Open issues in equity derivatives modelling

Lorenzo Bergomi

Equity Derivatives Quantitative Research
Société Générale
lorenzo.bergomi@sgcib.com
Talk Outline

- Equity derivatives at SG
- A brief history of equity derivative products
  - Prehistory  – 1997
  - History 1997 – 2003
    - Modern times 2003 –
- Modelling issues, algorithmic issues
- Risk measurement and management
- Conclusion
Equity derivatives at SG

- SG regarded by industry participants as No 1 in equity derivatives
A brief history of equity derivative products

Prehistory – 1997

Products
- Barrier options / Digitals
- Max / Min options
  \( \left( \max\left( S_t - K \right) \right)^+ \)
- Asian options
  \( \left( \frac{1}{N} \sum s_{t_i} - K \right)^+ \)
- Basket options
  \( \left( \frac{1}{N} \sum s_{iT}^T - K \right)^+ \)
- Volatility swaps
  \( \frac{1}{T} \sum \ln \left( \frac{S_k}{S_{k-1}} \right)^2 - \hat{\sigma}_K \)
- Simple cliquets
  \( \left( \frac{S_{i+1}}{S_{i_1}} - K \right)^+ \)

Risks
- Skew: level / dynamics (little)
- same
- Smile
- Correlation (level)
- Smile, VolOfVol
- Forward smile

Models / algos
- Black Scholes / local vol
- PDE / straight Monte Carlo
A brief history of equity derivative products

Capital-guaranteed products distributed by retail networks

- Everest 1997 5 years / 12 stocks
  \[ 100\% + \min \left( \frac{S^j_T}{S^j_0} \right) \]

- Emerald 2004 10 years / 20 stocks
  Every year, the stock whose performance since \( t = 0 \) is the largest gets frozen and removed from the basket, and its level is floored at 200% of its initial value.
  \[ 100\% + \text{maximum performance of yearly basket values since } t = 0, \text{floored at 0.} \]

... and many, many, many, other variations

\[ \Rightarrow \text{trying to find closed-form formulas for specific exotic payoffs now irrelevant and useless} \]
A brief history of equity derivative products

**History - 2  1997 – 2005**

- **Variance Swaps**  3 months \(\Rightarrow\) 5 years
  - Pays realized variance – usually measured using daily returns
    \[
    \sum_k \ln \left( \frac{S_k}{S_{k-1}} \right)^2 - \sigma^2 T
    \]
  - stocks / indices

- **Napoleon**  5 years / 1 index
  - Every year, pays coupon reduced by worst of 12 monthly performances of the index.
    \[
    \left( C + \min_k \left( \frac{S_k}{S_{k-1}} \right)^+ \right)
    \]

- **Accumulator**  3 years / 1 index
  - At maturity pays the sum – if it is positive – of the monthly performances, capped and floored.
    \[
    \left( \sum_k \min \left( \frac{S_k}{S_{k-1}} - 1,1\% ,-1\% \right) \right)^+
    \]
Modern times

- **Corridor variance swaps**
  Daily variance only counted when underlying is inside given interval
  \[
  \sum_{k} l_{S_k \epsilon [L,H]} \left( \ln \left( \frac{S_k}{S_{k-1}} \right)^2 - \sigma^2 \Delta t \right) \]
  \( [L,H], [L, +\infty), [0,H] \)

- **Correlation swaps**
  Pays realized correlation over 3 years by stocks of an index

- **Gap notes**
  Maturity = 1 year, a series of daily puts on daily returns of an index
  with strikes 85%, 90%

- **Options on realized variance**
  On indices, maturities: 3 months to 2 years
  \[
  \left[ \frac{1}{T} \sum_{i=1}^{T} \ln \left( \frac{S_i}{S_{i-1}} \right)^2 - \hat{\sigma}_k^2 \right]^+ \]

- **Timer options**
  Vanilla payoff, paid when realized variance \( Q_t \) reaches set level:
  \[
  Q_t = \sum_{j=1}^{t} \ln \left( \frac{S_{i+1}}{S_i} \right)^2 \]

- **Hybrids**
  Equities / Rates / Forex / Commodities  Arbitrary payoffs
Modelling issues – 1

• Why not just delta-hedge?
  • Variance of residual P&L too large ⇒ use other options
  ⇒ Options are hedged with options

• Once we start using options as hedging instruments
  • Less sensitivity to historical parameters, more sensitivity to implied parameters
  ⇒ Model the dynamics of implied parameters

• Example of simple cliquet
  \[
  \frac{S_{T_2}}{S_{T_1}} - 1 \Rightarrow \int_{T_1}^{T_2} \hat{\sigma}_{12} P(\hat{\sigma}_{12}, r, \cdots)
  \]

\[\begin{align*}
\text{Smile 3 mois K = 95 - K= 105}
\end{align*}\]
• **How should calibration be done? Do we really need to calibrate?**
  • *Not compulsory:* charge a hedging cost. We hedge parameter $p$ by trading instrument $O$ so that sensitivity to $p$ vanishes:

$$ \frac{dP}{dp} = \lambda \frac{dO}{dp} $$

• Model price $P$ is adjusted so as to include hedging cost:

$$ \text{Price} = P(\hat{p}) + \lambda (O(p_{\text{Market}}) - O(\hat{p})) $$

$$ \approx P(p = p_{\text{Market}}) $$

• **Then what is the point in calibrating?**
  • Ensures price factors in hedging costs incurred at $t = 0$ – *not future costs*!

$\Rightarrow$ **Necessary to calibrate model on relevant set of hedging instruments**

$\Rightarrow$ **Useless if one is unable to specify how to hedge the exotic with the hedge instruments**
Modelling issues – 3

Volatility risk – models

- « Old models »
  - Local volatility
  - Heston
  - SABR
  - Models based on process of instantaneous variance:
    \[ dS = \ldots dt + \sqrt{V} dW_s \]
  - Jump / Lévy
    \[ dV = \ldots dt + \ldots dW_v \]

Challenge: Build models that give control on joint dynamics of implied volatilities and spot:

- First step: model dynamics of curve of forward variances
- Next step: model dynamics of the implied volatility surface
  - Direct modelling of dynamics of implied volatilities is a dead end
  - Low-dimensional Markov representation desirable
  - How much freedom are we allowed?
Hybrids

- Equities
- Interest rates
- Forex
- Commodities

Hybrid models are not built by simply gluing together models for each asset class

- Passive hybrids: payoff involves one asset class only
  - Long-dated equity, Forex options
  - Credit / Equity: convertible bonds

- Active hybrids: payoff involves all asset classes
  - Require state-of-the-art models for each asset class
  - Even local vol calibration for equity smiles not easy when interest rates are stochastic
Modelling issues – 5

- Correlation – how do we put together correlation matrices?
  - How do we build the large correlation matrices needed in hybrid modelling?
  - Simpler question: imagine a 1-factor stoch. vol model and a payoff involving 2 securities
    - How do we set the cross-correlations?
  - Even simpler question – how do we measure correlations?
  - Example of European / Japanese stocks – no overlap

```
Europe
  o C o C o C o C
Japan
  o C o C o C o C
```

- Correlation – how do we measure correlation risk?
- Correlation – how to model correlation smile?
Algorithmic issues

- Monte Carlo

  - How can we speed up pricing?
  - Quasi-random numbers
  - Discretization of SDEs?

- Callable / putable options

- Computing sensitivities to
  - Initial conditions
  - Parameters of dynamics (volatilities / correlations, etc..)
Conclusion

- These are exciting times for doing quantitative finance
  - Lots of new instruments / product / algorithmic issues
  - Rich mathematical toolbox from which to pick