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Lévy processes

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Lévy Processes

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Some books

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Give us the tools, and we will finish the work.

Winston Churchill, February 9, 1941.

Definition and Main Properties of Lévy Processes

Definition

An $I\!R^d$-valued process X such that $X_0 = 0$ is a **Lévy process** if
a) for every $s, t, X_{t+s} - X_t$ is independent of $\mathcal{F}_t^X$
b) for every s, t the r.v's $X_{t+s} - X_t$ and X_s have the same law.
c) X is continuous in probability, i.e., $\mathbb{P}(|X_t - X_s| > \epsilon) \to 0$ when $s \to t$ for every $\epsilon > 0$.

The sum of two independent Lévy processes is a Lévy process.

Property c) implies that a Lévy process has no jumps at fixed time.

A Lévy process admits a càdlàg modification

(A process Y is said to be a modification of X is $\mathbb{P}(X_t = Y_t) = 1, \forall t)$

Let $T > 0$ be fixed. For any $\epsilon > 0$, the set $\{t \in [0,T] \,:\, |\Delta X_t| > \epsilon\}$ is finite

The set $\{t \in [0,T] \,:\, |\Delta X_t| > 0\}$ is countable

Examples

• Brownian motion

The standard Brownian motion is a process W with **continuous paths** such that

- for every s, t, $W_{t+s} - W_t$ is independent of $\mathcal{F}_t^W$,
- for every s, t, the r.v. $W_{t+s} - W_t$ has the same law as W_s.

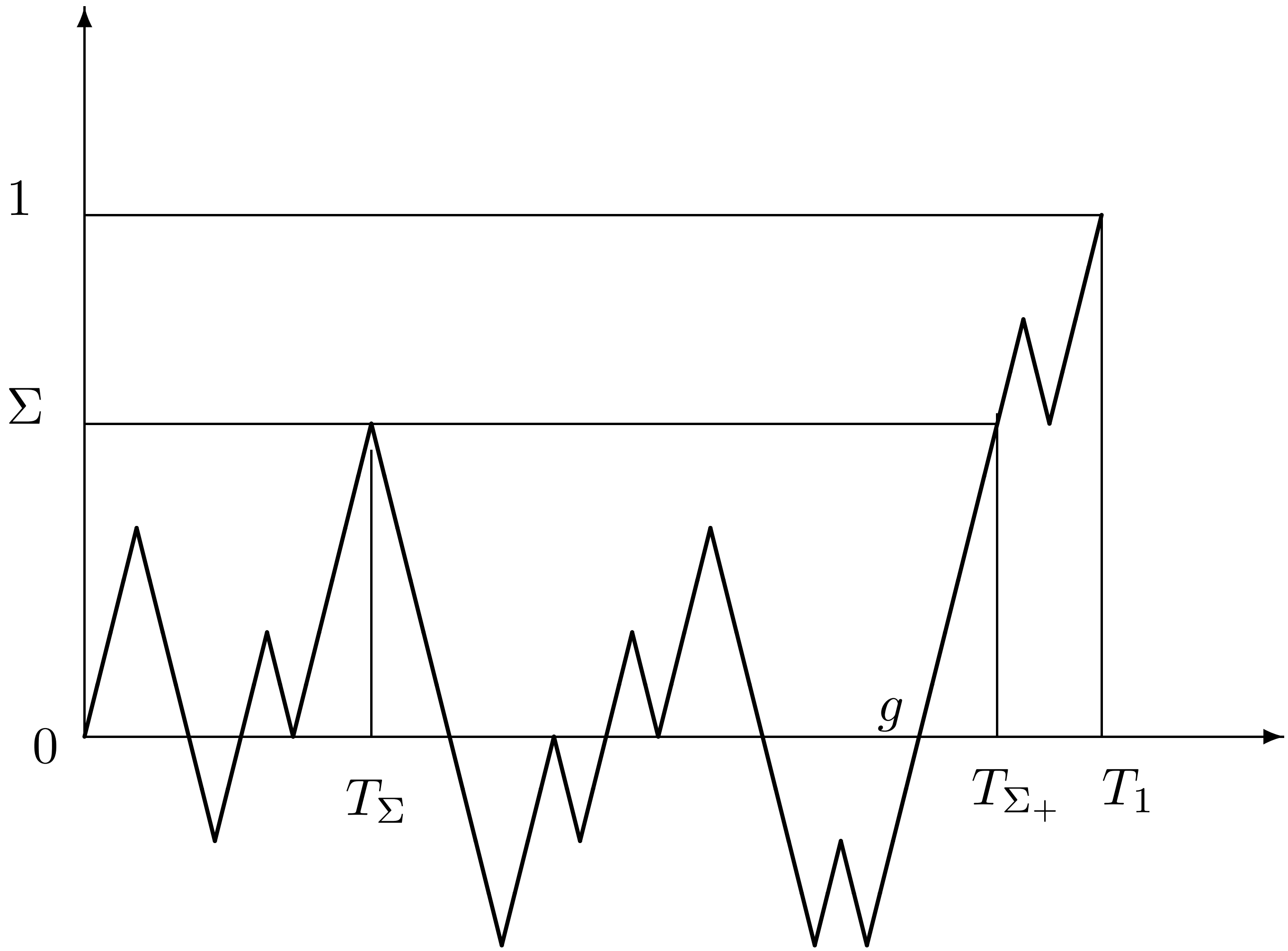
The law of W_s is $\mathcal{N}(0, s)$.

• **Brownian hitting time process**

Let W be a standard BM and for $a > 0$, define

$$T_a := \inf\{t : W_t = a\}$$

The process $(T_a, a \geq 0)$ is a Lévy process.



Non continuity of T_a

• **Poisson process**

The standard Poisson process is a **counting process** such that

- for every s, t, $N_{t+s} - N_t$ is independent of $\mathcal{F}_t^N$,

- for every s, t, the r.v. $N_{t+s} - N_t$ has the same law as N_s.

Then, the r.v. N_t has a Poisson law with parameter λt

$$\mathbb{P}(N_t = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

If X has a Poisson law with parameter $\theta > 0$, then

(i) for any $s \in \mathbb{R}$, $\mathbb{E}[s^X] = e^{\theta(s-1)}$.

(ii) $\mathbb{E}[X] = \theta$, $\mathrm{Var}\,(X) = \theta$.

(iii) for any $u \in \mathbb{R}$, $\mathbb{E}(e^{iuX}) = \exp(\theta(e^{iu} - 1))$

(iv) for any $\alpha \in \mathbb{R}$, $\mathbb{E}(e^{\alpha X}) = \exp(\theta(e^{\alpha} - 1))$

• Compound Poisson Process

Let λ be a positive number and $F(dy)$ be a probability law on $\mathbb{R}$ (we assume that $\mathbb{P}(Y_1 = 0) = 0$). A (λ, F)**-compound Poisson process** is a process $X = (X_t, t \geq 0)$ of the form

$$X_t = \sum_{k=1}^{N_t} Y_k$$

where N is a Poisson process with intensity $\lambda > 0$ and the $(Y_k, k \in \mathbb{N})$ are i.i.d. random variables, independent of N, with law
$F(dy) = \mathbb{P}(Y_1 \in dy)$.

If $\mathbb{E}(|Y_1|) < \infty$, for any t, $\mathbb{E}(X_t) = \lambda t \mathbb{E}(Y_1)$.

The characteristic function of the r.v. X_t is

$$\mathbb{E}[e^{iuX_t}] = e^{\lambda t(\mathbb{E}[e^{iuY_1}]-1)} = \exp\left(\lambda t \int_{I\!R} (e^{iuy} - 1)F(dy)\right).$$

Assume that $\mathbb{E}[e^{\alpha Y_1}] < \infty$. Then, the Laplace transform of the r.v. X_t is

$$\mathbb{E}[e^{\alpha X_t}] = e^{\lambda t(\mathbb{E}[e^{\alpha Y_1}]-1)} = \exp\left(\lambda t \int_{I\!R} (e^{\alpha y} - 1)F(dy)\right).$$

We shall note $\nu(dy) = \lambda F(dy)$ and say that X is a ν-compound Poisson process.

Martingales

Let X be a Lévy process.

- If $\mathbb{E}(|X_t|) < \infty$, the process $X_t - \mathbb{E}(X_t)$ is a martingale.
- For any u, the process $Z_t(u) := \frac{e^{iuX_t}}{\mathbb{E}(e^{iuX_t})}$ is a martingale.
- If $\mathbb{E}(e^{\lambda X_t}) < \infty$, the process $\frac{e^{\lambda X_t}}{\mathbb{E}(e^{\lambda X_t})}$ is a martingale

Examples

• **Brownian motion**

The standard Brownian motion is a martingale, the process $Y = \mathcal{E}(\lambda W)$ defined by $Y_t = e^{\lambda W_t - \frac{1}{2}\lambda^2 t}$ is a martingale.

The Doléans-Dade exponential $Y = \mathcal{E}(\lambda W)$ satisfies

$$dY_t = Y_t \lambda dW_t$$

• **Poisson process**

The process $M_t = N_t - \lambda t$ is a martingale.

For any α, the process

$$\exp(\alpha N_t - \lambda t(e^\alpha - 1)) = \exp(\alpha M_t - \lambda t(e^\alpha - 1 - \alpha)) = \mathcal{E}(\alpha M)_t$$

is a martingale.

For any β, the process $(1+\beta)^{N_t} e^{-\lambda \beta t}$ is a martingale

If X is a counting process and if, for some λ the process $M_t = N_t - \lambda t$ is a martingale, then X is a Poisson process

If X is a counting process with stationary and independent increments, then X is a Poisson process.

• Compound Poisson Processes

Assume that $\mathbb{E}(|Y_1|) < \infty$. Then, the process $(Z_t := X_t - t\lambda\mathbb{E}(Y_1), t \geq 0)$ is a martingale and in particular,

$$\mathbb{E}(X_t) = \lambda t \mathbb{E}(Y_1) = \lambda t \int_{-\infty}^{\infty} y F(dy) = t \int_{-\infty}^{\infty} y \nu(dy)$$

For any $\alpha \in I\!R$ such that $\int_{-\infty}^{\infty} |e^{\alpha x} - 1| F(dx) < \infty$, the process

$$\exp\left(\alpha X_t - t\lambda \int_{-\infty}^{\infty} (e^{\alpha x} - 1) F(dx)\right) = \exp\left(\alpha X_t - t \int_{-\infty}^{\infty} (e^{\alpha x} - 1) \nu(dx)\right)$$

is a martingale.

Random Measures

• **Counting process**: Let (T_n) be a sequence of random times, with

$$0 < T_1 < \cdots < T_n \ldots$$

and $N_t = \sum_{n \geq 1} \mathbb{1}_{T_n \leq t}$. Let A be a Borel set in $\mathbb{R}^+$ and

$$\mathbf{N}(\omega; A) := \text{Card } \{n \geq 1 \, : \, T_n(\omega) \in A\}$$

The measure $\mathbf{N}$ is a random measure and $N_t(\omega) = \mathbf{N}(\omega,]0, t])$.

For a Poisson process

$$\mathbb{E}(\mathbf{N}(A)) = \lambda \ \text{Leb}(A)$$

Let ν be a radon measure on E. A **random Poisson measure** $\mathbf{N}$ on E with intensity ν is a measure such that

- $\mathbf{N}(A)$ is an integer valued random measure,
- $\mathbf{N}(A) < \infty$ for A bounded Borel set,
- for disjoint sets A_i, the r.v's $\mathbf{N}(A_i)$ are independent
- the r.v. $\mathbf{N}(A)$ is Poisson distributed with parameter $\nu(A)$

• **Compound Poisson process**: Define $\mathbf{N} = \sum_n \delta_{T_n, Y_n}$ on $\mathbb{R}^+ \times \mathbb{R}$, i.e

$$\mathbf{N}(\omega, [0,t] \times A) = \sum_{n=1}^{N_t(\omega)} 1\!\!1_{Y_n(\omega) \in A}\,.$$

We shall also write $\mathbf{N}_t(dx) = \mathbf{N}([0,t], dx)$. The measure $\mathbf{N}$ is a random Poisson measure on $\mathbb{R}^+ \times \mathbb{R}$ with intensity $\lambda dt F(dx)$

We denote by $(f * \mathbf{N})_t$ the integral

$$\int_0^t \int_{\mathbb{R}} f(x) \mathbf{N}(ds, dx) = \int_{\mathbb{R}} f(x) \mathbf{N}_t(dx) = \sum_{k=1}^{N_t} f(Y_k) = \sum_{s \le t} f(\Delta X_s) 1\!\!1_{\Delta X_s \neq 0}\,.$$

In particular

$$X_t = \sum_{k=1}^{N_t} Y_k = \sum_{s \le t} \Delta X_s = \int_0^t \int_{\mathbb{R}} x \mathbf{N}(ds, dx)$$

If $\nu(|f|) < \infty$, the process

$$\begin{aligned} M_t^f : &= (f * \mathbf{N})_t - t\nu(f) = \int_0^t \int_{I\!R} f(x)(\mathbf{N}(ds, dx) - ds\nu(dx)) \\ &= \sum_{s \le t} f(\Delta X_s) \mathbb{1}_{\Delta X_s \neq 0} - t\nu(f) \end{aligned}$$

is a martingale.

PROOF: Indeed, the process $Z_t = \sum_{k=1}^{N_t} f(Y_k)$ is a $\widehat{\nu}$ compound Poisson process, where $\widehat{\nu}$, defined as

$$\widehat{\nu}(A) = \lambda \mathbb{P}(f(Y_n) \in A)$$

is the image of ν by f. Hence, if $\mathbb{E}(|f(Y_1)|) < \infty$, the process $Z_t - t\lambda\mathbb{E}(f(Y_1)) = Z_t - t\int f(x)\nu(dx)$ is a martingale.

Using again that Z is a compound Poisson process, it follows that the process

$$\exp\left(\sum_{k=1}^{N_t} f(Y_k) - t\int_{-\infty}^{\infty}(e^{f(x)}-1)\nu(dx)\right)$$
$$= \exp\left(\int_0^t \int_{\mathbb{R}} f(x)\mathbf{N}(ds,dx) - t\int_{-\infty}^{\infty}(e^{f(x)}-1)\nu(dx)\right)$$

is a martingale

If X is a pure jump process, if there exists λ and a probability measure σ such that $\sum_{s\leq t} f(\Delta X_s)\mathbb{1}_{\Delta X_s \neq 0} - t\lambda\sigma(f)$ is a martingale, then X is a compound Poisson process.

• Lévy Processes

The random variable $\mathbf{N}(]s,t] \times A) = \sum_{s<u\leq t} \mathbb{1}_A(\Delta X_u)$ represents the number of jumps in the time interval $]s,t]$ with jump size in A.

We define ν by

$$\nu(A) = \mathbb{E}(\mathbf{N}([0,1] \times A))$$

For A compact set such that $0 \notin A$, $\nu(A) < \infty$

The process

$$\mathbf{N}_t^A = \sum_{0<s\leq t} \mathbb{1}_A(\Delta X_s) = \mathbf{N}([0,t] \times A)$$

is a Poisson process with intensity $\nu(A)$.

The processes $\mathbf{N}^A$ and $\mathbf{N}^C$ are independent if $\nu(A \cap C) = 0$, in particular if A and C are disjoint.

Let A be a Borel set of $I\!R^d$ with $0 \notin \bar{A}$, and f a Borel function defined on A. We have

$$\int_A f(x)\mathbf{N}_t(\omega, dx) = \int_0^t \int_A f(x)\mathbf{N}(\omega, ds, dx) = \sum_{0<s\leq t} f(\Delta X_s(\omega))\mathbb{1}_A(\Delta X_s(\omega))\,.$$

The process

$$\int_A f(x)\mathbf{N}_t(\omega, dx)$$

is a Lévy process; if $\int_A |f(x)|\nu(dx) < \infty$, then

$$M_t^f = \int_A f(x)\mathbf{N}_t(\omega, dx) - t\int_A f(x)\nu(dx) = \int_0^t \int_A f(x)(\mathbf{N}(ds, dx) - \nu(dx)ds)$$

is a martingale.

If f is bounded and vanishes in a neighborhood of 0,

$$\mathbb{E}(\sum_{0<s\leq t} f(\Delta X_s)) = t\int_{I\!R^d} f(x)\nu(dx)\,.$$

The measure ν satisfies

$$\int (1 \wedge |x|^2)\nu(dx) < \infty$$

i.e. $\int_{|x|\geq 1} \nu(dx) < \infty$ and $\int_{|x|<1} |x|^2\nu(dx) < \infty$

Infinitely Divisible Random Variables

Definition

A random variable X taking values in $\mathbb{R}^d$ is **infinitely divisible** if its characteristic function satisfies

$$\hat{\mu}(u) = \mathbb{E}(e^{i(u \cdot X)}) = (\hat{\mu}_n(u))^n$$

where $\hat{\mu}_n$ is a characteristic function.

Examples: The **Gaussian law** $\mathcal{N}(m, \sigma^2)$ has the characteristic function $\exp(ium - u^2\sigma^2/2)$.

Cauchy laws. The standard Cauchy law has the characteristic function $\exp(-c|u|)$.

The **hitting time of the level a for a Brownian motion** has Laplace transform

$$\mathbb{E}[\exp(-\frac{\lambda^2}{2}T_a)] = \exp(-|a|\lambda)$$

Poisson laws. The Poisson law with parameter λ has characteristic function

$$\exp(c(e^{iu} - 1))$$

Poisson Random Sum. Let X_i i.i.d. r.v's with characteristic function φ and N a r.v. independent of the X_i's with a Poisson law. Let

$$X = X_1 + X_2 + \cdots + X_N$$

The characteristic function of X is

$$\exp(-\lambda(1 - \varphi(u)))$$

Gamma laws. The Gamma law $\Gamma(a, \nu)$ has density

$$\frac{\nu^a}{\Gamma(a)} x^{a-1} e^{-\nu x} \mathbb{1}_{x>0}$$

and characteristic function

$$(1 - iu/\nu)^{-a}$$

A **Lévy measure** ν is a positive measure on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{\mathbb{R}^d \setminus \{0\}} \min(1, \|x\|^2)\nu(dx) < \infty\,.$$

Lévy-Khintchine representation.

If X is an infinitely divisible random variable, there exists a triple (m, A, ν) where $m \in \mathbb{R}^d$, A is a non-negative quadratic form and ν is a Lévy measure such that

$$\hat{\mu}(u) = \exp\left(i(u{\cdot}m) - \frac{1}{2}(u{\cdot}Au) + \int_{\mathbb{R}^d} (e^{i(u\cdot x)} - 1 - i(u{\cdot}x)\mathbb{1}_{\{|x|\le 1\}})\nu(dx) \right).$$

The triple (m, A, ν) is called the ***characteristic triple***.

If $\int |x|\mathbb{1}_{\{|x|\le 1\}}\nu(dx) < \infty$, one writes the LK representation in a *reduced form*

$$\hat{\mu}(u) = \exp\left(i(u{\cdot}m_0) - \frac{1}{2}(u{\cdot}Au) + \int_{\mathbb{R}^d} (e^{i(u\cdot x)} - 1)\nu(dx) \right).$$

A first step is to prove that any function φ such that

$$\varphi(u) = \exp\left(i(u{\cdot}m) - \frac{1}{2}(u{\cdot}Au) + \int_{\mathbb{R}^d} (e^{i(u\cdot x)} - 1 - i(u{\cdot}x)\mathbb{1}_{\{|x|\leq 1\}})\nu(dx)\right) \qquad (*)$$

is a characteristic function (hence, i.d.).

If φ satisfies (*), one proves that

- it is continuous at 0
- it is the limit of characteristic functions.

The result follows.

Continuity: show that

$$\psi(u) = \int (e^{iux} - 1 - iux\mathbb{1}_{\{|x|\leq 1\}})\nu(dx)$$

is continuous.

$$\psi(u) = \int_{|x|\leq 1} (e^{iux} - 1 - iux)\nu(dx) + \int_{|x|>1} (e^{iux} - 1)\nu(dx)$$

Then, using the fact that

$$|e^{iux} - 1 - iux| \leq \frac{1}{2}u^2x^2$$

the result is obtained

Limit:

$$\int_{|x|\le 1}(e^{iux}-1-iux)\nu(dx) = \lim \int_{|x|\ge 1/n}(e^{iux}-1-iux)\nu(dx)$$

The right-hand side corresponds to compound Poisson process

If $\widehat{\mu}$ is i.d., then it satisfies LK.

If $\widehat{\mu}$ is i.d., then $\widehat{\mu}(u)$ does not vanish.

Then,

$$\widehat{\mu}(u) = (\widehat{\mu}_n(u))^n$$

implies that

$$\Phi_n(u) := \exp(n(\widehat{\mu}_n(u) - 1)) = \exp(n(e^{\frac{1}{n}\ln \widehat{\mu}(u)} - 1))$$

converges to $\widehat{\mu}(u)$.

$$\Phi_n(u) = \exp(n \int (e^{iux} - 1)\mu_n(dx))$$

is associated with a compound Poisson process.

Examples:

Gaussian laws. The Gaussian law $\mathcal{N}(m, \sigma^2)$ has the characteristic function $\exp(ium - u^2\sigma^2/2)$. Its characteristic triple is $(m, \sigma, 0)$.

Cauchy laws. The standard Cauchy law has the characteristic function

$$\exp(-c|u|) = \exp\left(\frac{c}{\pi}\int_{-\infty}^{\infty}(e^{iux} - 1)x^{-2}dx\right) .$$

Its reduced form characteristic triple is $(0, 0, \pi^{-1}x^{-2}dx)$.

Gamma laws. The Gamma law $\Gamma(a, \nu)$ has characteristic function

$$(1 - iu/\nu)^{-a} = \exp\left(a \int_0^\infty (e^{iux} - 1) e^{-\nu x} \frac{dx}{x} \right) .$$

Its reduced form characteristic triple is $(0, 0, \mathbb{1}_{\{x>0\}}\, a x^{-1} e^{-\nu x} dx)$.

Brownian hitting times. The first hitting time of $a > 0$ for a Brownian motion has characteristic triple (in reduced form)

$$(0, 0, \frac{a}{\sqrt{2\pi}}\, x^{-3/2} \mathbb{1}_{\{x>0\}} dx)\,.$$

Indeed $\mathbb{E}(e^{-\lambda T_a}) = e^{-a\sqrt{2\lambda}}$. Moreover

$$\sqrt{2\lambda} = \frac{1}{\sqrt{2}\Gamma(1/2)} \int_0^\infty (1 - e^{-\lambda x}) x^{-3/2} dx\,,$$

hence, using that $\Gamma(1/2) = \sqrt{\pi}$

$$\mathbb{E}(e^{-\lambda T_a}) = \exp\left(-\frac{a}{\sqrt{2\pi}} \int_0^\infty (1 - e^{-\lambda x})\, x^{-3/2} dx\right)\,.$$

Inverse Gaussian laws. The Inverse Gaussian law has density

$$\frac{a}{\sqrt{2\pi}} e^{a\nu} x^{-3/2} \exp\left(-\frac{1}{2}(a^2x^{-1} + \nu^2 x)\right) \mathbb{1}_{\{x>0\}}$$

This is the law of the first hitting time of a for a Brownian motion with drift ν. The Inverse Gaussian law has characteristic triple (in reduced form)

$$\left(0, 0, \frac{a}{\sqrt{2\pi x^3}} \exp\left(-\frac{1}{2}\nu^2 x\right) \mathbb{1}_{\{x>0\}} dx\right) .$$

Inverse Gaussian laws. The Inverse Gaussian law has density

$$\frac{a}{\sqrt{2\pi}} e^{a\nu} x^{-3/2} \exp\left(-\frac{1}{2}(a^2x^{-1} + \nu^2 x)\right) \mathbb{1}_{\{x>0\}}$$

This is the law of the first hitting time of a for a BM with drift ν.

The Inverse Gaussian law has characteristic triple (in reduced form)

$$\left(0, 0, \frac{a}{\sqrt{2\pi x^3}} \exp\left(-\frac{1}{2}\nu^2 x\right) \mathbb{1}_{\{x>0\}} dx\right).$$

Indeed

$$\begin{aligned}
&\exp\left(-\frac{a}{\sqrt{2\pi}} \int_0^\infty \frac{dx}{x^{3/2}} (1 - e^{-\lambda x})\, e^{-\nu^2 x/2}\right) \\
= \quad & \exp\left(-\frac{a}{\sqrt{2\pi}} \int_0^\infty \frac{dx}{x^{3/2}} \left((e^{-\nu^2 x/2} - 1) + (1 - e^{-(\lambda+\nu^2/2)x})\right)\right) \\
= \quad & \exp(-a(-\nu + \sqrt{\nu^2 + 2\lambda})
\end{aligned}$$

is the Laplace transform of the first hitting time of a for a BM with drift ν.

Stable Random Variables

A random variable is stable if for any $a > 0$, there exist $b > 0$ and $c \in I\!R$ such that $[\hat{\mu}(u)]^a = \hat{\mu}(bu)\, e^{icu}$.

X is stable if

$$\forall n,\ \exists(\beta_n, \gamma_n),\ \text{ such that }\ X_1^{(n)} + \cdots + X_n^{(n)} \stackrel{law}{=} \beta_n X + \gamma_n$$

where $(X_i^{(n)}, i \leq n)$ are i.i.d. random variables with the same law as X.

A stable law is infinitely divisible.

The characteristic function of a stable law can be written

$$\hat{\mu}(u) = \begin{cases} \exp(ibu - \frac{1}{2}\sigma^2 u^2), & \text{for } \alpha = 2 \\ \exp\left(-\gamma|u|^\alpha[1 - i\beta \operatorname{sgn}(u) \tan(\pi\alpha/2)]\right), & \text{for } \alpha \neq 1, \neq 2 \\ \exp\left(\gamma|u|(1 - i\beta v \ln|u|)\right), & \alpha = 1 \end{cases},$$

where $\beta \in [-1, 1]$. For $\alpha \neq 2$, the Lévy measure of a stable law is absolutely continuous with respect to the Lebesgue measure, with density

$$\nu(dx) = \begin{cases} c^+ x^{-\alpha-1} dx & \text{if } x > 0 \\ c^- |x|^{-\alpha-1} dx & \text{if } x < 0\,. \end{cases}$$

Examples: A Gaussian variable is stable with $\alpha = 2$. The Cauchy law is stable with $\alpha = 1$.

Lévy-Khintchine Representation

Let X be a Lévy process. Then, X_1 is i.d.

There exists $m \in I\!R^d$, a non-negative semi-definite quadratic form A, a Lévy measure ν such that for $u \in I\!R^d$

$$\mathbb{E}(\exp(i(u \cdot X_1))) =$$

$$\exp\left(i(u \cdot m) - \frac{1}{2}(u \cdot Au) + \int_{I\!R^d} (e^{i(u \cdot x)} - 1 - i(u \cdot x) \mathbb{1}_{|x| \leq 1}) \nu(dx) \right)$$

where ν is the Lévy measure.

• If $\nu(\mathbb{R} \setminus \{0\}) < \infty$, the process X has a finite number of jumps in any finite time interval. In finance, one refers to **finite activity**.

• If $\nu(\mathbb{R} \setminus \{0\}) < \infty$, the process X has a finite number of jumps in any finite time interval. In finance, one refers to **finite activity**.

• If $\nu(\mathbb{R} \setminus \{0\}) = \infty$, the process corresponds to **infinite activity**.

The complex valued continuous function Φ such that

$$\mathbb{E}\left[\exp(iuX_1)\right] = \exp(-\Phi(u))$$

is called the **characteristic exponent** (sometimes the Lévy exponent) of the Lévy process X.

The complex valued continuous function Φ such that

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is called the **characteristic exponent** (sometimes the Lévy exponent) of the Lévy process X.
If $\mathbb{E}\left[e^{\lambda X_1}\right] < \infty$ for any $\lambda > 0$, the function Ψ defined on $[0, \infty[$, such that

$$\mathbb{E}\left[\exp(\lambda X_1)\right] = \exp(\Psi(\lambda))$$

is called the **Laplace exponent** of the Lévy process X.

The complex valued continuous function Φ such that

$$\mathbb{E}\left[\exp(iuX_1)\right] = \exp(-\Phi(u))$$

is called the **characteristic exponent** (sometimes the Lévy exponent) of the Lévy process X.
If $\mathbb{E}\left[e^{\lambda X_1}\right] < \infty$ for any $\lambda > 0$, the function Ψ defined on $[0, \infty[$, such that

$$\mathbb{E}\left[\exp(\lambda X_1)\right] = \exp(\Psi(\lambda))$$

is called the **Laplace exponent** of the Lévy process X.
It follows that, if $\Psi(\lambda)$ exists,

$$\mathbb{E}\left[\exp(iuX_t)\right] = \exp(-t\Phi(u)), \qquad \mathbb{E}\left[\exp(\lambda X_t)\right] = \exp(t\Psi(\lambda))$$

and

$$\Psi(\lambda) = -\Phi(-i\lambda)\,.$$

From LK formula, the characteristic exponent and the Laplace exponent can be computed as follows:

$$\begin{aligned}\Phi(u) &= -ium + \frac{1}{2}\sigma^2 u^2 - \int (e^{iux} - 1 - iux\mathbb{1}_{|x|\le 1})\nu(dx)\\ \Psi(\lambda) &= \lambda m + \frac{1}{2}\sigma^2\lambda^2 + \int (e^{\lambda x} - 1 - \lambda x\mathbb{1}_{|x|\le 1})\nu(dx)\,.\end{aligned}$$

Martingales

• If $\mathbb{E}(|X_t|) < \infty$, i.e., $\int_{|x|\geq 1} |x|\nu(dx) < \infty$;
$\mathbb{E}(X_t) = t(m + \int_{|x|\geq 1} |x|\nu(dx))$, the process $X_t - \mathbb{E}(X_t)$ is a martingale.

Martingales

• If $\mathbb{E}(|X_t|) < \infty$, i.e., $\int_{|x|\geq 1} |x|\nu(dx) < \infty$, the process $X_t - \mathbb{E}(X_t)$ is a martingale and $\mathbb{E}(X_t) = t(m + \int_{|x|\geq 1} |x|\nu(dx))$.

• If $\Psi(\alpha)$ exists (i.e., if $\int_{|x|>1} e^x \nu(dx) < \infty$), the process

$$\frac{e^{\alpha X_t}}{\mathbb{E}(e^{\alpha X_t})} = e^{\alpha X_t - t\Psi(\alpha)}$$

is a martingale

More generally, for any bounded predictable process H

$$\mathbb{E}\left[\sum_{s\leq t} H_s f(\Delta X_s)\right] = \mathbb{E}\left[\int_0^t ds H_s \int f(x) d\nu(x)\right]$$

and if H is a predictable function (i.e. $H\ : \Omega \times I\!R^+ \times I\!R^d \to I\!R$ is $\mathcal{P} \times \mathcal{B}$ measurable)

$$\mathbb{E}\left[\sum_{s\leq t} H_s(\omega, \Delta X_s)\right] = \mathbb{E}\left[\int_0^t ds \int d\nu(x) H_s(\omega, x)\right] .$$

Both sides are well defined and finite if

$$\mathbb{E}\left[\int_0^t ds \int d\nu(x) |H_s(\omega, x)|\right] < \infty$$

(Exponential formula.) Let X be a Lévy process and ν its Lévy measure. For all t and all Borel function f defined on $\mathbb{R}^+ \times \mathbb{R}^d$ such that $\int_0^t ds \int |1 - e^{f(s,x)}|\nu(dx) < \infty$, one has

$$\mathbb{E}\left[\exp\left(\sum_{s\le t} f(s, \Delta X_s)\mathbb{1}_{\{\Delta X_s \neq 0\}}\right)\right] = \exp\left(-\int_0^t ds \int (1 - e^{f(s,x)})\nu(dx)\right) .$$

The above property does not extend to predictable functions.

Lévy-Itô's decomposition

If X is a R^d-valued Lévy process, it can be decomposed into $X = Y^{(0)} + Y^{(1)} + Y^{(2)} + Y^{(3)}$ where $Y^{(0)}$ is a affine function, $Y^{(1)}$ is a linear transform of a Brownian motion, $Y^{(2)}$ is a compound Poisson process with jump size greater than or equal to 1 and $Y^{(3)}$ is a Lévy process with jumps sizes smaller than 1. The processes $Y^{(i)}$ are independent.

More precisely

$$X_t = mt + \sigma W_t + X_t^1 + \lim_{\epsilon \to 0} \widetilde{X}_t^\epsilon$$

where

$$\begin{aligned} X_t^1 &= \int_0^t \int_{\{|x| \geq 1\}} x \mathbf{N}(dx, ds) = \sum_{s \leq t} \Delta X_s \mathbb{1}_{|\Delta X_s| \geq 1} \\ \widetilde{X}_t^\epsilon &= \int_0^t \int_{\{\epsilon \leq |x| < 1\}} x \left(\mathbf{N}(dx, ds) - \nu(dx) ds \right) \end{aligned}$$

The processes X^1 is a compound Poisson process, the process $\widetilde{X}^\epsilon$ is a compensated compound Poisson process, it is a martingale. Note that $\int_0^t \int_{\{\epsilon \leq |x| < 1\}} x \, \mathbf{N}(dx, ds)$ and $\int_0^t \int_{\{\epsilon \leq |x| < 1\}} x \, \nu(dx) ds)$ are well defined. However, these quantities do not converge as ϵ goes to 0.

Path properties

• The Lévy process X is continuous iff $\nu = 0$

• The Lévy process X with piecewise constant paths iff it is a compound Poisson process or iff $m = 0$, $\sigma = 0$ and $\int \nu(dx) < \infty$

• The Lévy process X is with finite variation path iff $\sigma = 0$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$. In that case,

$$ium + \int (e^{iux} - 1 - iux \mathbb{1}_{|x|\leq 1})\nu(dx)$$

can be written

$$ium_0 + \int (e^{iux} - 1)\nu(dx)$$

and

$$X_t = m_0 t + \sum_{s\leq t} \Delta X_s$$

- If $\int_{|x|\leq 1} |x|\nu(dx) = \infty$, the sum $\sum_{s\leq t} |\Delta X_s| \mathbb{1}_{|\Delta X_s|\leq\epsilon}$ diverges, however the compensated sum converges.

If X is a Lévy process with jumps bounded (by 1), it admits moments of any order, and, setting $Z_t = X_t - \mathbb{E}(X_t)$, $Z = Z^c + Z^d$ where Z^c is a continuous martingale,

$$Z_t^d = \int_{|x|<1} x \, (\mathbf{N}(dt, dx) - \nu(dx)dt)$$

and Z^c and Z^d are martingales and independent Lévy processes.

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If X is a Lévy process, it admits a decomposition as

$$dX_t = \alpha dt + \sigma dB_t + \int_{|x|<1} x \left(\mathbf{N}(dt, dx) - \nu(dx)dt\right) + \int_{|x|\geq 1} x\mathbf{N}(dt, dx)$$

The Lévy process is a semi-martingale, hence $\sum_{0<s\leq t} (\Delta X_s)^2 < \infty$

Some definitions on general stochastic processes

Local martingale

An adapted, right-continuous process M is an $\mathbf{F}$-**local martingale** if there exists a sequence of stopping times (T_n) such that

(i) The sequence T_n is increasing and $\lim_n T_n = \infty$, a.s.

(ii) For every n, the stopped process $M^{T_n} \mathbb{1}_{\{T_n > 0\}}$ is an $\mathbf{F}$-martingale.

Covariation of Martingales

• **Continuous local martingales:** Let X be a continuous local martingale. The **predictable quadratic variation** process of X is the continuous increasing process $\langle X \rangle$ such that $X^2 - \langle X \rangle$ is a local martingale.

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Let X and Y be two continuous local martingales.

• The predictable covariation process is the continuous finite variation process $\langle X, Y \rangle$ such that $XY - \langle X, Y \rangle$ is a local martingale. Note that $\langle X \rangle = \langle X, X \rangle$ and

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• Integration by parts formula

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t$$

• **General Local martingales:**

Let X and Y be two local martingales.

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▶ This covariation process is the limit in probability of $\sum_{i=1}^{p(n)} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$, for $0 < t_1 < \cdots < t_{p(n)} \leq t$ when $\sup_{i \leq p(n)} (t_i - t_{i-1})$ goes to 0.

▶ The covariation $[X, Y]$ of both processes X and Y can be also defined by polarisation

$$[X + Y, X + Y] = [X, X] + [Y, Y] + 2[X, Y]$$

Let X and Y be two local martingales.

▶ The **predictable covariation process** is the finite variation process $\langle X, Y \rangle$ such that

$XY - \langle X, Y \rangle$ is a local martingale

$\langle X, Y \rangle$ is predictable.

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$$XY - \langle X, Y\rangle \text{ is a local martingale}$$

$$\langle X, Y\rangle \text{ is predictable.}$$

The existence of the predictable covariation process requests some additional conditions on the local martingales ($[X, L]$ is $\mathbb{P}$-locally integrable).

If W is a Brownian motion $\langle W\rangle_t = [W]_t = t$.

If M is the compensated martingale of a Poisson process, $[M]_t = N_t$ and $\langle M\rangle_t = \lambda t$, and $[W, M] = 0$.

If $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, the covariation process under $\mathbb{P}$ and under $\mathbb{Q}$ are equal. This is not the case for the predictable covariation process.

Spaces of martingales

$\mathbf{H}^2$ is the set of **square integrable** martingales, i.e., martingales such that $\sup_{t<\infty} E(M_t^2) < \infty$

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For any martingale $M \in \mathbf{H}^2$, we denote by M^c its projection on $\mathbf{H}^{2,c}$ and by M^d its projection on $\mathbf{H}^{2,d}$. Then, $M = M^c + M^d$ is the decomposition of any martingale in $\mathbf{H}^2$ into its continuous and purely discontinuous parts.

Covariation of Semi-martingales

A semi-martingale is a càdlàg process X such that $X_t = M_t + A_t$, M martingale, A bounded variation process. One can write $X_t = M_t^c + M_t^d + A_t$ where M^c is continuous. We use the (usual) notation $X^c := M^c$.

▶ If X and Y are semi-martingales and if X^c, Y^c are their continuous martingale parts, their quadratic covariation is

$$[X,Y]_t = \langle X^c, Y^c \rangle_t + \sum_{s \le t} (\Delta X_s)(\Delta Y_s)\,.$$

▶ In the case where X, Y are continuous semi-martingales, their predictable covariation process is the predictable covariation process of their continuous martingale parts.

Stieltjes Integral

Let U be a càdlàg process with bounded variation (i.e., the difference between two increasing processes). The **Stieltjes integral** $\int_0^\infty \theta_s dU_s$ is defined for elementary processes θ of the form $\theta_s = \vartheta_a 1\!\!1_{]a,b]}(s)$, with ϑ_a a r.v. as $\int_0^\infty \theta_s dU_s = \vartheta_a (U(b) - U(a))$ and for θ such that $\int_0^\infty |\theta_s||dU(s)| < \infty$ by linearity and passage to the limit. (Hence, the integral is defined path-by-path.) Then, one defines the integral

$$\int_0^t \theta_s dU_s = \int_{]0,t]} \theta_s dU_s = \int_0^\infty 1\!\!1_{\{]0,t]\}} \theta_s dU_s \, .$$

Note that if U has a jump at time t_0, then $(\Theta_t := \int_0^t \theta_s dU_s,\ t \geq 0)$ has also a jump at time t_0 given as $\Delta\Theta_{t_0} = \Theta_{t_0} - \Theta_{t_0^-} = \theta_{t_0} \Delta U_{t_0}$.

Integration by Parts If U and V are two finite variation processes, Stieltjes' integration by parts formula can be written as follows

$$\begin{aligned} U_t V_t &= U_0 V_0 + \int_{]0,t]} V_s dU_s + \int_{]0,t]} U_{s-} dV_s \\ &= U_0 V_0 + \int_{]0,t]} V_{s-} dU_s + \int_{]0,t]} U_{s-} dV_s + \sum_{s \le t} \Delta U_s \, \Delta V_s \, . \end{aligned}$$

The summation $\sum_{s \le t} \Delta U_s \, \Delta V_s$ is in fact a summation over a denumerable number of times s, i.e., the times where U and V admit a common jump. As a partial check, one can verify that the jumps of the left-hand side, i.e., $U_t V_t - U_{t-} V_{t-}$, are equal to the jumps of the right hand side $V_{t-} \Delta U_t + U_{t-} \Delta V_t + \Delta U_t \, \Delta V_t$.

Chain Rule Let $F \in C^1$ and A a finite variation process. Then,

$$F(A_t) = F(A_0) + \int_0^t F'(A_{s-})dA_s + \sum_{s \le t} (F(A_s) - F(A_{s-}) - F'(A_{s-})\Delta A_s)$$

or,

$$F(A_t) = F(A_0) + \int_0^t F'(A_{s-})dA_s^c + \sum_{s \le t} F(A_s) - F(A_{s-})$$

where A^c is the continuous part of A.

Stochastic Integral

Let N be a counting process. The **stochastic integral**

$$\int_0^t C_s dN_s$$

is defined pathwise as a Stieltjes integral for every bounded measurable process (not necessarily $\mathbf{F}^N$-adapted) $(C_t, t \geq 0)$ by

$$(C \star N)_t \stackrel{def}{=} \int_0^t C_s dN_s = \int_{]0,t]} C_s dN_s \stackrel{def}{=} \sum_{n=1}^{\infty} C_{T_n} \mathbb{1}_{\{T_n \leq t\}} \, .$$

We emphasize that the integral $\int_0^t C_s dN_s$ is here an integral over the time interval $]0, t]$, where the upper limit t is included and the lower limit 0 excluded. This integral is finite since there is a finite number of jumps during the time interval $]0, t]$.

We shall also write

$$\int_0^t C_s dN_s = \sum_{s \le t} C_s \Delta N_s$$

where the right-hand side contains only a finite number of non-zero terms. The integral $\int_0^\infty C_s dN_s$ is defined as $\int_0^\infty C_s dN_s = \sum_{n=1}^\infty C_{T_n}$, when the right-hand side converges.

We shall also use the differential notation $d(C \star N)_t \stackrel{def}{=} C_t dN_t$.

Integration by parts formula for Poisson process

Let $(x_t, t \geq 0)$ and $(y_t, t \geq 0)$ be two predictable processes and let $X_t = x + \int_0^t x_s dN_s$ and $Y_t = y + \int_0^t y_s dN_s$. The jumps of X (resp. of Y) occur at the same times as the jumps of N and $\Delta X_s = x_s \Delta N_s, \Delta Y_s = y_s \Delta N_s$. Then

$$X_t Y_t = xy + \sum_{s \leq t} \Delta(XY)_s = xy + \sum_{s \leq t} X_{s-} \Delta Y_s + \sum_{s \leq t} Y_{s-} \Delta X_s + \sum_{s \leq t} \Delta X_s \, \Delta Y_s$$

The first equality is obvious, the second one is easy to check.

Integration by parts formula for Poisson process

Let $(x_t, t \geq 0)$ and $(y_t, t \geq 0)$ be two predictable processes and let $X_t = x + \int_0^t x_s dN_s$ and $Y_t = y + \int_0^t y_s dN_s$. The jumps of X (resp. of Y) occur at the same times as the jumps of N and $\Delta X_s = x_s \Delta N_s, \Delta Y_s = y_s \Delta N_s$. Then

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The first equality is obvious, the second one is easy to check.
Hence, from the definition of stochastic integrals

$$X_t Y_t = xy + \int_0^t Y_{s-} dX_s + \int_0^t X_{s-} dY_s + [X, Y]_t$$

where (note that $(\Delta N_t)^2 = \Delta N_t$)

$$[X, Y]_t := \sum_{s \leq t} \Delta X_s \, \Delta Y_s = \sum_{s \leq t} x_s y_s \Delta N_s = \int_0^t x_s \, y_s \, dN_s \, .$$

More generally, if $dX_t = \mu_t dt + x_t dN_t$ with $X_0 = x$ and $dY_t = \nu_t dt + y_t dN_t$ with $Y_0 = y$, one gets

$$X_t Y_t = xy + \int_0^t Y_{s-} dX_s + \int_0^t X_{s-} dY_s + [X, Y]_t$$

where $[X, Y]_t = \int_0^t x_s \, y_s \, dN_s$.

If x is a predictable (bounded) process, the integral

$$\int_0^t x_s dM_s$$

is a martingale.

This is no more the case if x is not predictable, even if the integral is well defined. The process $\int_0^t N_s dM_s$ is not a martingale.

In particular, from integration by parts formula, if $dX_t = x_t dM_t$ and $dY_t = y_t dM_t$, the process $X_t Y_t - [X, Y]_t$ is a local martingale.

Doláns-Dade exponential of a finite variation process

Let U be a càdlàg process with finite variation. The unique solution of

$$dY_t = Y_{t-} dU_t,\ Y_0 = y$$

is the stochastic exponential of U (the Doléans-Dade exponential of U) equal to

$$\begin{aligned} Y_t &= y \exp(U_t^c - U_0^c) \prod_{s \le t} (1 + \Delta U_s) \\ &= y \exp(U_t - U_0) \prod_{s \le t} (1 + \Delta U_s) e^{-\Delta U_s}\ . \end{aligned}$$

PROOF: Applying the integration by parts formula shows that it is a solution to the equation $dY_t = Y_{t-} dU_t$. As for the uniqueness, if $Y^i, i = 1, 2$ are two solutions, then, setting $Z = Y^1 - Y^2$ we get $Z_t = \int_0^t Z_{s-} dU_s$. Let $M_t = \sup_{s \le t} |Z_s|$, then, if V_t is the variation process of U_t

$$|Z_t| \le M_t V_t$$

which implies that

$$|Z_t| \le M_t \int_0^t V_{s-} dV_s = M_t \frac{V_t^2}{2} \, .$$

Iterating, we obtain $|Z_t| \le M_t \frac{V_t^n}{n!}$ and the uniqueness follows by letting $n \to \infty$. △

Itô's formula

Itô's Formula For Poisson processes

Let N be a Poisson process and f a bounded Borel function. The decomposition

$$f(N_t) = f(N_0) + \sum_{0<s\leq t} [f(N_s) - f(N_{s-})]$$

is trivial and is the main step to obtain Itô's formula for a Poisson process.

We can write the right-hand side as a stochastic integral:

$$\begin{aligned} \sum_{0<s\leq t} [f(N_s) - f(N_{s-})] &= \sum_{0<s\leq t} [f(N_{s-} + 1) - f(N_{s-})]\Delta N_s \\ &= \int_0^t [f(N_{s-} + 1) - f(N_{s-})] dN_s \,, \end{aligned}$$

hence, the canonical decomposition of $f(N_t)$ as the sum of a martingale and an absolute continuous adapted process is

$$f(N_t) = f(N_0) + \int_0^t [f(N_{s^-} + 1) - f(N_{s^-})] dM_s + \int_0^t [f(N_{s^-} + 1) - f(N_{s^-})] \lambda ds \,.$$

More generally, assume that N is an inhomogeneous Poisson process (i.e., N is a counting process and there exists a non-negative function λ such that $N_t - \int_0^t \lambda(s)ds$ is a martingale). Let h be an adapted process and g a predictable process such that $\int_0^t |h_s|ds < \infty$, $\int_0^t |g_s|\lambda_s ds < \infty$. Let $F \in C^{1,1}(I\!R^+ \times I\!R)$ and

$$dX_t = h_t dt + g_t dM_t = (h_t - g_t\lambda_t)dt + g_t dN_t$$

Then

$$\begin{aligned}
F(t, X_t) &= F(0, X_0) + \int_0^t \partial_t F(s, X_s)ds + \int_0^t \partial_x F(s, X_{s-})(h_s - g_s\lambda(s))ds \\
&\quad + \sum_{s \le t} F(s, X_s) - F(s, X_{s-}) \\
&= F(0, X_0) + \int_0^t \partial_t F(s, X_s)ds + \int_0^t \partial_x F(s, X_{s-})dX_s \\
&\qquad + \sum_{s \le t} \left[F(s, X_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-}) g_s \Delta N_s \right] .
\end{aligned}$$

$$
\begin{aligned}
F(t, X_t) &= F(0, X_0) + \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_s)(h_s - g_s \lambda(s)) ds \\
&\quad + \int_0^t [F(s, X_s) - F(s, X_{s-})] dN_s
\end{aligned}
$$

PROOF: Indeed, between two jumps, $dX_t = (h_t - \lambda_t g_t)dt$, and for $T_n < s < t < T_{n+1}$,

$$F(t, X_t) = F(s, X_s) + \int_s^t \partial_t F(u, X_u)du + \int_s^t \partial_x F(u, X_u)(h_u - g_u \lambda_u)du\,.$$

At jump times, $F(T_n, X_{T_n}) = F(T_n, X_{T_n-}) + \Delta F(\cdot, X)_{T_n}$. $\triangle$

Remark that, in the "ds" integrals, we can write X_{s-} or X_s, since, for any bounded Borel function f,

$$\int_0^t f(X_{s-})ds = \int_0^t f(X_s)ds\,.$$

Note that since dN_s a.s. $N_s = N_{s-} + 1$, one has

$$\int_0^t f(N_{s-})dN_s = \int_0^t f(N_s - 1)dN_s\,.$$

We shall use systematically use the form $\int_0^t f(N_{s-})dN_s$, even if the $\int_0^t f(N_s - 1)dN_s$ has a meaning.

The reason is that $\int_0^t f(N_{s-})dM_s = \int_0^t f(N_{s-})dN_s + \lambda \int_0^t f(N_{s-})ds$ is a martingale, whereas $\int_0^t f(N_s - 1)dM_s$ is not.

Check that the above formula can be written as

$$
\begin{aligned}
& F(t, X_t) - F(0, X_0) \\
= \quad & \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_s)(h_s - g_s \lambda(s)) ds \\
& \qquad + \int_0^t [F(s, X_s) - F(s, X_{s-})] dN_s \\
= \quad & \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_{s-}) dX_s \\
& \qquad + \int_0^t [F(s, X_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-}) g_s] dN_s
\end{aligned}
$$

$$
\begin{aligned}
& F(t, X_t) - F(0, X_0) \\
= & \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_{s-}) dX_s \\
& \qquad + \int_0^t [F(s, X_{s-} + g_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-}) g_s] dN_s \\
= & \int_0^t \left(\partial_t F(s, X_s) + [F(s, X_{s-} + g_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-}) g_s] \lambda \right) ds \\
& \qquad + \int_0^t [F(s, X_{s-} + g_s) - F(s, X_{s-})] dM_s
\end{aligned}
$$

Let X be a ν-compound Poisson process, and $Z_t = Z_0 + bt + X_t$. Then, using that $\mathbf{N} = \sum_{n=1}^{\infty} \delta_{T_n, Y_n}$, Itô's formula

$$
\begin{aligned}
f(Z_t) - f(Z_0) &= b \int_0^t f'(Z_s) ds + \sum_{k,\, T_k \le t} f(Z_{T_k}) - f(Z_{T_k -}) \\
&= b \int_0^t f'(Z_s) ds + \int_0^t \int_{I\!R} [f(Z_{s-} + y) - f(Z_{s-})]\, \mathbf{N}(ds, dy) \\
&= \int_0^t ds\, (\mathcal{L} f)(Z_s) + M(f)_t
\end{aligned}
$$

can be written as where $\mathcal{L} f(x) = b f'(x) + \int_{I\!R} (f(x+y) - f(x))\, \nu(dy)$ is the infinitesimal generator of Z and

$$
M(f)_t = \int_0^t \int_{I\!R} [f(Z_{s-} + y) - f(Z_{s-})]\, (\mathbf{N}(ds, dy) - ds\, \nu(dy))
$$

is a local martingale.

Let $\mathbb{Q}$ be equivalent to $\mathbb{P}$ on $\mathcal{F}_t$, for any t and $\mathbb{Q}|_{\mathcal{F}_t} = L_t \, \mathbb{P}|_{\mathcal{F}_t}$ where L is a strictly positive $\mathbb{P}$-martingale. Any $\mathbb{P}$-local martingale X is a $\mathbb{Q}$ semi-martingale and its semi-martingale decompositions are given by the following theorem:

(i)

$$\boxed{X_t - \int_0^t \frac{d[X,L]_s}{L_s} \quad \text{is a } \mathbb{Q}\text{-local martingale}}$$

(ii) If $[X,L]$ is $\mathbb{P}$-locally integrable, the process

$$\boxed{X_t - \int_0^t \frac{d\langle X,L\rangle_s}{L_{s^-}} \quad \text{is a } \mathbb{Q}\text{-local martingale}}$$

General case

Let X be a semi-martingale and $f \in C^{1,2}$ Then

$$\begin{aligned} df(t, X_t) &= \partial_t f(t, X_t)dt + \partial_x f(t, X_{t-})dX_t + \frac{1}{2}\partial_{xx} f(t, X_{t-})d[X^c]_t \\ &\quad + f(t, X_t) - f(t, X_{t-}) - \Delta X_t \, \partial_x f(t, X_{t-}) \end{aligned}$$

Back to Lévy processes

Begin at the beginning, and go on till you come to the end. Then, stop.

L. Carroll, Alice's Adventures in Wonderland

Covariation processes

Let X be a (m, σ^2, ν) real valued Lévy process. Then, $X_t^c = \sigma W_t$ and

$$[X]_t = \sigma^2 t + \int_0^t \int x^2 \mathbf{N}(ds, dx)$$

If $\int x^2 \nu(dx) < \infty$,

$$\langle X \rangle_t = \sigma^2 t + t \int x^2 \nu(dx)$$

Itô's formula

If X is a Lévy process, it admits a decomposition as

$$dX_t = \alpha dt + \sigma dB_t + \int_{|x|<1} x \left(\mathbf{N}(dt, dx) - \nu(dx)dt\right) + \int_{|x|\geq 1} x\mathbf{N}(dt, dx)$$

The Lévy process is a semi-martingale, hence $\sum_{0<s\leq t} (\Delta X_s)^2 < \infty$

$$
\begin{aligned}
f(X_t) &= f(X_0) + \frac{\sigma^2}{2}\int_0^t f''(X_s)ds + \int_0^t f'(X_{s-})dX_s \\
&\quad + \sum_{s\le t}(f(X_{s-} + \Delta X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-}))
\end{aligned}
$$

As a consequence of the semi-martingale property, if F is a C^2 function, then, the series

$$
\sum_{s\le t} f(X_{s-} + \Delta X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-}))
$$

converges, since

$$
|f(X_{s-} + \Delta X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-})| \le c(\Delta X_s)^2
$$

.

Let $Y_t = f(t, X_t)$, with f bounded with bounded derivatives. Then, Y is a semi-martingale

Its martingale part is

$$\partial_x f(t, X_t)\sigma dB_t + \int \left(f(t, X_{t^-} + x) - f(t, X_{t^-})\right)\left(\mathbf{N}(dt, dx) - \nu(dx)dt\right)$$

Its finite variation part is

$$\begin{aligned}\partial_t f(t, X_t) \quad &+ \quad \alpha\partial_x f(t, X_t) + \frac{1}{2}\sigma^2\partial_{xx} f(t, X_t) \\ &+ \int \left(f(t, X_{t^-} + x) - f(t, X_{t^-}) - x\partial_x f(t, X_{t^-})\mathbb{1}_{x\leq 1}\right)\nu(dx)\end{aligned}$$

Representation Theorem

Let X be a $\mathbb{R}^d$-valued Lévy process and $\mathbf{F}^X$ its natural filtration. Let M be a locally square integrable martingale with $M_0 = m$. Then, there exists a family (φ, ψ) of predictable processes such that

$$\int_0^t |\varphi_s^i|^2 ds < \infty, \text{ a.s.}$$

$$\int_0^t \int_{\mathbb{R}^d} |\psi_s(x)|^2 ds\, \nu(dx) < \infty, \text{ a.s.}$$

and

$$M_t = m + \sum_{i=1}^d \int_0^t \varphi_s^i dW_s^i + \int_0^t \int_{\mathbb{R}^d} \psi_s(x)(\mathbf{N}(ds, dx) - ds\, \nu(dx)) .$$

Change of measure

Poisson Process

Let N be a Poisson process with intensity λ, and $\mathbb{Q}$ be the probability defined by (with $\beta > -1$)

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = (1+\beta)^{N_t} e^{-\lambda\beta t}$$

Then, the process N is a $\mathbb{Q}$-Poisson process with intensity equal to $(1+\beta)\lambda$.

The process L defined as

$$L_t = (1+\beta)^{N_t} e^{-\lambda \beta t}$$

is a strictly positive martingale with expectation equal to 1. Then, from the definition of $\mathbb{Q}$, for any sequence $0 = t_1 < t_2 < \cdots < t_{n+1} = t$,

$$\mathbb{E}_Q \left(\prod_{i=1}^{n} x_i^{N_{t_{i+1}} - N_{t_i}} \right) = \mathbb{E}_P \left(e^{-\lambda \beta t} \prod_{i=1}^{n} ((1+\beta) x_i)^{N_{t_{i+1}} - N_{t_i}} \right)$$

The right-hand side is computed using that, under $\mathbb{P}$, the process N is a Poisson process (hence with independent increments) and is equal to

$$\begin{aligned} e^{-\lambda \beta t} \prod_{i=1}^{n} \mathbb{E}_P \left(((1+\beta) x_i)^{N_{t_{i+1} - t_i}} \right) &= e^{-\lambda \beta t} \prod_{i=1}^{n} e^{-\lambda (t_{i+1} - t_i)} \, e^{\lambda (t_{i+1} - t_i)(1+\beta) x_i} \\ &= \prod_{i=1}^{n} e^{(1+\beta)\lambda (t_{i+1} - t_i)(x_i - 1)} \, . \end{aligned}$$

$$\mathbb{E}_Q\left(\prod_{i=1}^{n} x_i^{N_{t_{i+1}}-N_{t_i}}\right)=\prod_{i=1}^{n} e^{(1+\beta)\lambda(t_{i+1}-t_i)(x_i-1)}\,.$$

In particular, for any j (take all the x_i's, except the jth one, equal to 1)

$$\mathbb{E}_Q\left(x_j^{N_{t_{j+1}}-N_{t_j}}\right)=e^{(1+\beta)\lambda(t_{j+1}-t_j)(x_j-1)}\,,$$

which establishes that, under $\mathbb{Q}$, the r.v. $N_{t_{j+1}}-N_{t_j}$ has a Poisson law with parameter $(1+\beta)\lambda$, then that

$$\mathbb{E}_Q\left(\prod_{i=1}^{n} x_i^{N_{t_{i+1}}-N_{t_i}}\right)=\prod_{i=1}^{n}\mathbb{E}_Q\left(x_i^{N_{t_{i+1}}-N_{t_i}}\right)$$

which is equivalent to the independence of the increments.

Compound Poisson process

Let X be a ν-compound Poisson process under $\mathbb{P}$, we present some particular probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that, under $\mathbb{Q}$, X is still a compound Poisson process.

Let $\widetilde{\nu}$ a positive finite measure on $I\!R$ absolutely continuous w.r.t. ν, and $\widetilde{\lambda} = \widetilde{\nu}(I\!R) > 0$. Let

$$L_t = \exp\left(t(\lambda - \widetilde{\lambda}) + \sum_{s \le t} \ln\left(\frac{d\widetilde{\nu}}{d\nu}\right)(\Delta X_s) \right) .$$

Recall that

$$\exp\left(\int_0^t \int_{I\!R} f(x) \mathbf{N}(ds, dx) - t \int_{-\infty}^{\infty} (e^{f(x)} - 1)\nu(dx) \right)$$

is a martingale

Applying this martingale property for $f = \ln\left(\frac{d\widetilde{\nu}}{d\nu}\right)$, the process L is a martingale. Set $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$. **Under $\mathbb{Q}$, the process X is a $\widetilde{\nu}$-compound Poisson process.**

PROOF: First we find the law of the r.v. X_t under $\mathbb{Q}$. From the definition of $\mathbb{Q}$

$$
\begin{aligned}
\mathbb{E}_Q(e^{iuX_t}) &= \mathbb{E}_P(e^{iuX_t} \exp\left(t(\lambda - \widehat{\lambda}) + \sum_{k=1}^{N_t} f(Y_k)\right) \\
&= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} e^{t(\lambda - \widehat{\lambda})} \left(\mathbb{E}_P(e^{iuY_1 + f(Y_1)})\right)^n \\
&= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} e^{t(\lambda - \widehat{\lambda})} \left(\mathbb{E}_P(\frac{d\widehat{\nu}}{d\nu}(Y_1) e^{iuY_1})\right)^n \\
&= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-t\widehat{\lambda}} \left(\frac{1}{\lambda} \int e^{iuy} d\widehat{\nu}(y)\right)^n = \exp t \int (e^{iuy} - 1) d\widehat{\nu}(y)
\end{aligned}
$$

It remains to check that X is with independent and stationary increments under $\mathbb{Q}$.

By Bayes formula, for $t > s$

$$\begin{aligned}
\mathbb{E}_Q(e^{iu(X_t - X_s)}|\mathcal{F}_s) &= \frac{1}{L_s}\mathbb{E}_P(L_t e^{iu(X_t - X_s)}|\mathcal{F}_s) \\
&= \exp\left((t-s)\int(e^{iux}-1)\widetilde{\nu}(dx)\right) .
\end{aligned}$$

$\triangle$

Esscher transform

We assume that $\mathbb{E}(e^{(\theta \cdot X_t)}) < \infty$. We define a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$ by the formula

$$\mathbb{Q}|_{\mathcal{F}_t} = \frac{e^{(\theta \cdot X_t)}}{\mathbb{E}(e^{(\theta \cdot X_t)})} \mathbb{P}|_{\mathcal{F}_t} .(*)$$

This particular choice of measure transformation, (called an Esscher transform) preserves the Lévy process property.

Let X be a $\mathbb{P}$-Lévy process with parameters (m, A, ν) where $A = R^T R$. Let θ be such that $\mathbb{E}(e^{(\theta \cdot X_t)}) < \infty$ and suppose $\mathbb{Q}$ is defined by (*).
Then X is a Lévy process under $\mathbb{Q}$

It is not difficult to prove that X has independent and stationary increments under $\mathbb{Q}$. The characteristic exponent of X under $\mathbb{Q}$ is $\Phi^{(\theta)}$ such that

$$\begin{aligned} e^{-t\Phi^{(\theta)}(u)} &= \mathbb{E}_{\mathbb{Q}}(e^{i(u \cdot X_t)}) = \mathbb{E}(e^{i(u \cdot X_t) + (\theta \cdot X_t)})e^{t\Phi(-i\theta)} \\ &= e^{-t(\Phi(u-i\theta)-\Phi(-i\theta))} \,. \end{aligned}$$

The characteristic exponent of X under $\mathbb{Q}$ is

$$\Phi^{(\theta)}(u) = \Phi(u - i\theta) - \Phi(-i\theta)\,.$$

If $\Psi(\theta) < \infty$, $\Psi^{(\theta)}(u) = \Psi(u + \theta) - \Psi(\theta)$ for $u \geq \min(-\theta, 0)$.

A simple computation leads to

$$
\begin{aligned}
\Phi(u-i\theta) \quad - \quad \Phi(-i\theta) &= -iu\cdot m + \frac{1}{2}u\cdot Au - \frac{1}{2}iu\cdot A\theta - \frac{1}{2}i\theta\cdot Au \\
&\quad - \int \left(e^{\theta\cdot x}(e^{iu\cdot x}-1) - iu\cdot x \mathbb{1}_{\{|x|\le 1\}}\right)\nu(dx) \\
= \quad & -iu\cdot\left(m + \frac{1}{2}(A+A^T)\theta + \int (e^{\theta\cdot x}-1)x\mathbb{1}_{\{|x|\le 1\}}\nu(dx)\right) \\
&\quad + \frac{1}{2}u\cdot Au + \int e^{\theta\cdot x}(e^{iu\cdot x} - 1 - iu\cdot x\mathbb{1}_{\{|x|\le 1\}})\nu(dx)\,.
\end{aligned}
$$

Hence, X_1 has the required Lévy-Khintchine representation under $\mathbb{Q}$.

$$
\begin{aligned}
\mathbb{E}_Q(\exp(i(u\cdot X_1))) &= \exp\bigg(i(u\cdot m^{(\theta)}) - \frac{1}{2}(u\cdot Au) \\
&\qquad + \int_{\mathbb{R}^d} (e^{i(u\cdot x)} - 1 - i(u\cdot x)\mathbb{1}_{|x|\le 1})\nu^{(\theta)}(dx)\bigg)
\end{aligned}
$$

with

$$
\begin{aligned}
m^{(\theta)} &= m + \frac{1}{2}(A + A^T)\theta + \int_{|x|\le 1} x(e^{\theta x} - 1)\nu(dx) \\
\nu^{(\theta)}(dx) &= e^{\theta x}\nu(dx)\,.
\end{aligned}
$$

General case

More generally, any density $(L_t, t \geq 0)$ which is a positive martingale can be used.

$$dL_t = \sum_{i=1}^{d} \widetilde{\varphi}_t^i dW_t^i + \int \widetilde{\psi}_t(x)[\mathbf{N}(dt, dx) - dt\nu(dx)]\,.$$

From the strict positivity of L, there exists φ, ψ such that $\widetilde{\varphi}_t = L_{t-}\varphi_t$, $\widetilde{\psi}_t = L_{t-}(e^{\psi(t,x)} - 1)$, hence the process L satisfies

$$dL_t = L_{t-}\left(\sum_{i=1}^{d} \varphi_t^i dW_t^i + \int (e^{\psi(t,x)} - 1)[\mathbf{N}(dt, dx) - dt\nu(dx)]\right) \;(**)$$

Let $\mathbb{Q}|_{\mathcal{F}_t} = L_t\,\mathbb{P}|_{\mathcal{F}_t}$ where L is defined in (**). With respect to $\mathbb{Q}$,

(i) $W_t^{\varphi} \stackrel{def}{=} W_t - \int_0^t \varphi_s ds$ is a Brownian motion

(ii) The process N is compensated by $e^{\psi(s,x)} ds\nu(dx)$ meaning that for any Borel function h such that

$$\int_0^T \int_{I\!R} |h(s,x)| e^{\psi(s,x)} ds\nu(dx) < \infty\,,$$

the process

$$\int_0^t \int_{I\!R} h(s,x) \left(\mathbf{N}(ds,dx) - e^{\psi(s,x)} ds\nu(dx)\right)$$

is a local martingale.

Fluctuation theory

Let $M_t = \sup_{s \leq t} X_s$ be the running maximum of the Lévy process X. The reflected process $M - X$ enjoys the strong Markov property.

Let θ be an exponential variable with parameter q, independent of X. Note that

$$\mathbb{E}(e^{iuX_\theta}) = q \int \mathbb{E}(e^{iuX_t}) e^{-qt} dt = q \int e^{-t\Phi(u)} e^{-qt} dt \,.$$

Using excursion theory, **the random variables M_θ and $X_\theta - M_\theta$ can be proved to be independent**, hence

$$\mathbb{E}(e^{iuM_\theta}) \mathbb{E}(e^{iu(X_\theta - M_\theta)}) = \frac{q}{q + \Phi(u)} \,.$$

This equality is known as the **Wiener-Hopf factorization**.

Let $m_t = \min_{s \leq t}(X_s)$. Then

$$m_\theta \stackrel{law}{=} X_\theta - M_\theta \,.$$

If $\mathbb{E}(e^{X_1}) < \infty$, using Wiener-Hopf factorization, Mordecki proves that the boundaries for perpetual American options are given by

$$b_p = K\mathbb{E}(e^{m_\theta}), b_c = K\mathbb{E}(e^{M_\theta})$$

where $m_t = \inf_{s \leq t} X_s$ and θ is an exponential r.v. independent of X with parameter r, hence $b_c b_p = \dfrac{rK^2}{1 - \ln \mathbb{E}(e^{X_1})}$.

Pecherskii-Rogozin Identity

For $x > 0$, denote by T_x the first passage time above x defined as

$$T_x = \inf\{t > 0 \,:\, X_t > x\}$$

and by $K_x = X_{T_x} - x$ the so-called overshoot.

For every triple of positive numbers (α, β, q),

$$\int_0^\infty e^{-qx}\mathbb{E}(e^{-\alpha T_x - \beta K_x})dx = \frac{\kappa(\alpha, q) - \kappa(\alpha, \beta)}{(q-\beta)\kappa(\alpha, q)}$$

where κ is the Laplace exponent of the ladder process defined as

$$e^{-\ell\kappa(\alpha,\beta)} = E(\exp(-\alpha\tau_\ell - \beta H_\ell))\,,$$

where H is defined in terms of M and the local time of $M - X$.

Exponential Lévy Processes

Let X be a Lévy process.

The process X is a martingale iff $\int_{|x|\geq 1} |x|\nu(dx) < \infty$ and $b + \int_{|x|\geq 1} x\nu(dx) = 0$.

The process e^X is a martingale iff $\int_{|x|\geq 1} e^x \nu(dx)$ and $b + \frac{1}{2}\sigma^2 + \int (e^x - 1 - x 1\!\!1_{|x|\leq 1}\nu(dx) = 0$

Let $C(t, S)$ be a $C^{1,2}$ function and $S_t = S_0 e^{rt + X_t}$ where $\int_{|x|\geq 1} e^{2x}\nu(dx) < \infty$.

The process $e^{-rt}C(t, S_t)$ is a martingale iff

$$\begin{aligned} \frac{\partial C}{\partial t} &+ rS\frac{\partial C}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} - rC \\ &+ \int \nu(dx)(C(t, Se^x) - C(t, S) - S(e^x - 1)\frac{\partial C}{\partial S}) = 0 \end{aligned}$$

Exponential and stochastic exponential of Lévy Processes

Doléans-Dade Exponential Let X be a real-valued (m, σ^2, ν)-Lévy process. The solution of

$$dZ_t = Z_{t-} dX_t,\ Z_0 = 1$$

is

$$Z_t = e^{X_t - \frac{1}{2}\sigma^2 t} \prod_{s \le t} (1 + \Delta X_s) e^{-\Delta X_s}$$

Proof: in a first step, we prove that

$$V_t = \prod_{s \le t} (1 + \Delta X_s) e^{-\Delta X_s}$$

is well-defined and is a finite variation process.

$$V_t = \underbrace{\prod_{s \le t,\, |\Delta X_s| \le 1/2} (1 + \Delta X_s) e^{-\Delta X_s}}_{V_t^1} \underbrace{\prod_{s \le t,\, |\Delta X_s| > 1/2} (1 + \Delta X_s) e^{-\Delta X_s}}_{V_t^2}$$

The product in V^2 contains a finite number of terms

The process V^1 is non-negative and

$$\ln(V_t^1) = \sum_{s \le t,\, |\Delta X_s| \le 1/2} (\ln(1 + \Delta X_s) - \Delta X_s)$$

Using

$$0 \ge (\ln(1 + \Delta X_s) - \Delta X_s) \ge -(\Delta X_s)^2$$

we check that V^1 is well defined and with bounded variation.

Then, we apply Itô's formula. Let $Z_t = e^{X_t - \frac{1}{2}\sigma^2 t} V_t$. Then

$$\begin{aligned} dZ_t &= -\frac{\sigma^2}{2} Z_{t-} dt + Z_{t-} dX_t + e^{X_{t-} - \frac{1}{2}\sigma^2 t} dV_t + \frac{\sigma^2}{2} Z_{t-} dt \\ &\quad + (Z_t - Z_{t-}) - Z_{t-} \Delta X_t - e^{X_{t-} - \frac{1}{2}\sigma^2 t} \Delta V_t \qquad (\dagger) \\ &= Z_{t-} dX_t + e^{X_{t-} - \frac{1}{2}\sigma^2 t} (V_t e^{\Delta X_t} - V_{t-} - V_{t-} \Delta X_t) \\ &= Z_{t-} dX_t \end{aligned}$$

More generally, the solution of the SDE

$$dS_t = S_{t-}(b(t)dt + \sigma(t)dX_t)$$

is

$$\begin{aligned} S_t &= S_0 \exp\left(\int_0^t \sigma(s)dX_s + \int_0^t (b(s) - \frac{\sigma^2(s)}{2}ds\right) \\ &\quad \prod_{0<s\leq t} (1 + \sigma(s)\Delta X_s) \exp(-\sigma(s)\Delta X_s) \,. \end{aligned}$$

Exponentials of Lévy Processes

Let X be a real-valued (m, σ^2, ν)-Lévy process.

Let $S_t = e^{X_t}$ be the ordinary exponential of the process X. The stochastic logarithm of S (i.e., the process Y which satisfies $S_t = \mathcal{E}(Y)_t$) is a Lévy process and is given by

$$Y_t := \mathcal{L}(S)_t = X_t + \frac{1}{2}\sigma^2 t - \sum_{0<s\leq t} \left(1 + \Delta X_s - e^{\Delta X_s}\right) .$$

The Lévy characteristics of Y are

$$\begin{aligned}
m_Y &= m + \frac{1}{2}\sigma^2 + \int \left((e^x - 1)\mathbb{1}_{\{|e^x-1|\leq 1\}} - x\mathbb{1}_{\{|x|\leq 1\}}\right)\nu(dx) \\
\sigma_Y^2 &= \sigma^2 \\
\nu_Y(A) &= \nu(\{x : e^x - 1 \in A\}) = \int \mathbb{1}_A(e^x - 1)\,\nu(dx) .
\end{aligned}$$

The process $Y_t = X_t + \frac{1}{2}\sigma^2 t - \sum_{0<s\leq t}\left(1 + \Delta X_s - e^{\Delta X_s}\right)$ is a Lévy process, $\sigma_Y^2 = \sigma^2$, and $\Delta Y_t = e^{\Delta X_t} - 1$.

This implies the form of $\nu_Y(dx)$. $\quad (\dagger)$

we obtain that the Lévy-Itô decomposition of Y is

$$\begin{aligned}
Y_t &= X_t + \frac{1}{2}\sigma^2 t - \sum_{0<s\le t}\left(1+\Delta X_s - e^{\Delta X_s}\right) \\
&= mt + \sigma B_t + \int_0^t \int_{\{|x|\le 1\}} x\tilde{\mathbf{N}}(ds,dx) + \int_0^t \int_{\{|x|>1\}} x\mathbf{N}(ds,dx) + \frac{1}{2}\sigma^2 t \\
&\qquad - \int_0^t \int (1+x-e^x)\mathbf{N}(ds,dx) \\
&= m_Y t + \sigma B_t + \int_0^t \int (e^x-1)\mathbb{1}_{\{|e^x-1|\le 1\}}\tilde{\mathbf{N}}(ds,dx) \\
&\qquad + \int_0^t \int (e^x-1)\mathbb{1}_{\{|e^x-1|>1\}}\mathbf{N}(ds,dx) \\
&= m_Y t + \sigma B_t + \int_0^t \int y\mathbb{1}_{\{|y|\le 1\}}\tilde{\mathbf{N}}_Y(ds,dy) + \int_0^t \int y\mathbb{1}_{\{|y|>1\}}\mathbf{N}_Y(ds,dy)\,.
\end{aligned}$$

The result follows.

Let $Z_t = \mathcal{E}(X)_t$ the Doléans-Dade exponential of X. If $Z > 0$, the ordinary logarithm of Z is a Lévy process L given by

$$L_t := \ln(Z_t) = X_t - \frac{1}{2}\sigma^2 t + \sum_{0<s\leq t} (\ln(1+\Delta X_s) - \Delta X_s) \ .$$

Its Lévy characteristics are

$$\begin{aligned} m_L &= m - \frac{1}{2}\sigma^2 + \int \left(\ln(1+x)1\!\!1_{\{|\ln(1+x)|\leq 1\}} - x 1\!\!1_{\{|x|\leq 1\}}\right) \nu(dx) \\ \sigma_L^2 &= \sigma^2 \\ \nu_L(A) &= \nu(\{x \,:\, \ln(1+x) \in A\}) = \int 1\!\!1_A(\ln(1+x))\,\nu(dx) \end{aligned}$$

Exponentials of Lévy Processes

Let $S_t = xe^{X_t}$ where X is a (m, σ^2, ν) real valued Lévy process.

Let us assume that $\mathbb{E}(e^{-\alpha X_1}) < \infty$, for $\alpha \in [-\epsilon, , \epsilon]$. This implies that X has finite moments of all orders.

A particular case

Let W be a Brownian motion and Z be a ν-compound Poisson process independent of W of the form $Z_t = \sum_{n=1}^{N_t} Y_n$. Let

$$dS_t = S_{t^-}\left(\mu dt + \sigma dW_t + dZ_t\right), \tag{0.1}$$

where μ and σ are constants. The process $(S_t e^{-rt}, t \geq 0)$ is a martingale if and only if $E(|Y_1|) < \infty$ and $\mu + \lambda E(Y_1) = r$.
If $Y_1 \geq -1$ a.s., the process S can be written in an exponential form as

$$S_t = S_0 e^{X_t},\ X_t = bt + \sigma W_t + V_t$$

where $b = \mu - \frac{1}{2}\sigma^2$, V is the $(\lambda, \widetilde{F})$-compound Poisson process

$$V_t = \sum_{n=1}^{N_t} \ln(1 + Y_n) = \sum_{n=1}^{N_t} U_n$$

with $\widetilde{F}(u) = F(e^u - 1)$.

Option pricing with Esscher Transform

Let $S_t = S_0 e^{rt+X_t}$ where X is a Lévy process under the historical probability $\mathbb{P}$. Assume that $\Psi(\alpha) = \mathbb{E}(e^{\alpha X_1}) < \infty$ on some open interval (a, b) with $b - a > 1$ and that there exists a real number θ such that $\Psi(\theta) = \Psi(\theta + 1)$.

The process $e^{-rt}S_t = S_0 e^{X_t}$ is a martingale under the probability $\mathbb{Q}$ defined as $\mathbb{Q} = Z_t \mathbb{P}$ where $Z_t = \dfrac{e^{\theta X_t}}{\Psi(\theta)}$

Hence, the value of a contingent claim $h(S_T)$ can be obtained, assuming that the emm chosen by the market is $\mathbb{Q}$ as

$$V_t = e^{-r(T-t)}\mathbb{E}_Q(h(S_T)|\mathcal{F}_t) = e^{-r(T-t)}\frac{1}{\Psi(\theta)}\mathbb{E}_P(h(ye^{r(T-t)+X_{T-t}}e^{\theta X_{T-t}})\big|_{y=S_t}$$

A Differential Equation for Option Pricing

Let $S_t = \exp(rt + X_t)$ where X is a (m, σ^2, ν) Lévy process with $\int e^{2x}\nu(dx) < \infty$. Then

$$dS_t = rS_t dt + \sigma S_t dW_t + \int (e^x - 1)S_{t-}(\mathbf{N}(dt, dx) - dt\nu(dx))$$

$$\partial_t C(t, x) + rx\partial_x C + \frac{1}{2}\sigma^2 \partial_{xx} C + \int \nu(dy)(C(t, xe^y) - C(t, x) - x(e^y - 1)\partial_x C) = 0$$

and a terminal condition $C(T, x) = h(x)$

Put-call Symmetry

Let us study a financial market with a riskless asset with constant interest rate r, and a price process (a currency) $S_t = S_0 e^{X_t}$ where X is a Q-Lévy process such that $(Z_t = e^{-(r-\delta)t} S_t / S_0, t \geq 0)$ is a Q-strictly positive martingale with initial value equal to 1. The Q-characteristic triple (m, σ^2, ν) of X is such that

$$m = r - \delta - \sigma^2/2 - \int (e^y - 1 - y \mathbb{1}_{\{|y| \leq 1\}}) \nu(dy) \,.$$

Then,

$$\begin{aligned} E_Q(e^{-rT}(S_T - K)^+) &= E_Q(e^{-\delta T} Z_T (S_0 - K S_0/S_T)^+) \\ &= E_{\widehat{Q}}(e^{-\delta T}(S_0 - K S_0/S_T)^+) \end{aligned}$$

with $\widehat{Q}|_{\mathcal{F}_t} = Z_t Q|_{\mathcal{F}_t}$. The process X is a $\widehat{Q}$-Lévy process, with characteristic exponent $\Psi(\lambda+1) - \Psi(1)$. The process $S_0/S_t = e^{-X_t}$ is the exponential of the Lévy process, $Y = -X$ which is the dual of the Lévy process X and the characteristic exponent of Y is $\widetilde{\Psi}(\lambda) = \Psi(1-\lambda) - \Psi(1)$. Hence, the following symmetry between call and put prices holds:

$$C_E(S_0, K, r, \delta, T, \Psi) = P_E(K, S_0, \delta, r, T, \widetilde{\Psi})$$

Subordinators

A Lévy process which takes values in $[0, \infty[$ (i.e. with increasing paths) is a subordinator.

$$X_t = bt + \int_0^t \int x\mathbf{N}(dx, ds)$$

In this case, the parameters in the Lévy-Khintchine decomposition are $m \geq 0, \sigma = 0$ and the Lévy measure ν is a measure on $]0, \infty[$ with $\displaystyle\int_{]0,\infty[} (1 \wedge x)\nu(dx) < \infty$.

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Let Z be a subordinator and X an independent Lévy process. Then, $\widetilde{X}_t = X_{Z_t}$ is a Lévy process, called subordinated Lévy process.

Compound Poisson process. A compound Poisson process with $Y_k \geq 0$ is a subordinator.

Gamma process. The Gamma process $G(t;\gamma)$ is a subordinator which satisfies

$$G(t+h;\gamma) - G(t;\gamma) \stackrel{law}{=} \Gamma(h;\gamma)\,.$$

Here $\Gamma(h;\gamma)$ follows the Gamma law. The Gamma process is an increasing Lévy process, hence a subordinator, with one sided Lévy measure

$$\frac{1}{x}\exp(-\frac{x}{\gamma})\mathbb{1}_{x>0}\,.$$

Hitting times Let W be a BM, and

$$T_r = \inf\{t \geq 0 \,:\, W_t \geq r\}\,.$$

The process $(T_r, r \geq 0)$ is a stable $(1/2)$ subordinator, its Lévy measure is $\dfrac{1}{\sqrt{2\pi}\,x^{3/2}}\mathbb{1}_{x>0}dx$. Let B be a BM independent of W. The process B_{T_t} is a Cauchy process, its Lévy measure is $dx/(\pi x^2)$.

Changes of Lévy characteristics under subordination

Let X be a (a^X, A^X, ν^X) Lévy process and Z be a subordinator with drift β and Lévy measure ν^Z, independent of X.The process $\widetilde{X}_t = X_{Z_t}$ is a Lévy process with characteristic exponent

$$\Phi(u) = i(\widetilde{a}\cdot u) + \frac{1}{2}\widetilde{A}(u) - \int (e^{i(u\cdot x)} - 1 - i(u\cdot x)\mathbb{1}_{|x|\le 1})\widetilde{\nu}(dx)$$

with

$$\begin{aligned}
\widetilde{a} &= \beta a^X + \int \nu^Z(ds)\mathbb{1}_{|x|\le 1} x\mathbb{P}(X_s \in dx) \\
\widetilde{A} &= \beta A^X \\
\widetilde{\nu}(dx) &= \beta \nu^X dx + \int \nu^Z(ds)\mathbb{P}(X_s \in dx)\,.
\end{aligned}$$

"Vous leur conseillerez donc de faire le calcul. Elles [les grandes personnes] adorent les chiffres: ça leur plaira. Mais ne perdez pas votre temps à ce pensum. C'est inutile. Vous avez confiance en moi."

Le petit prince, A. de St Exupéry. Gallimard. 1946. p. 59.

Variance-Gamma Model

The **variance Gamma process** is a Lévy process where X_t has a Variance Gamma law VG(σ, ν, θ). Its characteristic function is

$$\mathbb{E}(\exp(iuX_t)) = \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-t/\nu}.$$

The Variance Gamma process can be characterized as a **time changed BM** with drift as follows: let W be a BM, $\gamma(t)$ a $G(1/\nu, 1/\nu)$ process. Then

$$X_t = \theta\gamma(t) + \sigma W_{\gamma(t)}$$

is a VG(σ, ν, θ) process.

The variance Gamma process is a finite variation process. Hence it is the difference of two increasing processes. Madan et al. showed that it is the difference of two independent Gamma processes

$$X_t = G(t; \mu_1, \gamma_1) - G(t; \mu_2, \gamma_2)\,.$$

Indeed, the characteristic function can be factorized

$$\mathbb{E}(\exp(iuX_t)) = \left(1 - \frac{iu}{\nu_1}\right)^{-t/\gamma} \left(1 + \frac{iu}{\nu_2}\right)^{-t/\gamma}$$

with

$$\begin{aligned} \nu_1^{-1} &= \frac{1}{2}\left(\theta\nu + \sqrt{\theta^2\nu^2 + 2\nu\sigma^2}\right) \\ \nu_2^{-1} &= \frac{1}{2}\left(\theta\nu - \sqrt{\theta^2\nu^2 + 2\nu\sigma^2}\right) \end{aligned}$$

The Lévy density of X is

$$\begin{aligned} &\frac{1}{\gamma}\,\frac{1}{|x|}\exp(-\nu_1|x|) \qquad \text{for } x<0 \\ &\frac{1}{\gamma}\,\frac{1}{x}\exp(-\nu_2 x) \qquad \text{for } x>0\,. \end{aligned}$$

The density of X_1 is

$$\frac{2e^{\frac{\theta x}{\sigma^2}}}{\gamma^{1/\gamma}\sqrt{2\pi}\sigma\Gamma(1/2)}\left(\frac{x^2}{\theta^2+2\sigma^2/\gamma}\right)^{\frac{1}{2\gamma}-\frac{1}{4}} K_{\frac{1}{\gamma}-\frac{1}{2}}\big(\frac{1}{\sigma^2}\sqrt{x^2(\theta^2+2\sigma^2/\gamma)}\big)$$

where K_α is the modified Bessel function.

Stock prices driven by a Variance-Gamma process have dynamics

$$S_t = S_0 \exp\left(rt + X(t;\sigma,\nu,\theta) + \frac{t}{\nu}\ln(1-\theta\nu-\frac{\sigma^2\nu}{2})\right)$$

From $\mathbb{E}(e^{X_t}) = \exp\left(-\frac{t}{\nu}\ln(1-\theta\nu-\frac{\sigma^2\nu}{2})\right)$, we get that $S_t e^{-rt}$ is a martingale. The parameters ν and θ give control on skewness and kurtosis.

The CGMY model, introduced by Carr et al. is an extension of the Variance-Gamma model. The Lévy density is

$$\begin{cases} \dfrac{C}{x^{Y+1}} e^{-Mx} & x > 0 \\ \dfrac{C}{|x|^{Y+1}} e^{Gx} & x < 0 \end{cases}$$

with $C > 0, M \geq 0, G \geq 0$ and $Y < 2, Y \notin \mathbb{Z}$.
If $Y < 0$, there is a finite number of jumps in any finite interval, if not, the process has infinite activity. If $Y \in [1,2[$, the process is with infinite variation. This process is also called KoBol.

Double Exponential Model

The Model A particular Lévy model is the double exponential jumps model, introduced by Kou and Kou and Wang. In this model

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\,,$$

where W is a Brownian motion independent of N and $\sum_{i=1}^{N_t} Y_i$ is a compound Poisson process. The r.v's Y_i are i.i.d., independent of N and W and the density of the law of Y_1 is

$$f(x) = p\eta_1 e^{-\eta_1 x} 1\!\!1_{\{x>0\}} + (1-p)\eta_2 e^{\eta_2 x} 1\!\!1_{\{x<0\}}\,.$$

The Lévy measure of X is $\nu(dx) = \lambda f(x)dx$.

Here, η_i are positive real numbers, and $p \in [0,1]$. With probability p (resp. $(1-p)$), the jump size is positive (resp. negative) with exponential law with parameter η_1 (resp η_2).

It is easy to prove that

$$\mathbb{E}(Y_1) = \frac{p}{\eta_1} - \frac{1-p}{\eta_2}, \text{ var } (Y_1) = \frac{p}{\eta_1^2} + \frac{1-p}{\eta_2^2} + p(1-p)\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)^2$$

and that, for $\eta_1 > 1$, $\mathbb{E}(e^{Y_1}) = p\frac{\eta_1}{\eta_1 - 1} + (1-p)\frac{\eta_2}{1+\eta_2}$. Moreover

$$\mathbb{E}(e^{iuX_t}) = \exp\left(t\left\{-\frac{1}{2}\sigma^2 u^2 + ibu + \lambda\left(\frac{p\eta_1}{\eta_1 - iu} + \frac{(1-p)\eta_2}{\eta_2 + iu} - 1\right)\right\}\right),$$

where $b = \mu + \lambda \mathbb{E}(Y_1 \mathbb{1}_{|Y_1| \leq 1}) =$
$\mu + \lambda p\left(\frac{1-e^{-\eta_1}}{\eta_1} - e^{-\eta_1}\right) - \lambda(1-p)\left(\frac{1-e^{-\eta_2}}{\eta_2} - e^{-\eta_2}\right)$. The Laplace exponent of X, i.e., the function Ψ such that $\mathbb{E}(e^{\beta X_t}) = \exp(\Psi(\beta)t)$ is defined for $-\eta_2 < \beta < \eta_1$ as

$$\Psi(\beta) = \beta\mu + \frac{1}{2}\beta^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - \beta} + \frac{(1-p)\eta_2}{\beta + \eta_2} - 1\right).$$

Change of probability Let $S_t = S_0 e^{rt+X_t}$ where $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$. Then, setting $V_i = e^{Y_i}$, using an Escher transform, the process $S_t e^{-rt}$ will be a $\mathbb{Q}$ martingale with $\mathbb{Q}|_{\mathcal{F}_t} = L_t d\mathbb{P}|_{\mathcal{F}_t}$ and $L_t = \frac{e^{\alpha X_t}}{\mathbb{E}(e^{\alpha X_t})}$, for α such that $\Psi(\alpha) = \Psi(\alpha+1)$. Under $\mathbb{Q}$, the Lévy measure of X is $\widehat{\nu}(dx) = e^{\alpha x}\nu(dx) = e^{\alpha x}\lambda f(x)dx = \widehat{\lambda}\widehat{f}(x)dx$ where, after some standard computations

$$
\begin{aligned}
\widehat{f}(x) &= \left(\widehat{p}\,\widehat{\eta}_1 e^{-\widehat{\eta}_1 x}\mathbb{1}_{\{x>0\}} + (1-\widehat{p})\widehat{\eta}_2 e^{\widehat{\eta}_2 x}\mathbb{1}_{\{x<0\}}\right) . \\
\widehat{\eta}_1 &= \eta_1 - \alpha\,, \qquad \widehat{\eta}_2 = \eta_2 + \alpha \\
\widehat{\lambda} &= \lambda\left(\frac{p\eta_1}{\eta_1-\alpha} + \frac{(1-p)\eta_2}{\eta_2+\alpha}\right) \\
\widehat{p} &= p\eta_1 \frac{\eta_2+\alpha}{\alpha p\eta_1 + \eta_2(\eta_1 - \alpha + \alpha p\eta_1)}
\end{aligned}
$$

In particular, the process X is a double exponential process under $\mathbb{Q}$.

Hitting times For any $x > 0$

$$\mathbb{P}(\tau_b \leq t,\, X_{\tau_b} - b \geq x) = e^{-\eta_1 x}\, \mathbb{P}(\tau_b \leq t, X_{\tau_b} - b \geq 0)$$

PROOF:

The infinitesimal generator of X is

$$\mathcal{L}f = \frac{1}{2}\sigma^2 \partial_{xx} f + \mu \partial_x f + \lambda \int_{I\!R} (f(x+y) - f(x))\nu(dx)$$

Let $T_x = \inf\{t \,:\, X_t \geq x\}$. Then Kou and Wang establish that, for $r > 0$ and $x > 0$,

$$\begin{aligned}
\mathbb{E}(e^{-rT_x}) &= \frac{\eta_1 - \beta_1}{\eta_1} \frac{\beta_2}{\beta_2 - \beta_1} e^{-x\beta_1} + \frac{\beta_2 - \eta_1}{\eta_1} \frac{\beta_1}{\beta_2 - \beta_1} e^{-x\beta_2} \\
\mathbb{E}(e^{-rT_x} 1\!\!1_{X_{T_x} - x > y}) &= e^{\eta_1 y} \frac{\eta_1 - \beta_1}{\eta_1} \frac{\beta_2 - \eta_1}{\beta_2 - \beta_1} \left(e^{-x\beta_1} - e^{-x\beta_2}\right) \\
\mathbb{E}(e^{\theta X_{T_x} - rT_x}) &= e^{\theta x} \left(\frac{\eta_1 - \beta_1}{\beta_2 - \beta_1} \frac{\beta_2 - \theta}{\eta_1 - \theta} e^{-x\beta_1} + (\frac{\beta_2 - \eta_1}{\beta_2 - \beta_1} \frac{\beta_1 - \theta}{\eta_1 - \theta} e^{-x\beta_2} \right)
\end{aligned}$$

where $0 < \beta_1 < \eta_1 < \beta_2$ are roots of $G(\beta) = r$. The method is based on finding an explicit solution of $\mathcal{L}u = ru$ where $\mathcal{L}$ is the infinitesimal generator of the process X.

Thank you for your attention.