x/\eta_+} 1_{x>0} + \frac{\lambda p}{\eta_-} e^{-\|x\|/\eta_-} 1_{x<0}. \] (3)

Loss probability is then given by

\[ P[\exists t \in [0, T] : V_t \leq B_t] = 1 - \exp \left( -Tp\lambda \left( 1 - 1/m \right)^{1/\eta_-} \right). \]
Expected loss

**Proposition 2** Assume

\[ \int_1^\infty x \nu(dx) < \infty. \]

Then

\[ E[C^*_T | \tau \leq T] = \frac{\lambda^* + m \int_{-1/m}^{1/m} x \nu(dx)}{(1 - e^{-\lambda^* T})(\psi(-i) - \lambda^*)} (e^{-\lambda^* T} \phi_T(-i) - 1). \]

Proof: \( L = L^1 + L^2 \) where \( L^2 \) is a process with piecewise constant trajectories and \( \Delta L^2_t \leq -1/m \) and \( L^1 \) is a process with \( \Delta L^1_t > -1/m \).

\( L^1 \) has Lévy measure \( \nu(dx)1_{x>1/m} \) and \( L^2 \) has Lévy measure \( \nu(dx)1_{x\leq1/m} \), no diffusion component

\( \tau \) time of first jump of \( L^2 = \) exponential random variable with intensity \( \lambda^* := \nu((-\infty, -1/m]) \)
$\phi_t$ characteristic function of Lévy process $\log \mathcal{E}(mL^1)_t$ and

$\psi(u) = \frac{1}{t} \log \phi_t(u)$.

$$C_T^* = \mathcal{E}(mL^1)_{\tau \wedge T}(1 + m\tilde{L}^2)_{\tau \leq T} = \mathcal{E}(mL^1)_T 1_{\tau > T} + \mathcal{E}(mL^1)_\tau (1 + m\tilde{L}^2)_{\tau \leq T}. \quad (4)$$

Since $L^1$ and $L^2$ are Lévy processes, $\tau$, $\tilde{L}^2$ and $L^1$ are independent. Since

$$E[\mathcal{E}(mL^1)_t] = \phi_t(-i),$$

we have

$$E[C_T^*|\tau \leq T] = \frac{E[1 + m\tilde{L}^2]}{1 - e^{-\lambda^* T}} \int_0^T \lambda^* e^{-\lambda^* t} E[\mathcal{E}(mL^1)_t] dt$$

$$= (\lambda^* + m \int_{-1}^{-1/m} x \nu(dx)) \frac{1}{1 - e^{-\lambda^* T}} \int_0^T e^{-\lambda^* t} \phi_t(-i) dt.$$

and the result follows.
Expected gain conditional on success

Expected gain conditional on the fact that the floor is not broken

\[
E[C_T^* | \tau > T] = E[\mathcal{E}(mL^1)_T] = \phi_T(-i)
\]

\[
= \exp \left\{ Tm\gamma + Tm \int_{z > -1/m} z\nu(dz) \right\}.
\]

Like the Black-Scholes case, conditional expected gain in an exponential Lévy model is increasing with the multiplier, provided the underlying has a positive expected growth rate.
Expected loss: Kou model

Hazard rate

\[ \lambda^* = c_-(1 - 1/m)^\lambda_-, \]

\[ 1 + \frac{m}{\lambda^*} \int_{-1}^{-1/m} x \nu_L(dx) = -\frac{m - 1}{\lambda_- + 1}, \]

Expected gain conditional on success

\[ E[C_T^*|\tau \leq T] = -\frac{(m - 1)(1 - e^{-\lambda^*T + \psi(-i)T})\lambda^*}{(\lambda_- + 1)(1 - e^{-\lambda^*T})(\lambda^* - \psi(-i))}. \]
Loss distribution

Idea: similar to Fourier method for option pricing.

Let $X$ and $X^*$ be real-valued random variables with respective distribution functions $F$ and $F^*$, characteristic functions $\phi$ and $\phi^*$ and bounded densities. Then

$$F(x) - F^*(x) = \frac{1}{2\pi} \int e^{-iux} \frac{\phi^*(u) - \phi(u)}{iu}.$$

Define

$$\tilde{\phi} := \frac{1}{\lambda^*} \int_{-\infty}^{-1/m} e^{iu \log(-1-mx)} \nu(dx)$$

denotes the characteristic function of $\log(-1 - m\tilde{L}^2)$.

**Proposition 3** Choose $X^*$ with characteristic function $\phi^*$, where
\[ E[|X^*|] < \infty \text{ and } \frac{\phi^*(u)}{1+|u|} \in L^1. \text{ If} \]

\[ \frac{|	ilde{\phi}(u)|}{(1+|u|)|\lambda^* - \psi(u)|} \in L^1 \]

\[ \int_{\mathbb{R}\setminus[-\varepsilon, \varepsilon]} |\log |1 + mx|| \nu(dx) < \infty \]

for some \( \varepsilon \), then for every \( x < 0 \),

\[ P[C_T^* < x | \tau \leq T] = P[-e^{X^*} < x] \]

\[ + \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iu \log(-x)} \left( \frac{\lambda^* \tilde{\phi}(u)}{iu(\lambda^* - \psi(u))} \frac{1 - e^{-\lambda^* T + \psi(u)T}}{1 - e^{-\lambda^* T}} - \frac{\phi^*(u)}{iu} \right) du. \]
Data sets

Daily returns, December 1st 1996 to December 1st 2006

1. General Motors Corporation (GM)

2. Microsoft Corporation (MSFT)

3. Shanghai Composite index (SSE)

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<th>$p$</th>
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Table 1: Kou model parameters estimated from MSFT, GM and SSE time series.
Figure 1: Logarithm of the density for MSFT time series. Solid line: kernel density estimator. Dashed line: Kou model with parameters estimated via empirical characteristic function.
Figure 2: Probability of loss as a function of the multiplier.
Figure 3: Expected loss over $T = 3$ years as a function of the multiplier, for nominal $N = 1000$ and $r = 0.04$. Left: expected loss conditional on a loss having occurred. Right: unconditional expected loss.
Figure 4: Probability of loss of a given size as a function of loss size (distribution function of losses).
Stochastic volatility

\[
\frac{dS_t}{S_t} = \sigma_t dW_t
\]

has the same law as a time-changed Geometric Brownian motion

\[
S_t = e^{-\frac{v_t}{2} + W_v} = \mathcal{E}(W)_{v_t}, \quad \text{where} \quad v_t = \int_0^t \sigma_s^2 ds,
\]
Stochastic volatility via time change

Time-changed Lévy process (Carr Geman Madan Yor 2007)

\[ S^*_t = \mathcal{E}(L)_{v_t}, \quad v_t = \int_0^t \sigma^2_s ds \]

Example: CIR time change

\[ d\sigma_t^2 = k(\theta - \sigma_t^2)dt + \delta \sigma_t dW. \quad (8) \]

Key ingredient: Laplace transform of integrated variance \( v \):

\[ \mathcal{L}(\sigma, t, u) := E[e^{-uv_t} | \sigma_0 = \sigma] = \frac{\exp\left(\frac{k^2 \theta t \gamma^t}{\delta^2}\right)}{\left(\cosh \frac{\gamma^t}{2} + \frac{k}{\gamma} \sinh \frac{\gamma^t}{2}\right)^{\frac{2k\theta}{\delta^2}}} \exp\left(-\frac{2\sigma_0^2 u}{k + \gamma \coth \frac{\gamma^t}{2}}\right) \]

with \( \gamma := \sqrt{k^2 + 2\delta^2 u} \).
Probability of loss

By first conditioning on the volatility process we obtain

$$P[\exists s \in [t, T] : V_s \leq B_s | \mathcal{F}_t] = 1 - E[\exp \left( -\nu_T \int_{-\infty}^{-1/m} \nu(dx) \right) \lambda^*]$$

$$= 1 - \mathcal{L}(\sigma_t, T - t, \lambda^*)$$ (9)

If the initial loss probability was 5%, and the volatility increases by a factor of 2, the loss probability changes to about 19%, leading to a much bigger cost for the bank in terms of regulatory capital.
Adjusting the leverage

\( (m_t) \) continuous process adapted to the filtration generated by the volatility process \( (\sigma_t) \). Then

\[
C^*_t = \mathcal{E} \left( \int_0^\tau m_s dL_{v_s} \right)_{\tau \wedge T}, \quad \tau = \inf\{t \geq 0 : m_t \Delta L_{v_t} \leq -1\},
\]

and by conditioning on the trajectory of the volatility process

\[
P[\tau \leq T] = 1 - E \left[ \exp \left( - \int_0^T dt \sigma_t^2 \int_{-1/m_t}^{-\infty} \nu(dx) \right) \right].
\]
The random time $\tau$ is thus characterized by a hazard rate $\lambda_t$ given by

$$
\lambda_t = \sigma_t^2 \int_{-\infty}^{-1/m_t} \nu(dx) \quad (10)
$$

To maintain a constant exposure to gap risk we can choose the leverage $m_t$ as

$$
\sigma_t^2 \int_{-\infty}^{-1/m_t} \nu(dx) = \sigma_0^2 \int_{-\infty}^{-1/m_0} \nu(dx),
$$

which leads to decreasing leverage when volatility increases.
Loss distribution

Can be computed by Fourier inversion of

\[
E[e^{iu \log(-C_T^*)}|\tau \leq T, \mathcal{F}_t] = \frac{E[e^{iu \log(-C_T^*)}1_{\tau \leq T}|\mathcal{F}_t]}{P[\tau \leq T|\mathcal{F}_t]}
\]

\[
= \frac{\tilde{\phi}(u)(1 - \mathcal{L}(\sigma_t, T - t, \lambda^* - \psi(u)))}{(\lambda^* - \psi(u))(1 - \mathcal{L}(\sigma_t, T - t, \lambda^*))}.
\]
The bank arranging the deal usually insures the client against the gap risk and has to reimburse the loss if the fund breaks the floor.

The payoff of this insurance is equal to $C_T 1_{\tau \leq T}$.

Its cost is given by the expected loss computed under risk-neutral probability calibrated to market-quoted option prices, because an approximate hedge with OTM puts may be constructed.
Hedging the gap risk

• First, we show that in a discretely rebalanced CPPI, a perfect hedge with OTM puts can be constructed.

• Suppose that the portfolio is rebalanced $n$ times, and that at each date the market quotes puts with time to maturity $h = T/n$.

• We denote by $C_{k}^{*,n}$ the discounted cushion at rebalancing date $k$.

• The cushion is then given by

\[
C_{k+1}^{*,n} = mC_{k}^{*,n} \frac{S_{k+1}^{*}}{S_{kh}^{*}} + (1 - m)C_{k}^{*,n} = C_{k}^{*,n} \left( m \frac{S_{k+1}^{*}}{S_{kh}^{*}} + 1 - m \right).
\]
Hedging the gap risk

- The cushion becomes negative if \( S^{(k+1)h} < \frac{m-1}{m} S_{kh}^* \).

- To hedge this risk, we therefore buy put options with strike \( \frac{m-1}{m} e^{rh} S_{kh} \) expiring at date \((k + 1)h\).

- One such option has pay-off \( e^{rh} S_{kh} \left( \frac{m-1}{m} - \frac{S^{(k+1)h}}{S_{kh}^*} \right)^+ \), and if we buy \( \frac{mC_k^n}{S_{kh}} \) units, the discounted pay-off is exactly equal to \( C_{k,n}^* \left( m - 1 - m \frac{S^{(k+1)h}}{S_{kh}^*} \right)^+ \).
Hedging the gap risk

• The discounted cushion in presence of hedging satisfies

\[ C_{k+1}^{*,n} = C_k^{*,n} \left( 1 + m \left( \frac{S_{(k+1)h}^*}{S_{kh}^*} - 1 \right) \right)^+, \]

\[ C_n^{*,n} = C_0^* \prod_{k=0}^{n-1} \left( 1 + m \left( \frac{S_{(k+1)h}^*}{S_{kh}^*} - 1 \right) \right)^+. \]
Hedging the gap risk

- The cost of hedging is given by the sum of prices of all put options necessary for hedging:

\[
\text{Cost}^n = \sum_{k=0}^{n-1} E^Q \left[ C_{k}^{*,n} \left( -1 - m \left( \frac{S_{(k+1)h}^*}{S_{kh}^*} - 1 \right) \right)^+ \right] \\
= E^Q [C_{n}^{*,n}] - C_0^*.
\]

Proposition: suppose the Lévy measure has no atom at \(-1/m\) and \(\int_1^\infty x^m \nu(dx) < \infty\). Then,

\[
\lim_{n \to \infty} C_{n}^{*,n} = C_T^* \mathbb{1}_{\tau > T} \quad \text{a.s. and} \quad \lim_{n \to \infty} \text{Cost}^n = -E^Q [C_T^* \mathbb{1}_{\tau \leq T}].
\]
Conclusions

• Taking into account price jumps leads to substantially different and more realistic conclusions when discussing issues like hedging and portfolio insurance: hedge ratio $\neq$ sensitivity, impact of leverage on portfolio risk,...

• Jump-diffusion models can allow for computationally tractable solutions to such problems.

• In many instances where diffusion models (unrealistically) indicate a zero risk for various hedging and portfolio management strategies, jump-diffusion models allow, to analyze the residual risk of such strategies and compare various alternatives quantitatively.