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When discontinuity matters: portfolio insurance in presence of jumps

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Constant Proportion Portfolio Insurance

Black & Jones (1987) Rouhani (1986)

Ingredients:

Risky asset S: fund, index

Guarante: value of fund at maturity V_T must be $V_T > N$

Reserve asset: bond B_t (here: zero coupon with maturity T)

Self-financing strategy whose goal is to leverage the returns of a risky asset (traded fund or index) through dynamic trading while guaranteeing a fixed amount N of capital at maturity T.

Rule-based trading strategy

A fraction of the wealth is invested into the risky asset S_t and the remainder is invested in bonds (ex: zero-coupon bond with maturity T and nominal N) B_t. Denoting the value of the fund by V_t,

- if $V_t > B_t$, the exposure to the risky asset (wealth invested into the risky asset) is given by $mC_t \equiv m(V_t - B_t)$, where C_t is the 'cushion' and $m > 1$ is a constant multiplier.

- if $V_t \leq B_t$, the entire portfolio is invested into the zero-coupon.

CPPI: Black Scholes model

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad \frac{dB_t}{B_t} = r dt$$

Cushion $C_t = V_t - B_t$ satisfies

$$\frac{dC_t}{C_t} = (m\mu + (1-m)r)dt + m\sigma dW_t,$$

$$C_T = C_0 \exp\left(rT + m(\mu - r)T + m\sigma W_T - \frac{m^2\sigma^2 T}{2}\right).$$

$$V_T = N + (V_0 - Ne^{-rT}) \exp\left(rT + m(\mu - r)T + m\sigma W_T - \frac{m^2\sigma^2 T}{2}\right)$$

$$V_T = N + (V_0 - Ne^{-rT}) \exp\left(rT + m(\mu - r)T + m\sigma W_T - \frac{m^2\sigma^2 T}{2} \right)$$

Interpretation: in the Black-Scholes model with continuous trading, a CPPI strategy is equivalent to a long position in a zero-coupon bond with nominal N to guarantee the capital at maturity and investing the remaining sum into a (fictitious) risky asset which has m times the excess return and m times the volatility of S and is perfectly correlated with S.

No risk, expected returns increasing with leverage m:

$$E[V_T] = N + (V_0 - Ne^{-rT}) \exp(rT + m(\mu - r)T).$$

CPPI: stochastic volatility case

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \quad \frac{dB_t}{B_t} = r dt$$

Cushion $C_t = V_t - B_t$ still satisfies

$$\frac{dC_t}{C_t} = (m\mu_t + (1-m)r)dt + m\sigma_t dW_t = dX_t,$$

so $C_t = C_0 \mathcal{E}(X)$ where X is a continuous semimartingale.

In particular $C_t > 0$: no risk of going below the floor.

Stochastic exponentials

X semimartingale, $Y_0 > 0$

$$Y_t = Y_0 \mathcal{E}(X)_t \iff dY_t = Y_{t-} dX_t$$

Solution:

$$Y_t = Y_0 \exp(X_t - \frac{1}{2}[X]_t) \prod_{0<s\leq t} (1 + \Delta X_s) e^{-\Delta X_s}$$

If X continuous $\Rightarrow Y$ continuous and $Y_t > 0$

X discontinuous: $Y > 0 a.s. \iff [\forall s > 0, \Delta X_s > -1] a.s.$

Model setup

$$\frac{dS_t}{S_{t-}} = dZ_t \quad \text{and} \quad \frac{dB_t}{B_{t-}} = dR_t,$$

where Z Lévy process with Lévy measure ν, R continuous semimartingale.

Ex. $R_t = rT$

Ex. 2 Vasicek model $dr_t = (\alpha - \beta r_t)dt + \sigma dW_t$
$B_t = B(t,T) = E[e^{-\int_t^T r_s ds}]$ follows

$$\frac{dB_t}{B_t} = r_t dt - \sigma \frac{1 - e^{-\beta(T-t)}}{\beta} dW_t.$$

$S > 0$ so $\Delta Z_t > -1$ almost surely.

CPPI strategy: discontinuous case

Time at which floor is reached $\tau = \inf\{t : V_t \leq B_t\}$.

CPPI is self-financing: for $t \leq \tau$

$$dV_t = m(V_{t-} - B_t)\frac{dS_t}{S_{t-}} + \{V_{t-} - m(V_{t-} - B_t)\}\frac{dB_t}{B_t},$$

Cushion $C_t = V_t - B_t$

$$\frac{dC_t}{C_{t-}} = m dZ_t + (1-m) dR_t,$$

Dynamics of the cushion

Solution by change of numeraire: $C_t^* = \frac{C_t}{B_t}$

Ito formula:

$$\frac{dC_t^*}{C_{t-}^*} = m(dZ_t - d[Z,R]_t - dR_t + d[R]_t), \qquad (1)$$

Define $L_t \equiv Z_t - [Z,R]_t - R_t + [R]_t$

$$C_t^* = C_0^* \mathcal{E}(mL)_t, t \le \tau$$

For $t > \tau$, C^* remains constant so

$$C_t^* = C_0^* \mathcal{E}(mL)_{t\wedge\tau},$$

$$\frac{V_t}{B_t} = 1 + \left(\frac{V_0}{B_0} - 1\right) \mathcal{E}(mL)_{t\wedge\tau}. \qquad (2)$$

Gap risk

$$C_t^* = C_0^* \mathcal{E}(mL)_t, t \le \tau \qquad L = Z - [Z, R] - R + [R]$$

Since L is discontinuous, C can go negative as soon as $\Delta L_t < -1/m$

Proposition 1 *The probability of going below the floor is given by*

$$P[\exists t \in [0, T] : V_t \le B_t] = 1 - \exp\left(-T \int_{-\infty}^{-1/m} \nu(dx)\right).$$

Proof: since $\mathcal{E}(mL)_t$ does negative as soon as $m\Delta L_t < -1$,

$$\tau \le T \iff \exists t \le T, \Delta L_t < -\frac{1}{m}$$

The number of jumps of the Lévy process L^j in the interval $[0, T]$, whose sizes fall in $(-\infty, -1/m]$ is a Poisson random variable with intensity $T\nu((-\infty, -1/m])$.

Example: Kou model

$$\nu(x) = \frac{\lambda(1-p)}{\eta_+} e^{-x/\eta_+} 1_{x>0} + \frac{\lambda p}{\eta_-} e^{-|x|/\eta_-} 1_{x<0}. \tag{3}$$

Loss probability is then given by

$$P[\exists t \in [0,T] : V_t \leq B_t] = 1 - \exp\left(-Tp\lambda \left(1 - 1/m\right)^{1/\eta_-}\right).$$

Expected loss

Proposition 2 *Assume*

$$\int_1^\infty x\nu(dx) < \infty.$$

Then

$$E[C_T^*|\tau \le T] = \frac{\lambda^* + m\int_{-1}^{-1/m} x\nu(dx)}{(1-e^{-\lambda^* T})(\psi(-i)-\lambda^*)}(e^{-\lambda^* T}\phi_T(-i) - 1).$$

Proof: $L = L^1 + L^2$ where L^2 is a process with piecewise constant trajectories and $\Delta L_t^2 \le -1/m$ and L^1 is a process with $\Delta L_t^1 > -1/m$.

L^1 has Lévy measure $\nu(dx)1_{x>-1/m}$ and L^2 has Lévy measure $\nu(dx)1_{x\le -1/m}$, no diffusion component

τ time of first jump of L^2 = exponential random variable with intensity $\lambda^* := \nu((-\infty, -1/m])$

ϕ_t characteristic function of Lévy process $\log \mathcal{E}(mL^1)_t$ and
$\psi(u) = \frac{1}{t} \log \phi_t(u)$.

$$C_T^* = \mathcal{E}(mL^1)_{\tau \wedge T}(1 + m\tilde{L}^2 1_{\tau \leq T}) = \mathcal{E}(mL^1)_T 1_{\tau > T} + \mathcal{E}(mL^1)_\tau (1 + m\tilde{L}^2) 1_{\tau \leq T}. \tag{4}$$

Since L^1 and L^2 are Lévy processes, τ, $\tilde{L}^2$ and L^1 are independent.
Since

$$E[\mathcal{E}(mL^1)_t] = \phi_t(-i),$$

we have

$$\begin{aligned} E[C_T^* | \tau \leq T] &= \frac{E[1 + m\tilde{L}^2]}{1 - e^{-\lambda^* T}} \int_0^T \lambda^* e^{-\lambda^* t} E[\mathcal{E}(mL^1)_t] dt \\ &= (\lambda^* + m \int_{-1}^{-1/m} x\nu(dx)) \frac{1}{1 - e^{-\lambda^* T}} \int_0^T e^{-\lambda^* t} \phi_t(-i) dt. \end{aligned}$$

and the result follows.

Expected gain conditional on success

Expected gain conditional on the fact that the floor is not broken

$$E[C_T^*|\tau > T] = E[\mathcal{E}(mL^1)_T] = \phi_T(-i)$$

$$= \exp\left\{Tm\gamma + Tm\int_{z>-1/m} z\nu(dz)\right\}.$$

Like the Black-Scholes case, conditional expected gain in an exponential Lévy model is increasing with the multiplier, provided the underlying has a positive expected growth rate.

Expected loss: Kou model

Hazard rate

$$\lambda^* = c_-(1 - 1/m)^{\lambda_-},$$

$$1 + \frac{m}{\lambda^*}\int_{-1}^{-1/m} x\nu_L(dx) = -\frac{m-1}{\lambda_- + 1},$$

Expected gain conditional on success

$$E[C_T^*|\tau \le T] = -\frac{(m-1)(1 - e^{-\lambda^* T + \psi(-i)T})\lambda^*}{(\lambda_- + 1)(1 - e^{-\lambda^* T})(\lambda^* - \psi(-i))}.$$

Loss distribution

Idea: similar to Fourier method for option pricing.

Let X and X^* be real-valued random variables with respective distribution functions F and F^*, characteristic functions ϕ and ϕ^* and bounded densities. Then

$$F(x) - F^*(x) = \frac{1}{2\pi} \int e^{-iux} \frac{\phi^*(u) - \phi(u)}{iu}.$$

Define

$$\tilde{\phi} := \frac{1}{\lambda^*} \int_{-\infty}^{-1/m} e^{iu \log(-1-mx)} \nu(dx)$$

denotes the characteristic function of $\log(-1 - m\tilde{L}^2)$.

Proposition 3 *Choose X^* with characteristic function ϕ^*, where*

$E[|X^*|] < \infty$ *and* $\frac{|\phi^*(u)|}{1+|u|} \in L^1$. *If*

$$\frac{|\tilde{\phi}(u)|}{(1+|u|)|\lambda^* - \psi(u)|} \in L^1 \tag{5}$$

$$\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} |\log|1+mx||\nu(dx) < \infty \tag{6}$$

for some ε, *then for every* $x < 0$,

$$P[C_T^* < x|\tau \leq T] = P[-e^{X^*} < x]$$
$$+\frac{1}{2\pi}\int_{\mathbb{R}} e^{-iu\log(-x)} \left(\frac{\lambda^*\tilde{\phi}(u)}{iu(\lambda^* - \psi(u))}\frac{1-e^{-\lambda^* T+\psi(u)T}}{1-e^{-\lambda^* T}} - \frac{\phi^*(u)}{iu}\right) du. \tag{7}$$

Data sets

Daily returns, December 1st 1996 to December 1st 2006

1. General Motors Corporation (GM)
2. Microsoft Corporation (MSFT)
3. Shanghai Composite index (SSE)

Series	μ	σ	λ	p	η_+	η_-
MSFT	-0.473	0.245	99.9	0.230	0.0153	0.0256
GM	-0.566	0.258	104	0.277	0.0154	0.0204
SSE	0.101	0.161	39.1	0.462	0.0167	0.0175

Table 1: Kou model parameters estimated from MSFT, GM and SSE time series.

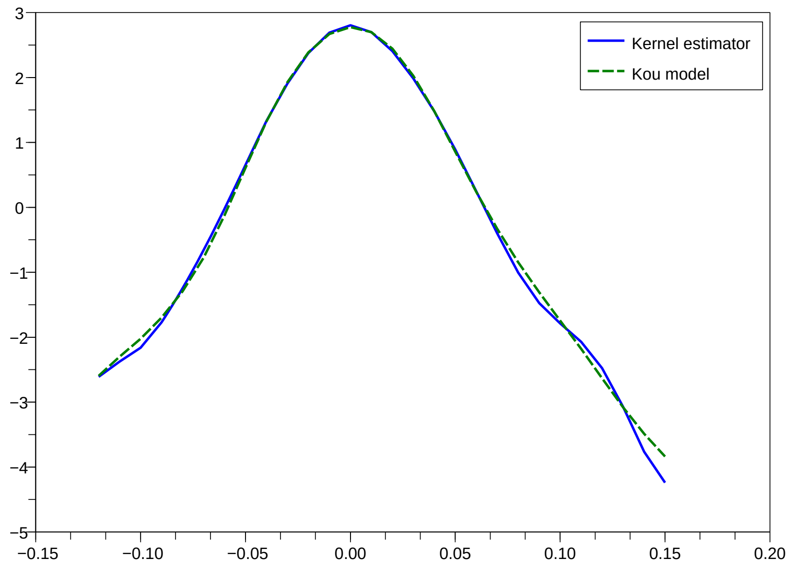

Figure 1: Logarithm of the density for MSFT time series. Solid line: kernel density estimator. Dashed line: Kou model with parameters estimated via empirical characteristic function.

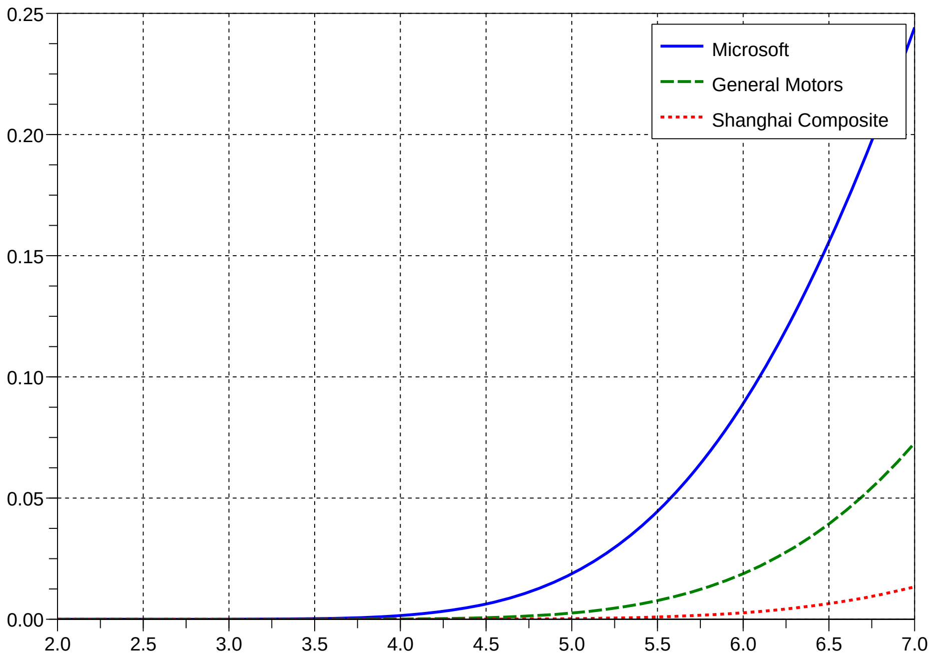

Figure 2: Probability of loss as a function of the multiplier.

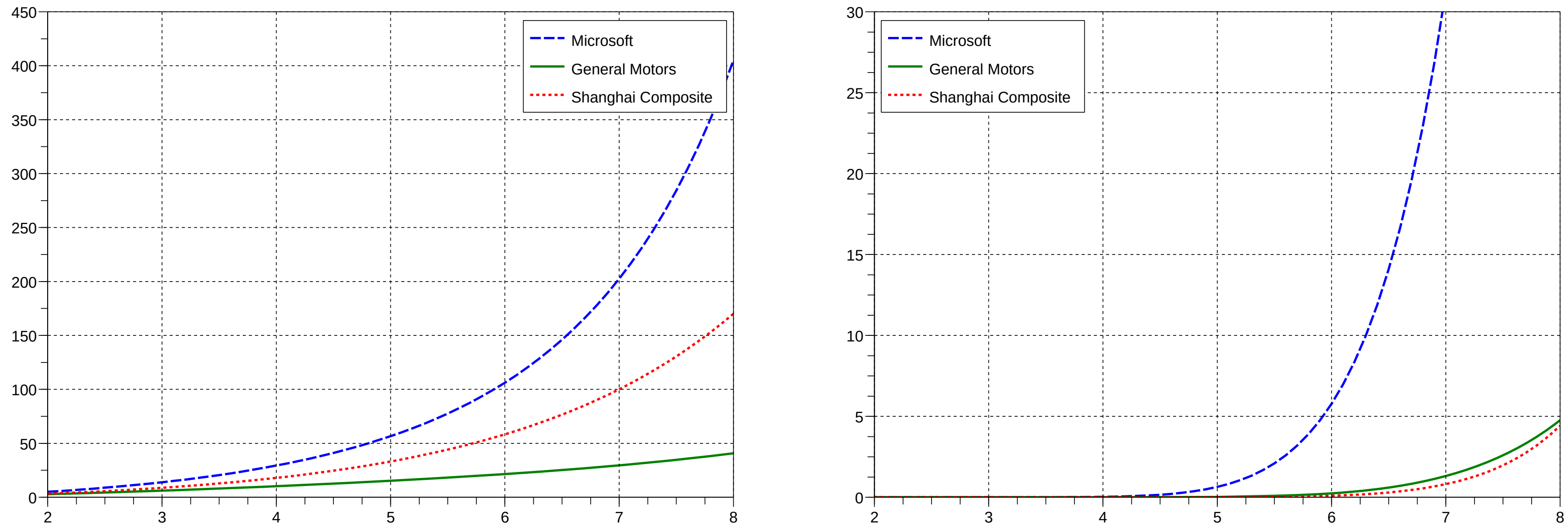

Figure 3: Expected loss over $T = 3$ years as a function of the multiplier, for nominal $N = 1000\$$ and $r = 0.04$. Left: expected loss conditional on a loss having occured. Right: unconditional expected loss.

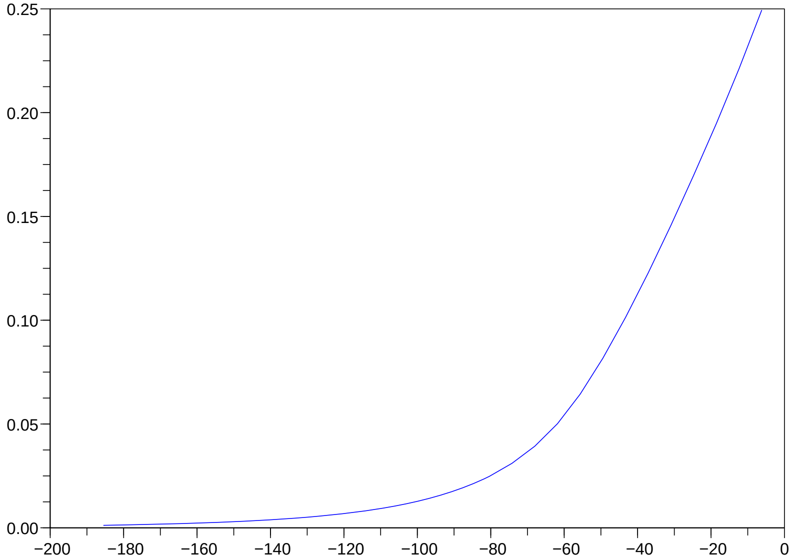

Figure 4: Probability of loss of a given size as a function of loss size (distribution function of losses).

Stochastic volatility

$$\frac{dS_t}{S_t} = \sigma_t dW_t$$

has the same law as a time-changed Geometric Brownian motion

$$S_t = e^{-\frac{v_t}{2} + W_{v_t}} = \mathcal{E}(W)_{v_t}, \quad \text{where} \quad v_t = \int_0^t \sigma_s^2 ds,$$

Stochastic volatility via time change

Time-changed Lévy process (Carr Geman Madan Yor 2007)

$$S_t^* = \mathcal{E}(L)_{v_t}, \qquad v_t = \int_0^t \sigma_s^2 ds$$

Example: CIR time change

$$d\sigma_t^2 = k(\theta - \sigma_t^2)dt + \delta\sigma_t dW. \tag{8}$$

Key ingredient: Laplace transform of integrated variance v:

$$\mathcal{L}(\sigma, t, u) := E[e^{-uv_t}|\sigma_0 = \sigma] = \frac{\exp\left(\frac{k^2\theta t}{\delta^2}\right)}{\left(\cosh\frac{\gamma t}{2} + \frac{k}{\gamma}\sinh\frac{\gamma t}{2}\right)^{\frac{2k\theta}{\delta^2}}} \exp\left(-\frac{2\sigma_0^2 u}{k + \gamma\coth\frac{\gamma t}{2}}\right)$$

with $\gamma := \sqrt{k^2 + 2\delta^2 u}$.

Probability of loss

By first conditioning on the volatility process we obtain

$$P[\exists s \in [t,T] : V_s \leq B_s | \mathcal{F}_t] = 1 - E[\exp\left(-v_T \overbrace{\int_{-\infty}^{-1/m} \nu(dx)}^{\lambda^*}\right)] = 1 - \mathcal{L}(\sigma_t, T-t, \lambda^*) \qquad (9)$$

If the initial loss probability was 5%, and the volatility increases by a factor of 2, the loss probability changes to about 19%, leading to a much bigger cost for the bank in terms of regulatory capital.

Adjusting the leverage

(m_t) continuous process adapted to the filtration generated by the volatility process (σ_t). Then

$$C_t^* = \mathcal{E}\left(\int_0^\cdot m_s dL_{v_s}\right)_{\tau\wedge T}, \quad \tau = \inf\{t \geq 0 : m_t \Delta L_{v_t} \leq -1\},$$

and by conditioning on the trajectory of the volatility process

$$P[\tau \leq T] = 1 - E\left[\exp\left(-\int_0^T dt\ \sigma_t^2 \int_{-\infty}^{-1/m_t} \nu(dx)\right)\right].$$

The random time τ is thus characterized by a hazard rate λ_t given by

$$\lambda_t = \sigma_t^2 \int_{-\infty}^{-1/m_t} \nu(dx) \tag{10}$$

To maintain a constant exposure to gap risk we can choose the leverage m_t as

$$\sigma_t^2 \int_{-\infty}^{-1/m_t} \nu(dx) = \sigma_0^2 \int_{-\infty}^{-1/m_0} \nu(dx),$$

which leads to decreasing leverage when volatility increases.

Loss distribution

Can be computed by Fourier inversion of

$$E[e^{iu\log(-C_T^*)}|\tau \leq T, \mathcal{F}_t] = \frac{E[e^{iu\log(-C_T^*)}1_{\tau \leq T}|\mathcal{F}_t]}{P[\tau \leq T|\mathcal{F}_t]}$$

$$= \frac{\tilde{\phi}(u)(1 - \mathcal{L}(\sigma_t, T-t, \lambda^* - \psi(u)))}{(\lambda^* - \psi(u))(1 - \mathcal{L}(\sigma_t, T-t, \lambda^*))}.$$

Pricing and hedging the gap risk

- The bank arranging the deal usually insures the client against the gap risk and has to reimburse the loss if the fund breaks the floor.
- The payoff of this insurance is equal to $C_T 1_{\tau \leq T}$.
- Its cost is given by the expected loss computed under risk-neutral probability calibrated to market-quoted option prices, because an approximate hedge with OTM puts may be constructed.

Hedging the gap risk

- First, we show that in a discretely rebalanced CPPI, a perfect hedge with OTM puts can be constructed.
- Suppose that the portfolio is rebalanced n times, and that at each date the market quotes puts with time to maturity $h = T/n$.
- We denote by $C_k^{*,n}$ the discounted cushion at rebalancing date k.
- The cusion is then given by

$$C_{k+1}^{*,n} = mC_k^{*,n} \frac{S^*_{(k+1)h}}{S^*_{kh}} + (1-m)C_k^{*,n} = C_k^{*,n} \left(m \frac{S^*_{(k+1)h}}{S^*_{kh}} + 1 - m \right).$$

Hedging the gap risk

- The cusion becomes negative if $S^*_{(k+1)h} < \frac{m-1}{m} S^*_{kh}$.
- To hedge this risk, we therefore buy put options with strike $\frac{m-1}{m} e^{rh} S_{kh}$ expiring at date $(k+1)h$.
- One such option has pay-off $e^{rh} S_{kh} \left(\frac{m-1}{m} - \frac{S^*_{(k+1)h}}{S^*_{kh}} \right)^+$, and if we buy $\frac{mC^n_k}{S_{kh}}$ units, the discounted pay-off is exactly equal to $C^{*,n}_k \left(m - 1 - m \frac{S^*_{(k+1)h}}{S^*_{kh}} \right)^+$.

Hedging the gap risk

- The discounted cushion in presence of hedging satisfies

$$C^{*,n}_{k+1} = C^{*,n}_{k} \left(1 + m\left(\frac{S^*_{(k+1)h}}{S^*_{kh}} - 1\right)\right)^+,$$

$$C^{*,n}_{n} = C^*_0 \prod_{k=0}^{n-1} \left(1 + m\left(\frac{S^*_{(k+1)h}}{S^*_{kh}} - 1\right)\right)^+.$$

Hedging the gap risk

- The cost of hedging is given by the sum of prices of all put options necessary for hedging:

$$\text{Cost}^n = \sum_{k=0}^{n-1} E^Q\left[C_k^{*,n}\left(-1 - m\left(\frac{S^*_{(k+1)h}}{S^*_{kh}} - 1\right)\right)^+\right]$$
$$= E^Q[C_n^{*,n}] - C_0^*.$$

Proposition: suppose the Lévy measure has no atom at $-1/m$ and $\int_1^\infty x^m \nu(dx) < \infty$. Then,

$$\lim_{n\to\infty} C_n^{*,n} = C_T^* 1_{\tau > T} \quad \text{a.s. and} \quad \lim_{n\to\infty} \text{Cost}^n = -E^Q[C_T^* 1_{\tau \le T}].$$

Conclusions

- Taking into account price jumps leads to *substantially different* and more realistic conclusions when discussing issues like hedging and portfolio insurance: hedge ratio $\neq$ sensitivity, impact of leverage on portfolio risk,...
- Jump-diffusion models can allow for *computationally tractable* solutions to such problems.
- In many instances where diffusion models (unrealistically) indicate a zero risk for various hedging and portfolio management strategies, jump-diffusion models allow, to analyze the residual risk of such strategies and compare various alternatives quantitatively.