Exotic Risks Mitigation

Christophe Michel

Head of Interest Rates and Hybrid Quantitative Research

Agenda

Interest Rate Quantitative Research

Some Facts

Open Questions

Proposed Way

Quantitative Research : Missions

50%

Innovative Quantitative Developments (Models, Methods, Tools)
Complete and Optimize the range of models for each asset class and develop models for joined risks and multi-assets dynamics. Develop new numerical method to explore more models

15%

Support (Trading, Structuring, IT, Risk Department)
Ensure that all users are aware and trained on all quant developments. Understand local needs to drive future developments. Help IT to plug libraries.

15%

Risk Studies (for Management, Trading, Quants)
Produce complete documentation on case study available for all traders. Build, share and spread quantitative IRD culture. Guide future developments. Speed up validation process.

7%

Relations with Academic World
Track innovation. Identify and select good profiles.

13%

Client Advisory Quantitative Research
Provide tools to costumers to quantify exposition and opportunities, to understand products Behaviour, to simulate client's financial problematic. Strengthen relations with clients.

Quantitative Research : Yield Curve Models

Yield curve models are used in several contexts

- Build the yield curve itself

  Use markets prices of vanilla instruments (bonds, swaps) to build the ideal concept of yield curve

- Model instantaneous yield curve dynamic

  Capture the joint behavior of rates for macro hedging or arbitrage purpose

- Evaluate standard and exotic derivatives

  Explicit and model the dynamic of sources of market risks embedded in exotic products

- Evaluate a global future exposure

  Quantify future portfolio MtM distribution

- Build risk indicators in simulation tools

  Value-at-Risk, Cost-at-Risk, …

Quantitative Research : Pricing Exotic Products

The main focus is on pricing exotic products

- For pricing purpose, the target is to use a model which capture all sources of risk embedded in the product.
  - Some risks are liquid : the model have to reflect them (e.g. rates, volatility level, smile)
  - Some risks are illiquid (exotic) : the model have to state them explicitly and conservatively if possible (e.g. correlation, term structure of volatility, smile)
  - Some are difficult to identify : the model state them implicitly (e.g. smile dynamic, correlation dynamic)
- Tractability is key
  - Number of factor necessary to specify the future
  - Specification : avoid numerical black-box and explicit good leverage
  - Exact simulation
  - Explicit formulas for liquid products
  - From factors to zero-coupon bonds

Models are specialized/specified by product categories

Some Facts : Modeling Forward Swap Rates Dynamic

- The most standard IR derivatives are swaps and swaptions
- If we consider forward swap rates as underlying, each standard derivative is associated with a single underlying and the IR problematic becomes very similar to the one of other asset classes and all models

$$
\begin{aligned}
E^Q\left[e^{-\int_0^{T_0} r_s ds}\left(\sum_{n=1}^{N}\delta_n\left(Libor\left(T_0,T_{n-1}\right)-K\right)B\left(T_0,T_n\right)\right)_+\right] &= E^Q\left[e^{-\int_0^{T_0} r_s ds}\left(B\left(T_0,T_0\right)-B\left(T_0,T_N\right)-K\sum_{n=1}^{N}\delta_n B\left(T_0,T_n\right)\right)_+\right] \\
&= E^Q\left[e^{-\int_0^{T_0} r_s ds}\left(\frac{B\left(T_0,T_0\right)-B\left(T_0,T_N\right)}{\sum_{n=1}^{N}\delta_n B\left(T_0,T_n\right)}-K\right)_+\sum_{n=1}^{N}\delta_n B\left(T_0,T_n\right)\right] \\
&= \sum_{n=1}^{N}\delta_n B\left(0,T_n\right)E^{Q^{LVL}}\left[\left(S_{T_0}-K\right)_+\right]
\end{aligned}
$$

- The models are in fact simpler since forward swaps are martingale under their LVL probability and options on forward swap are not liquid
- In the other hand, the link with yield curve dynamic is just not questioned in this approach

Some Facts : Use Forward Swap Model

- CMS options are fairly liquid

  A CMS option pays max(S(t)-K;0)

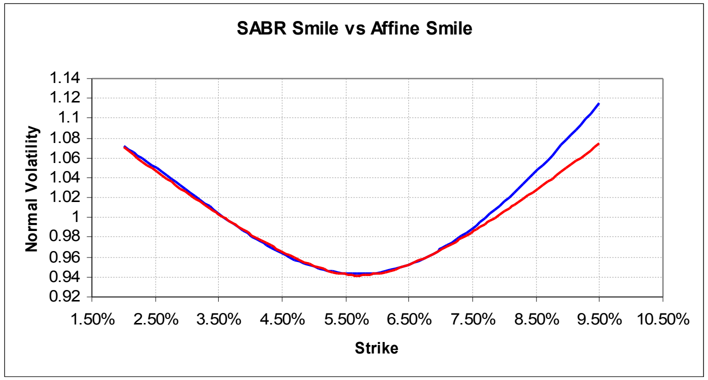
$$E^Q\left[e^{-\int_0^{T_p} r_s ds}\left(S_{T_0}-K\right)_+\right] = \sum_{n=1}^{N} \delta_n B(0,T_n) E^{Q^{LVL}}\left[\left(S_{T_0}-K\right)_+ \frac{B(T_0,T_p)}{\sum_{n=1}^{N} \delta_n B(T_0,T_n)}\right]$$

$$\cong \sum_{n=1}^{N} \delta_n B(0,T_n) E^{Q^{LVL}}\left[\left(S_{T_0}-K\right)_+ g\left(S_{T_0}\right)\right]$$

  The replication formula gives:

$$E_t\left[F\left(S_T\right)\right] = F(H) + \int_H^{+\infty} F''(x) \times E_t\left[\left(S_T - x\right)_+\right]dx + \int_{-\infty}^{H} F''(x) \times E_t\left[\left(x - S_T\right)_+\right]dx$$

- Hence the smile extrapolation is fairly liquid
- Exotic prices bring liquidity but the market is often one way
- The best model may just be the most conservative



Some Facts : CMS Spread Options

- CMS spread options are very (too) popular and became fairly (but not enough) liquid
  - A CMS Spread option pays max(a1*S1(t)-a2*S2(t)-K;0)

$$E^Q\left[e^{-\int_0^{T_p} r_s ds}\left(a_1 \times S_1(T_0) - a_2 \times S_2(T_0) - K\right)_+\right]$$

  - Consistency with CMS options leads to inconsistency (1 factor assumption)
  - There is a link between CMS spread options and term structure of volatility:

$$S_{1,N}(T_0) = \frac{B(T_0,T_0) - B(T_0,T_N)}{\sum_{i=1}^{N} \delta_i B(T_0,T_i)} = \frac{B(T_0,T_0) - B(T_0,T_n)}{\sum_{i=1}^{n} \delta_i B(T_0,T_i)} \frac{\sum_{i=1}^{n} \delta_i B(T_0,T_i)}{\sum_{i=1}^{N} \delta_i B(T_0,T_i)} + \frac{B(T_0,T_n) - B(T_0,T_N)}{\sum_{i=n+1}^{N} \delta_i B(T_0,T_i)} \frac{\sum_{i=n+1}^{N} \delta_i B(T_0,T_i)}{\sum_{i=1}^{N} \delta_i B(T_0,T_i)}$$

$$= S_{1,n}(t)\frac{LVL_{1,n}(T_0)}{LVL_{1,N}(T_0)} + S_{n+1,N}(t)\frac{LVL_{n+1,N}(T_0)}{LVL_{1,N}(T_0)}$$

  - Hence

$$E^Q\left[e^{-\int_0^{T_0} r_s ds}\left(S_{n+1,N}(T_0) - K\right)_+ LVL_{n+1,N}(T_0)\right] = LVL_{n+1,N}(0)E^{Q^{LVLn+1,N}}\left[\left(S_{n+1,N}(T_0) - K\right)_+\right]$$

$$\cong LVL_{n+1,N}(0)E^{Q^{LVLn+1,N}}\left[\left(\frac{LVL_{1,N}(0)}{LVL_{n+1,N}(0)} S_{1,N}(T_0) - \frac{LVL_{1,n}(0)}{LVL_{n+1,N}(0)} S_{1,n}(T_0) - K\right)_+\right]$$

  - Volatility term structure is revealed by these CMS spread options

First Conclusions

1. The forward swap approach allows to focus on the main risks of first generation derivatives

3. It allows to treat interest rates like any other underling (using the same framework)

5. The market of first generation derivatives is very developed which brings information on exotic risks

7. This information may result from a one way exotic market which make it distorted

9. The forward swap approach ignore the yield curve which leads to inconsistencies in joint behaviors or probability changes.

11. Don’t forget that the forward swap isn’t traded itself but modeled from traded assets

Some Facts : Yield Curve Models Approach

- Exotics need yield curve models and liquidity leads to sophistication

$$B(t,T) = E^{Q}\left[e^{-\int_0^T r_s ds} \right]$$

Start with Vasicek (equilibrium)

$$dr_t = \lambda(\theta - r_t)dt + \sigma dW_t$$

Observe the yield curve

$$dr_t = \lambda(\theta_t - r_t)dt + \sigma dW_t$$

Observe Caps and Floors…

$$dr_t = \lambda(\theta_t - r_t)dt + \sigma_t dW_t$$

… and Swaptions

$$dr_t = \lambda_t(\theta_t - r_t)dt + \sigma_t dW_t$$

$$B(t,T) = e^{-\int_0^T f(t,u)du}$$

$$df(t,T) = \left(\sigma_t^2 e^{-\int_t^T \lambda_s ds} \int_t^T e^{-\int_t^u \lambda_s ds} du \right) dt + \sigma_t e^{-\int_t^T \lambda_s ds} dW_t$$

Spread Options, …

$$df(t,T) = \left(\vec{v}(t,T) \cdot \int_t^T \vec{v}(t,u)du \right) dt + \vec{v}(t,T) \cdot d\vec{W}_t$$

Smile
Exotics

$$\begin{cases} df(t,T) = \left(\vec{v}(t,T,f) \cdot \int_t^T \vec{v}(t,u,f)du \right) dt + \sigma_t \vec{v}(t,T,f) \cdot d\vec{W}_t \\ d\sigma_t^2 = \lambda(\theta - \sigma_t^2)dt + \sigma_t \vec{\eta} \cdot d\vec{W} \end{cases}$$

- Sophistication leads to withdraw stationarity

Some Facts : Pricing a Callable Swap

- A callable swap is a swap plus the right to call the swap at each fixing date of the floating flows.
- Its price depends upon the joint dynamic of embedded forward swaps and the forward swap approach cannot straightforwardly be used.
- One could choose a pricing model for its econometrical properties but liquid market prices give arbitrage constraints. For instance:

  callable swap price > max(embedded swaption's price)

  callable swap price < associated cap price
- The best model isn't necessarily the most sophisticated
  - Since the joint dynamic of forward swaps isn't fixed by liquid market products, it is worthless to be fully calibrated on all of them.
  - Since the main risk of such a product lies on the term structure of volatility, it is useful to explicit the position of the model on that risk
  - Most often, the more sophisticated the model is, the less robust the calibration process becomes.

Some Facts : Around the Calibration Set

- Calibration Set allows to take into account arbitrage constraints on the exotic price BUT exotic product isn't a fixed linear combination of liquid products of the calibration set

- Calibration Set imply a hedging strategy BUT it may be arbitrary and the efficiency of the implicit hedging strategy is to be tested

- The dynamic of Calibration Set products is defined by the model used BUT it may not be disconnected with actual dynamic

- The Calibration Set have to be perfectly fitted in order to avoid numerical noise in the explanation of the exotic product's price when liquid product's prices move

- Calibration Set is defined for an exotic product BUT it may be complex to attain or inconsistent with the model used

Some Facts : Capture Swap Models' Sophistication

Some models are built based on the Calibration Set

- **Brace-Gatarek-Musiela:**

  Introduce Forward Libor $L_k(T_i) = \frac{1}{\delta_k}\left(\frac{B(T_i,T_k)}{B(T_i,T_{k+1})} - 1\right)$

  Deduce ZC prices $B(T_i,T_j) = \prod_{k=i}^{j-1}\left(1+\delta_k L_k(T_i)\right)^{-1}$

  Use Forward Libor Model $dL_k(t) = \sigma_k(\cdot)^* dW_t^{Q^{T_{k+1}}}$ (possible extension to fwd swap)

  In practice, there are a lot of numerical complexity in pricing and calibration

- **Hunt-Kennedy-Pelsser**

  Introduce any IR asset : N(t) and note Q^N the associated probability

  Introduce any markovian factor under Q^N: $dx_t = \mu(t,x_t)dt + \sigma(t,x_t)dW_t^{Q^N}$

  Assume the functional relation : $N_t = f(t,x_t)$

  Deduce ZC prices $B(t,T) = N_t E_t^{Q_N}\left[\frac{1}{N_T}\right] = f(t,x_t)E_t^{Q_N}\left[\frac{1}{f(T,x_T)}\right] = g(t,T,x_t)$

  In practice, extension to more factors is tricky and path-dependent products are numerically complex to price

But even hidden, exotic risks remain

Some Facts : Explicit Exotic Risks

- Risk diversification consists in multiplying sources of risks in order to take advantage of exotic risks and reduce the risk on a global portfolio.
- Risk Mitigation consists in minimizing the impact of Exotic parameter's
- Exotic parameters are the parameters of the model which are not calibrated with the Calibration Set and their level define the price of the exotic product within the constraints of the Calibration Set and the possibilities let by the model
- A measure the impact of exotic parameter is a risk indicator which allows to define a dynamic pricing policy for each product depending on the positive or negative impact of the product on the global position.
- It is of course necessary to mitigate risk based on (at least) consistent risk indicators.
- It is crucial to keep in mind that in most cases risk indicators don't reflect all sources of risk and remaining risks have to be studied carefully.

Some Facts : Back Testing

- A way to check whether a model is consistent with market prices or not is back-testing.
- Back testing involves putting in place the hedging strategy consistent with the pricing model used (and then with all assumptions made including exotic risks).
- When doing so, one tests all assumptions implied by the pricing model. All assumptions are tested globally which means there is no specific focus on exotic risks.
- It gives information on the consistency between model assumptions and historical market behavior.
- It doesn’t hinder assumptions made about future “behavior changes”.

Other Conclusions

1. To replicate market prices, yield curve models are more and more sophisticated but less and less stationary

3. To price an exotic product one need to choose a model, a consistent calibration set and tractable numerical solutions

5. An exotic product's price will depend upon liquid market prices and (preferably) explicit exotic parameter. These liquid products will guide the hedging strategy

7. Back-Testing is a way to test the pricing/hedging process

9. Exposure to exotic risks is complex to mitigate since exotic products are rarely priced with the same triplet (Model, Calibration Set, Exotic Parameters)

Open Questions

Is it possible to define a model consistent with all market prices?

Is it possible to build a model consistent with all products of the portfolio of an investment bank?

Is it possible to build a model consistent with liquid market prices and the pricing theory?

Proposed Way : Affine Model

❖ Affine models are commonly used in finance because of their tractability

Affine factors are such that $dX_t = (A_t X_t + b_t)dt + \Sigma(X_t)dW_t$ with

$\Sigma(X_t)\Sigma(X_t)^* = V(X_t)$ and $[V(X_t)]_{ij} = \alpha_{ij}(t) + \beta_{ij}(t)^* X_t$

We have the standard property of affine models

$$E_t\left[e^{-\int_t^T \delta_0(t)+\delta_t^* X_s ds + \Delta_t^* X_T}\right] = e^{\Phi(t,T)+\Psi(t,T)^* X_t}$$

With

$$\begin{cases} \dfrac{d\Phi(t,T)}{dt} = -\Psi(t,T)^* b_t - \dfrac{1}{2}\Psi(t,T)^*[\alpha(t)]\Psi(t,T) + \delta_0(t) \\ \dfrac{d\Psi(t,T)}{dt} = -A_t^*\Psi(t,T) - \dfrac{1}{2}\sum_{ij}\Psi_i(t,T)\Psi_j(t,T)\beta_{ij}(t) + \delta_t \end{cases} \qquad \begin{cases} \Phi(T,T) = 0 \\ \Psi(T,T) = \Delta_T \end{cases}$$

And the stationary case

$$\begin{cases} \dfrac{d\Phi(\tau)}{d\tau} = \Psi(\tau)^* b + \dfrac{1}{2}\Psi(\tau)^*[\alpha]\Psi(\tau) - \delta_0 \\ \dfrac{d\Psi(\tau)}{d\tau} = A^*\Psi(\tau) + \dfrac{1}{2}\sum_{ij}\Psi_i(\tau)\Psi_j(\tau)\beta_{ij} - \delta \end{cases} \qquad \begin{cases} \Phi(0) = 0 \\ \Psi(0) = \Delta \end{cases}$$

These equations can be solved either analytically (Riccati) or numerically (Runge-Kutta)

Proposed Way : Affine Yield Curve Model

- An affine yield curve model is based on affine factors and the following assumption :

$$r_t = \delta_0(t) + \delta(t)^* X_t$$

- Hence the zero-coupon Bond is obtained with the affine property

$$B(t,T) = E_t\left[e^{-\int_t^T \delta_0(t)+\delta_t^* X_s ds}\right] = e^{\Phi(t,T)+\Psi(t,T)^* X_t}$$

- A lot of well known models are special cases of this general framework
  - Vasicek
  - Hull-White
  - CIR
  - Gaussian Quadratic
  - LGM with stochastic volatility
  - Wishart
  - …
- Market prices are generally captured by parameters (calibration)

Proposed Way : Admissibility

- Canonical representation (Duffie-Kan, Dai-Singleton)

$$dX_t = \begin{bmatrix} A^{1,1}_{m\times m} & 0_{m\times(N-m)} \\ A^{1,2}_{(N-m)\times m} & A^{2,2}_{(N-m)\times(N-m)} \end{bmatrix} \left(\begin{pmatrix} b^1_{m\times 1} \\ 0_{(N-m)\times 1} \end{pmatrix} - X_t \right) dt + \Sigma \sqrt{S_t}\, dW_t$$

$$[S_t]_{ii} = \alpha_i + \beta_i^* X_t \qquad \beta = \begin{bmatrix} I_{m\times m} & \beta^{1,2}_{m\times(N-m)} \\ 0_{(N-m)\times m} & 0_{(N-m)\times(N-m)} \end{bmatrix} \quad \alpha = \begin{pmatrix} 0_{m\times 1} \\ 1_{(N-m)\times 1} \end{pmatrix}$$

$$1 \le i \le m : \sum_{j=1}^{m} A_{ij} b_j > 0 \quad 1 \le j \ne i \le m : A_{ij} \le 0 \quad \beta_{ij} \ge 0 \quad b_i \ge 0$$

- Wishart (Bru, Gourieroux-Sufana)

$$dY_t = \left(\Omega\,\Omega^* + MY_t + Y_t M^*\right)dt + \sqrt{Y_t}\,[dW_t]\,Q + Q^*\,[dW_t]^*\sqrt{Y_t}$$

$$\Omega\,\Omega^* = \beta \times Q^* Q, \beta > n-1$$

Proposed Way : Remaining Stationary

❖ Let's propose another way to capture market prices

$$dr_t = \lambda\left(\theta - r_t\right)dt + \sigma\, dW_t$$

$$dr_t = \lambda\left(\theta_t - r_t\right)dt + \sigma\, dW_t$$

$$dr_t = \lambda_t\left(\theta_t - r_t\right)dt + \sigma_t dW_t$$

Standard way: use parameter

$$\begin{cases} dr_t = \lambda_r\left(\alpha_r + \theta_t - r_t\right)dt + \sigma_r dW_r(t) \\ d\theta_t = \lambda_\theta\left(\alpha_r - \theta_t\right)dt + \sigma_\theta\, dW_\theta(t) \end{cases}$$

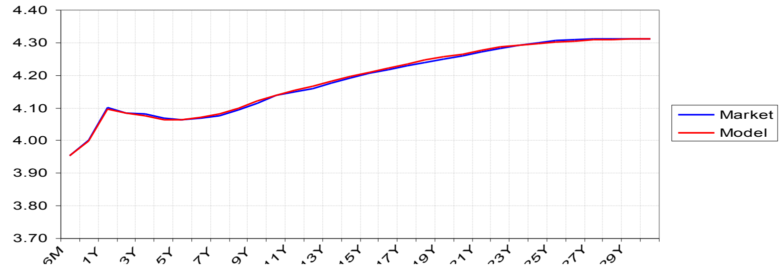
$$r_t = \delta_0 + \delta^* X_t$$

Proposed way: use factors

The idea is to assume that if a market price isn't "consistent enough" with the model then one should add a factor to the model. Doing so, tractability is weaker but readability is better and the model remain stationary

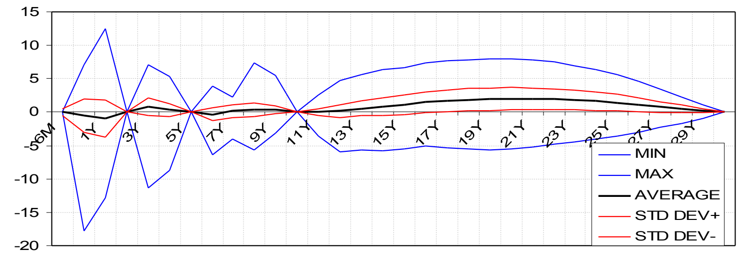
Proposed Way : a Practical Example

- In practice, one can use five liquid market prices to implicitly deduce factors realizations. Choosing for instance 6M, 2Y, 5Y, 10Y and 30Y we obtain :



- Having specified once the model, it is possible to back-test it. For instance we can compute the spread between the actual curve and the theoretical curve. In our specific example we find :

Historical Spread Study



- More generally, once specified, the model can be tested either statistically (law of realized factors vs model with an affine assumption for risk premium) or implicitly (model prices vs market prices, macro hedging efficiency)

Proposed Way : About Risks

- Instead of using market prices to define the model, the affine stationary model uses market prices to reveal factor's level
- Once current value of risk factors is deduced from (few) liquid prices, all prices are explained by these initial values and the dynamic properties of the stationary model
- The target isn't to fit perfectly all market prices but to be reasonably good for all basic products
- Basic products to calibrate factors are:
  - Swaps
  - Caps/Floors/Swaptions
  - Spread Options/Forward swap options
  - …
- Risks to measure are on
  - Market risks based on basic liquid (or illiquid) products : factor realization
  - Exotic/model risks : model definition/level of parameters

Proposed Way : About Numerical Issues

- Basic products can be efficiently evaluated
  - Zero-coupon bonds are directly solutions of EDO hence swaps and Bonds can be deduced straightforwardly
  - Zero-coupon bond options (Caps and Floors) can be evaluated since the characteristic function of the underlying is known (FFT, Moment matching, …)
  - Swaption and Forward swaption can be evaluated with standard approximations

- For exotic products, Monte Carlo method is *a priori* the best candidate
  - Exact simulation of a general affine model remain a good problem to optimize
  - As usually with Monte Carlo method, early exercise feature are complex to evaluate and the complexity increase with the number of factors
  - To enhance the method, it useful to consider adapted techniques like control variable, quantization, Portfolio approximation, …

Proposed Way : Practical Potential Applications

- Build the yield curve itself from few market prices
- Model instantaneous yield curve dynamic and provide macro-hedging solutions and even statistical arbitrage ideas
- Evaluate standard and exotic derivatives within a unique model and then allow a consistent risk mitigation policy
- Evaluate a global future exposure with a stationary model benchmarked on historical data
- Build risk indicators in simulation tools based on readable model and explicit exotic risks
  - Value-at-Risk, Cost-at-Risk, …