2. Classical RMT results

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Spectral Transforms

- Stieltjes transform, Green and Blue functions

\[ \rho(\lambda) = N^{-1} \sum_i \delta(\lambda - \lambda_i) \]

- Stieltjes transform:

\[ S(z) = \int d\lambda \frac{\rho(\lambda)}{\lambda - z} = \frac{1}{N} \text{Tr} \left[ (H - zI)^{-1} \right] \]

- Green function:

\[ G(z) \equiv -S(z); \quad \rho(\lambda) = \lim_{\epsilon \to 0} \frac{1}{\pi} \Im \left( G(\lambda - i\epsilon) \right) \]

- Blue function: \[ B[G(z)] = z \]

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Spectral Transforms

• R-transforms and S-transforms

  – R-transform:  $R(z) = B(z) - z^{-1}$

  – Properties:

    $R_{aH}(z) = aR_H(az)$

    $R(z) = \sum_{k=1}^{\infty} c_k z^{k-1}$  \hspace{1cm} c_k:  \quad \text{Generalized cumulants}$

  – S-transform:

    $\eta(y) \equiv -\frac{1}{y} G \left( -\frac{1}{y} \right)$; \quad $S(z) = -\frac{1+z}{z} \eta^{-1}(1 + z)$

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Spectral Transforms

- **Example 1: Wigner semi-circle**

\[
G(z) = \frac{z \pm \sqrt{z^2 - 4}}{2} \quad R(z) = z
\]

- **Example 2: Marcenko-Pastur** \( Q = T/N, \ q = 1/Q \)

\[
\rho(\lambda) = (1-Q)^+ \delta(\lambda) + \frac{\sqrt{4\lambda q - (\lambda + q - 1)^2}}{2\pi\lambda q} \quad \lambda \in [(1-\sqrt{q})^2, (1+\sqrt{q})^2]
\]

\[
G(z) = \frac{(z + q - 1) - \sqrt{(z + q - 1)^2 - 4zq}}{2zq}, \ R(z) = \frac{1}{1 - qz}, \ S(z) = \frac{1}{1 + qz}
\]
Four-ways to the semi-circle

• 1. From the full multivariate density. The GOE measure:

\[ P(H) \propto \exp[-\frac{1}{2} \text{Tr} H^2] \rightarrow \rho(\lambda_1, \lambda_2, \ldots \lambda_N) = Z^{-1} \prod_{i<j} |\lambda_i - \lambda_j| \exp[-\frac{1}{2} \sum_i \lambda_i^2] \]

• Transform Van der Monde determinant with Orthogonal Polynomials, compute \( \rho(\lambda) \) by integrating over \( N - 1 \) variables, take the large \( N \) limit \( \rightarrow \rho(\lambda) = \sqrt{4 - \lambda^2}/2\pi \)

• Use the analogy with the partition function of a charged gaz with logarithmic interactions, confined by a parabolic potential. At equilibrium, force on each particle is zero:

\[ \lambda = \int d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'} \]
Four-ways to the semi-circle

Tricomi’s equation, solved by Wigner’s semi-circle

- Note: Dyson’s Brownian motion: add a small Gaussian matrix and use second order perturbation theory and rescale to keep a fixed variance:

$$d\lambda_i = [-\lambda_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}] dt + dB_t$$
Four-ways to the semi-circle

• 2. From a recursion relation on the Green function $G(z) = (zI - H)^{-1}$

• Start from an $N \times N$ sym. matrix with IID entries and add a row and a column of IID elements.

• Expand twice the inverse of a matrix in terms of minors. One easily gets:

$$\frac{1}{G_{00}^{N+1}(z)} = z - H_{00} - \sum_{i,j} H_{0i}H_{0j}G_{ij}^{N}(z)$$

• Find a similar recursion for $G_{0i}^{N+1}$ which shows that off-diagonal elements are $O(1/\sqrt{N})$ whereas diagonal elements are of order one.
Four-ways to the semi-circle

- Hence:

\[
\frac{1}{G_{00}^{N+1}(z)} \approx z - \sum_{i}^{N} H_{0i}^2 G_{ii}^{N}(z) + O(1/\sqrt{N})
\]

- Since \(H_{0i}\) and \(G_{ii}\) are independent, one can use the law of large numbers to get, for large \(N\):

\[
\overline{G}^{-1} = z - \overline{G}^{-1} \rightarrow \overline{G} = \frac{1}{2}(z \pm \sqrt{z^2 - 4})
\]
Four-ways to the semi-circle

3. The REPLICA method

- Use a Gaussian integral representation of the inverse:

\[
A^{-1}_{ii} = \frac{\int [\prod_j d\phi_j] \phi_i^2 \exp[-\frac{1}{2} \sum_{jk} \phi_j A_{kj} \phi_k]}{\int [\prod_j d\phi_j] \exp[-\frac{1}{2} \sum_{jk} \phi_j A_{kj} \phi_k]}
\]

The “Replica Trick” is to write this as:

\[
A^{-1}_{ii} = \lim_{n \to 0} \int [\prod_j d\phi_j] \frac{1}{n} \sum_{a=1}^n \phi_i a^2 \exp[-\frac{1}{2} \sum_{a=1}^n \sum_{jk} \phi_j A_{kj} \phi_k]
\]

- Use this with \( A = z1 - H \), trace over \( i \) and average over Gaussian \( H_{ij} \), leading to:

\[
G(z) = \lim_{n \to 0} \frac{2\partial}{n \partial z} \int [\prod_j d\phi_j] \exp[-z \sum_{a} \sum_{j} \phi_i a^2 + \frac{1}{4N} \sum_{a,b} \sum_{ij} \phi_i a^b \phi_j a^b \phi_i a^b \phi_j a^b]
\]

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Four-ways to the semi-circle

- ‘Replica sym.’ ansatz: Introduce two ‘order parameters’

\[ q_0(z) = \frac{1}{N} \sum_i \phi_i^a \phi_i^a, \quad q_1(z) = \frac{1}{N} \sum_i \phi_i^a \phi_i^b, \]

enforced by two \( \delta \) functions, expressed in Fourier transform. Integration over \( \phi \) becomes Gaussian again.

- The remaining integral finally reads:

\[
\int dq_0 \, dq_1 \, d\lambda \, d\mu \exp[NnF(q_0, q_1, \lambda, \mu)]
\]

with:

\[
F(q_0, q_1, \lambda, \mu) = (\lambda + \frac{z}{2})q_0 + \mu(n-1)q_1 - \frac{1}{2} \ln \lambda - \frac{1}{2n} \ln \frac{\lambda + n\mu}{\lambda} + \frac{q_0^2}{4} + \frac{(n-1)q_1^2}{4}
\]
Four-ways to the semi-circle

- Work at finite $n$ and let $N \to \infty$: saddle point calculation leading to the following equation:

$$\lambda^* + \frac{z}{2} + \frac{q_0^*}{2}, \quad \lambda^* q_0^* = 1, \quad \rightarrow G(z) = q_0^* = \frac{1}{2}(z \pm \sqrt{z^2 - 4})$$
Four-ways to the semi-circle

- 4. Free convolution (more below)

- If $H_1$ and $H_2$ are two IID large ($N \to \infty$) Gaussian matrices, then:

  $$R_{H_1 + H_2}(z) = R_{H_1}(z) + R_{H_2}(z)$$

- But: $H_1 + H_2 \overset{L}{=} \sqrt{2}H$ – Hence:

  $$R_{\sqrt{2}H}(z) = \sqrt{2}R_H(\sqrt{2}z) = 2R_H(z) \longrightarrow R_H(z) = az$$

- Generalized CLT

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Four-ways to the semi-circle: pros and cons

1. Rigorous, access to multi-point correlations but specific (Gaussian ensemble) and relatively heavy

2. Makes explicit the CLT character of the semi-circle, can be extended to Lévy variables, no access to multi-point correlations

3. Non rigorous but very flexible, leads to solution for more complicated RM ensembles, can be extended to multi-point correlations

4. Elegant and powerful (see below) but no access to multi-point correlations and structure of eigenvectors
Other classic RMT results

1. Structure of eigenvectors

The GOE is invariant under rotations, hence there cannot be any localisation of eigenvectors. Therefore, the inverse participation ratio (Hirfindahl index) of any eigenvector is zero:

$$\sum_i w_i^\alpha^2 = \frac{1}{N}, \quad w_i^\alpha = |\langle i | \alpha \rangle|^2$$

More precisely, for a given $\alpha$,

$$P(w) = N \exp[-Nw]$$

(Porter-Thomas distribution)

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Other classic RMT results

- 2. Finite $N$ results

- Note: The density of states is *self-averaging*

- Convergence of the averaged density of states for Gaussian elements:

  $$|E[\rho_N] - \rho_\infty| \leq \kappa N^{-2/3}$$

  Note: comes from the edge scaling spill-over:

  $$\rho_N(\lambda = 2 + \epsilon) = N^{-2/3} f(\epsilon N^{-2/3})$$

  which itself can be guessed from:

  $$\int_{2-\epsilon}^{2} du \sqrt{2-u} = \frac{1}{N} \propto \epsilon^{3/2}.$$  

- For a fixed realization:

  $$|\rho_N - \rho_\infty| = \xi N^{-2/5}$$
Other classic RMT results

- Can be extended to non Gaussian elements provided high enough moments exist.

- More precise results about the largest eigenvalue: Tracy-Widom, see below.
Other classic RMT results

• 3. Universal correlations

• Universal level repulsion: degeneracies are of co-dimension 1
  – For example for a \(2 \times 2\) matrix:

\[
\Delta = 0 \quad \text{when} \quad (H_{11} - H_{22})^2 + H_{12}^2 = 0
\]

This implies, for the level spacing distribution:

\[
P(s) \sim s \rightarrow 0 \ s
\]

• Wigner surmise (for a \(2 \times 2\) Gaussian matrix):

\[
P(s) \propto s \exp[-s^2]
\]

extremely good fit for the exact large \(N\) result, and universal
(resists change of global densities, etc.) – for example, true
Other classic RMT results

for the Marcenko-Pastur problem, or for quantized classically chaotic systems

- Universal two-level density on a local scale (after rescaling such that $\rho(\lambda) = 1$):

  $$\rho(\lambda, \lambda + \frac{u}{N}) = 1 - \frac{\sin \pi u}{\pi u}$$

- Incompressibility

  $$\langle n^2(\Delta) \rangle - \langle n(\Delta) \rangle^2 \sim \frac{2}{\pi^2} \ln \Delta \ll \Delta$$
Other classic RMT results

• 4. Lévy matrices

- $H_{ij}$: IID with power-law tails, exponent $\mu < 2$ such that the variance diverges, rescaled by $N^{1/\mu}$

- $\rho(\lambda)$ can be exactly computed and has no edge [for a rigorous proof, see Ben-Arous-Guionnet]

- $\rho(\lambda) \sim |\lambda| \to \infty |\lambda|^{-1-\mu}$

- Interesting structure of eigenvectors (no rotational symmetry) – transition between localized and extended

- Structure of correlations and level spacing unknown

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