Financial Modeling under Illiquidity

Part II:

Viability of Market Impact Models and Optimal Execution

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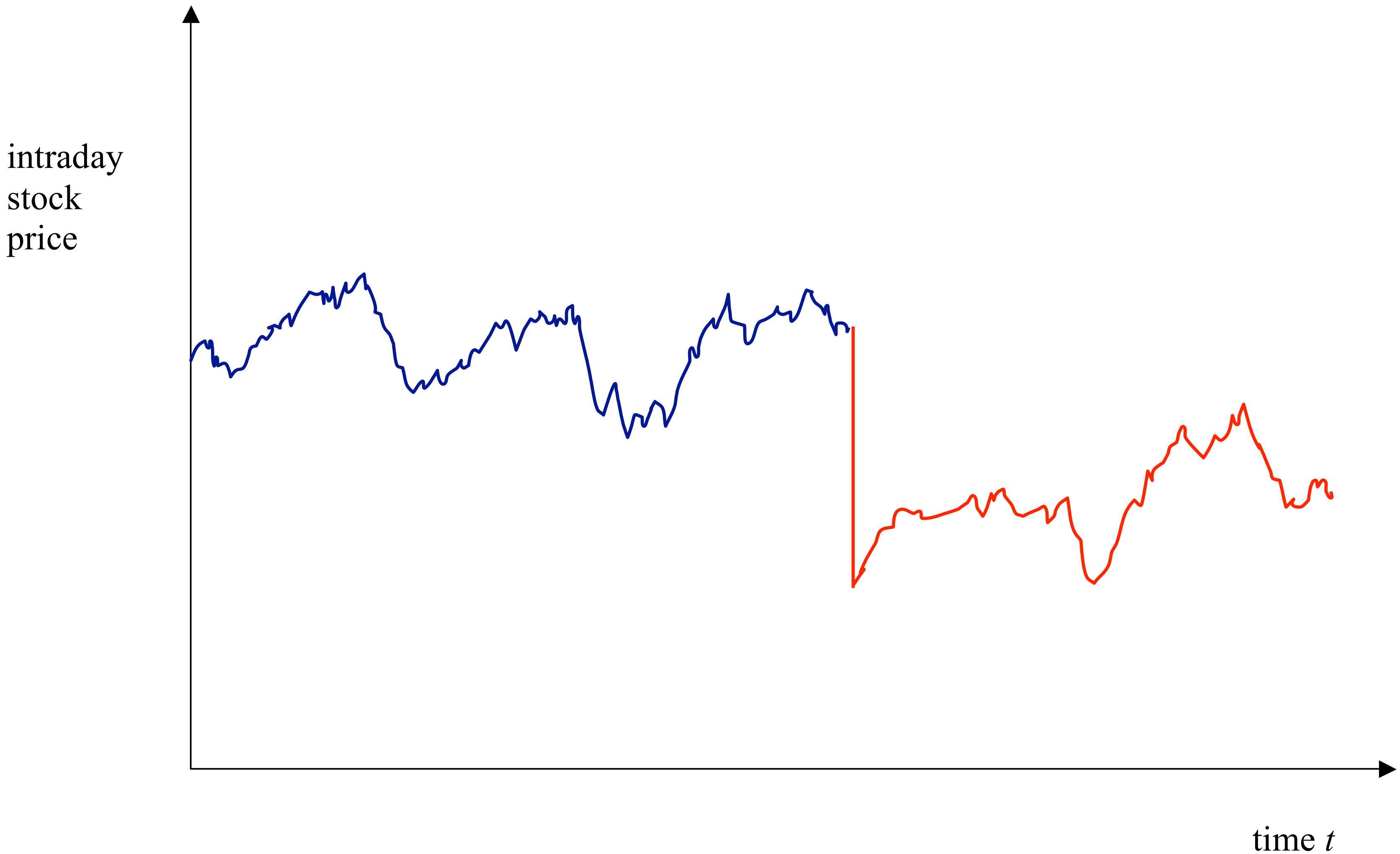
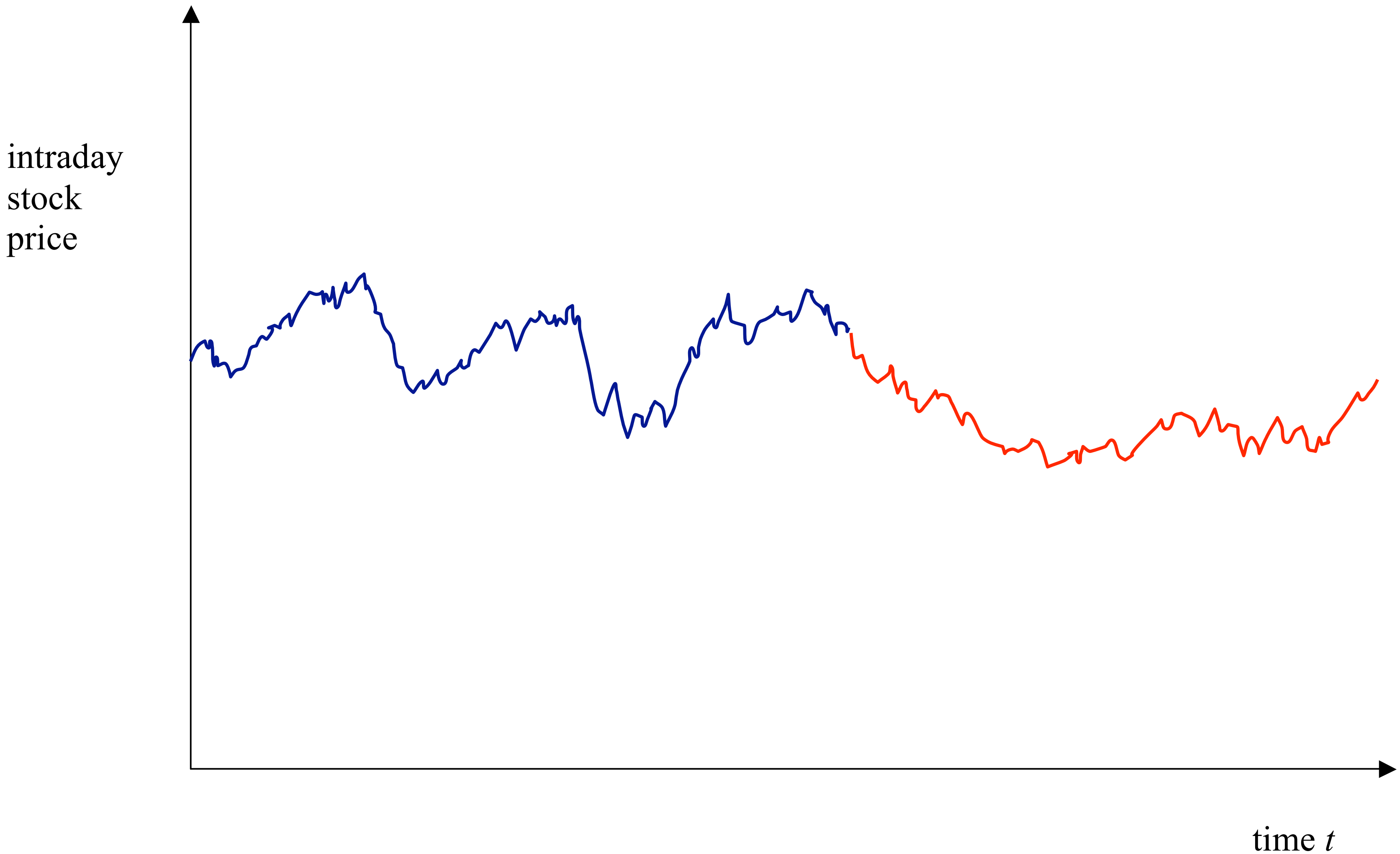
Large trades can significantly impact prices



Spreading the order can reduce the overall price impact



How to execute a single trade of selling X_0 shares?

Interesting because:

- **Liquidity/market impact risk in its purest form**
  - development of realistic market impact models
  - checking viability of these models
  - building block for more complex problems
- **Relevant in applications**
  - real-world tests of new models
- **Interesting mathematics**

Overview:

I. Order book models

II. The qualitative effects of risk aversion

III. Multi-agent equilibrium

Overview:

I. Order book models

Microscopic: Emphasis on single trades

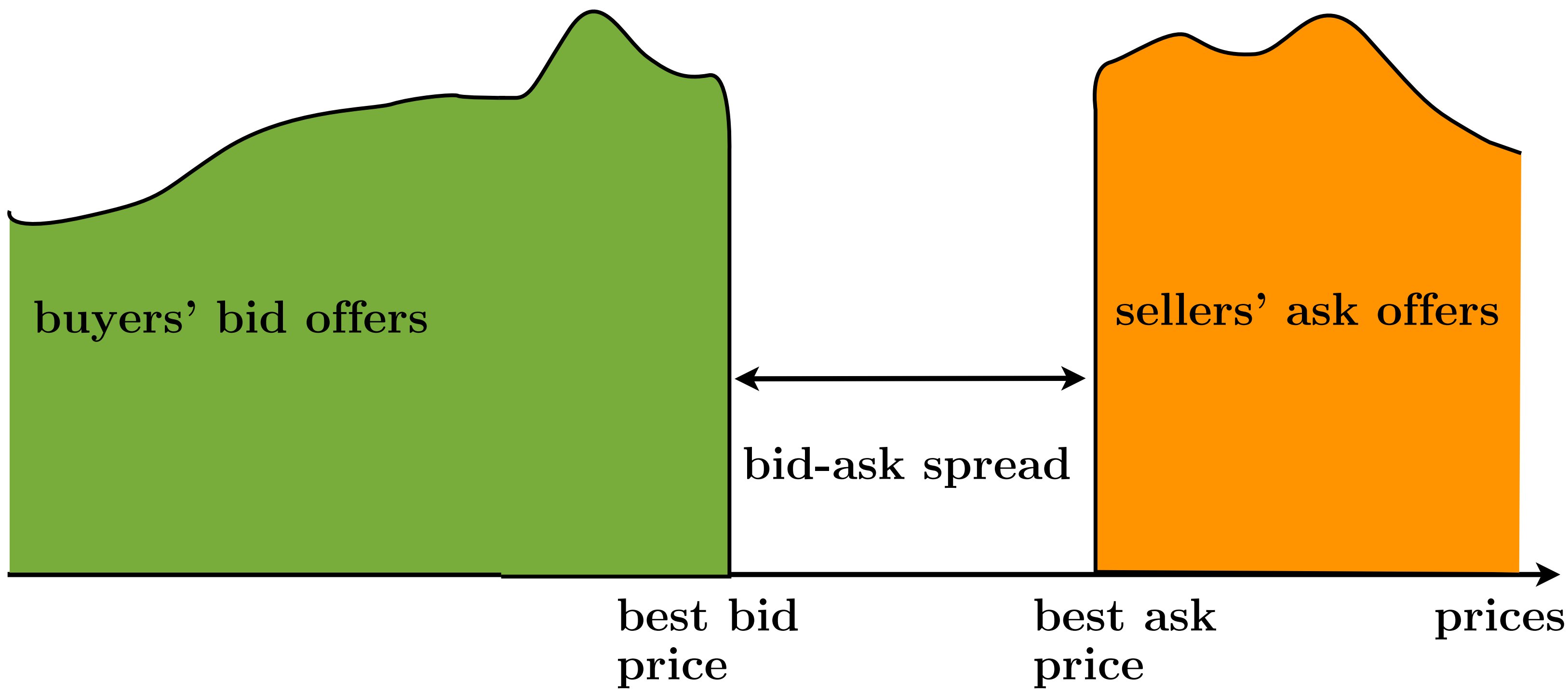
II. The qualitative effects of risk aversion

Mesoscopic: Emphasis on trajectory of trades

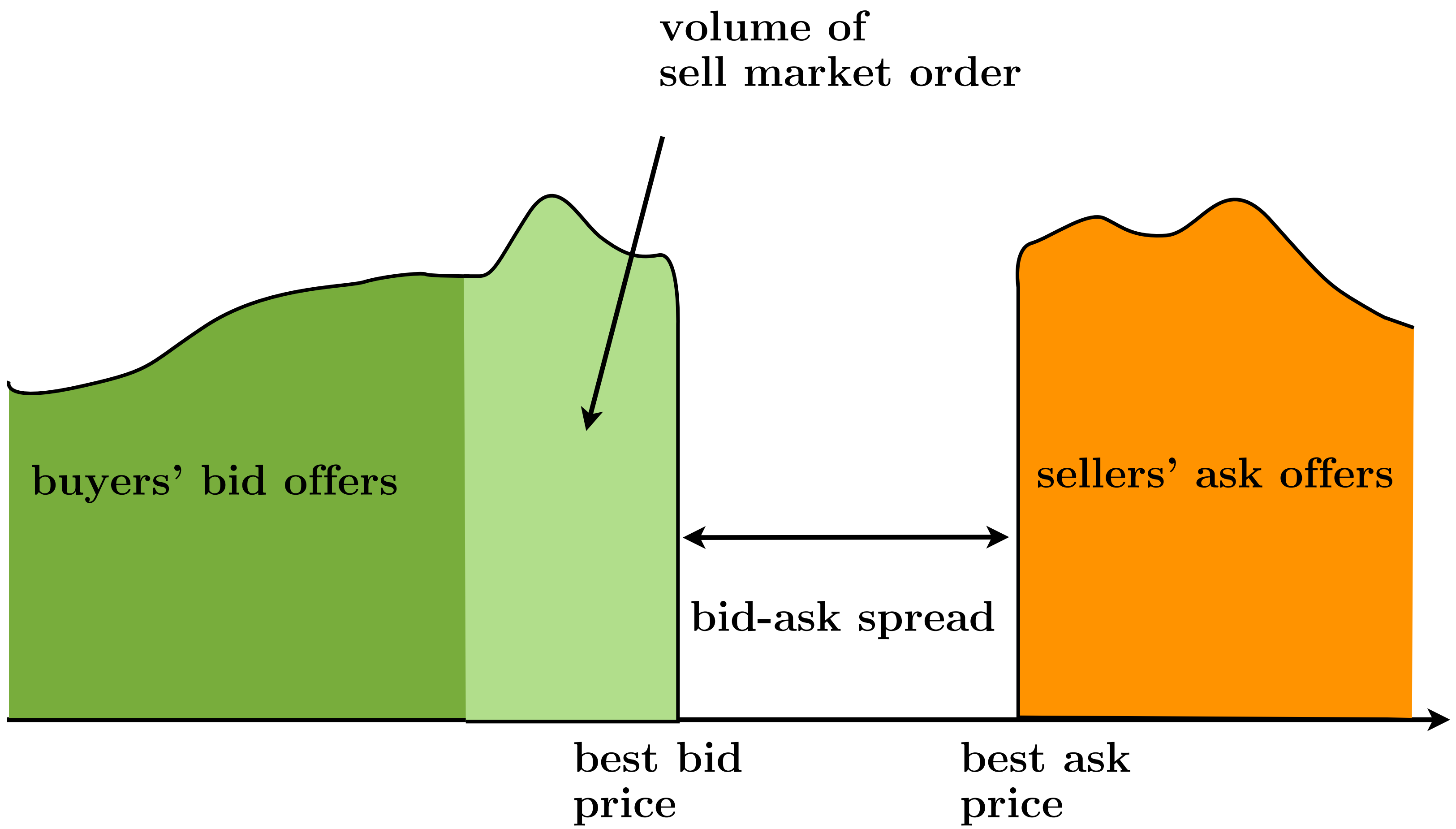
III. Multi-agent equilibrium

Macroscopic: Emphasis on interaction with competitors

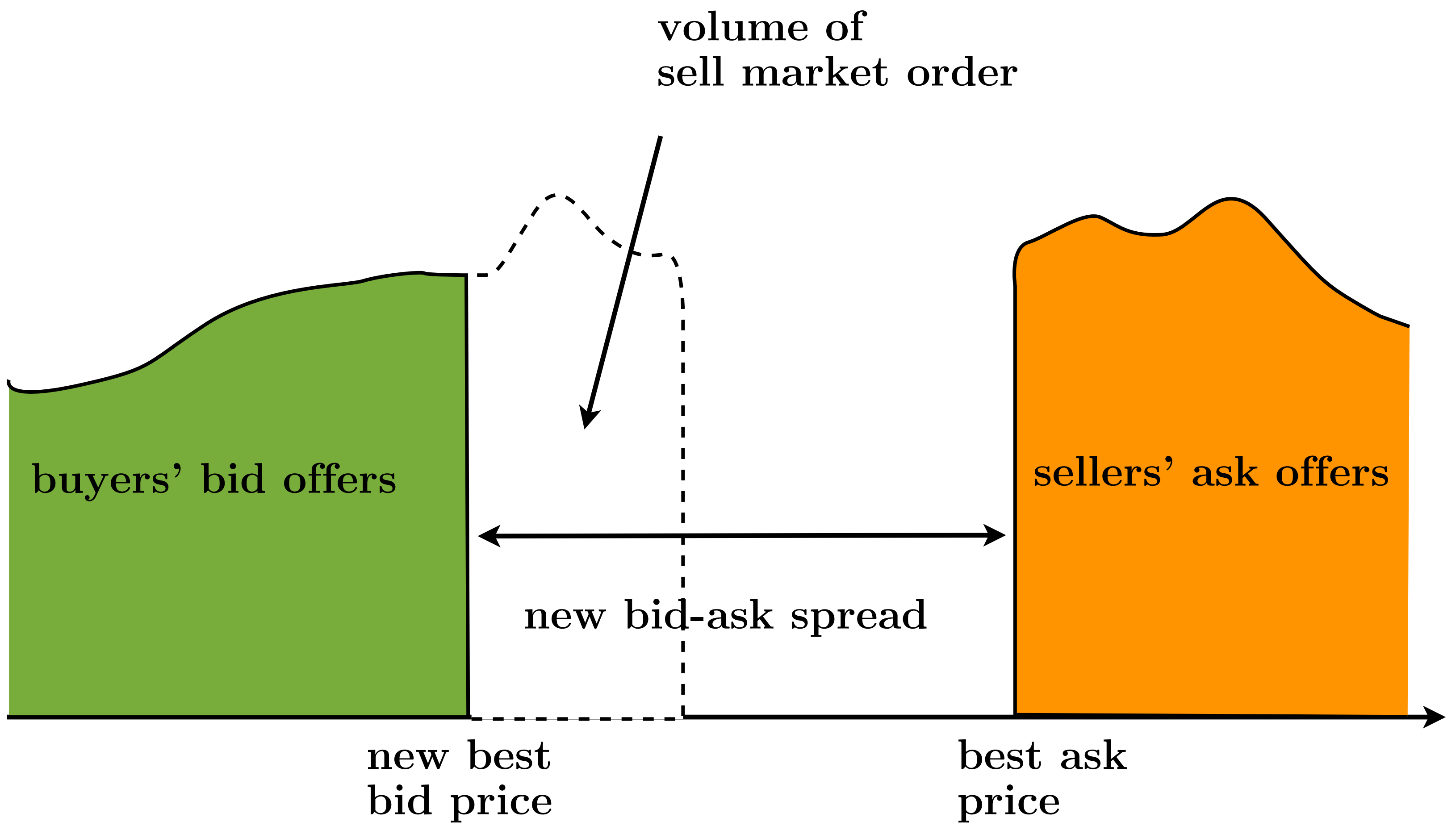
Limit order book before market order



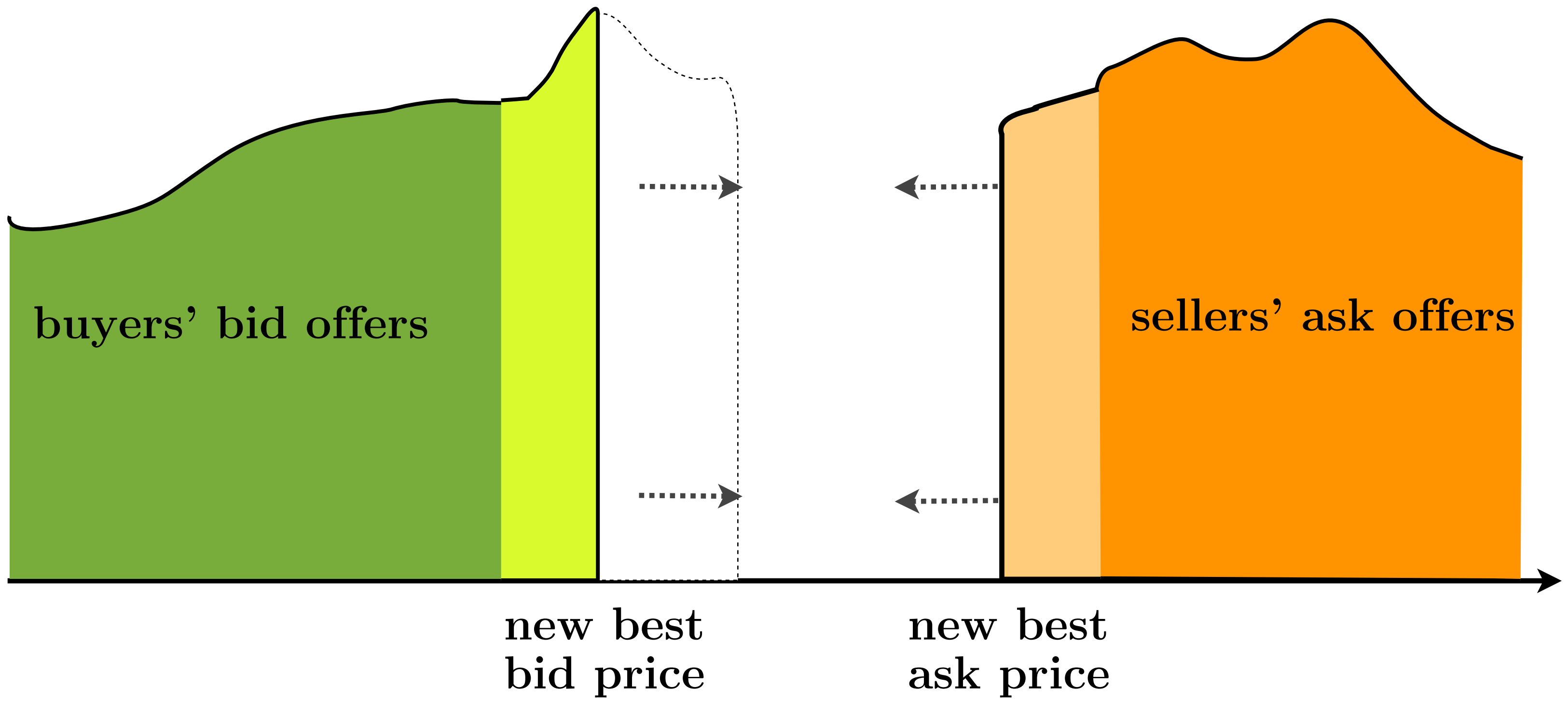
Limit order book before market order



Limit order book after market order



Resilience of the limit order book after market order



I. Order book models

1. **Linear impact, general resilience**

2. **Nonlinear impact, exponential resilience**

3. **Gatheral's model**

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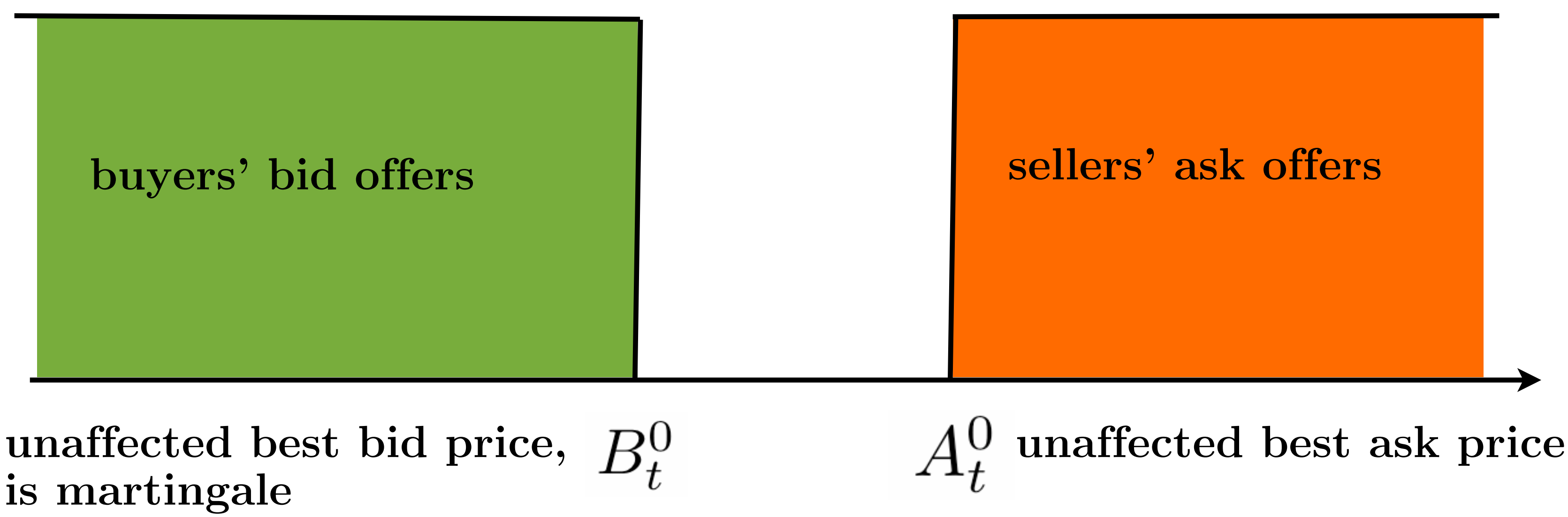
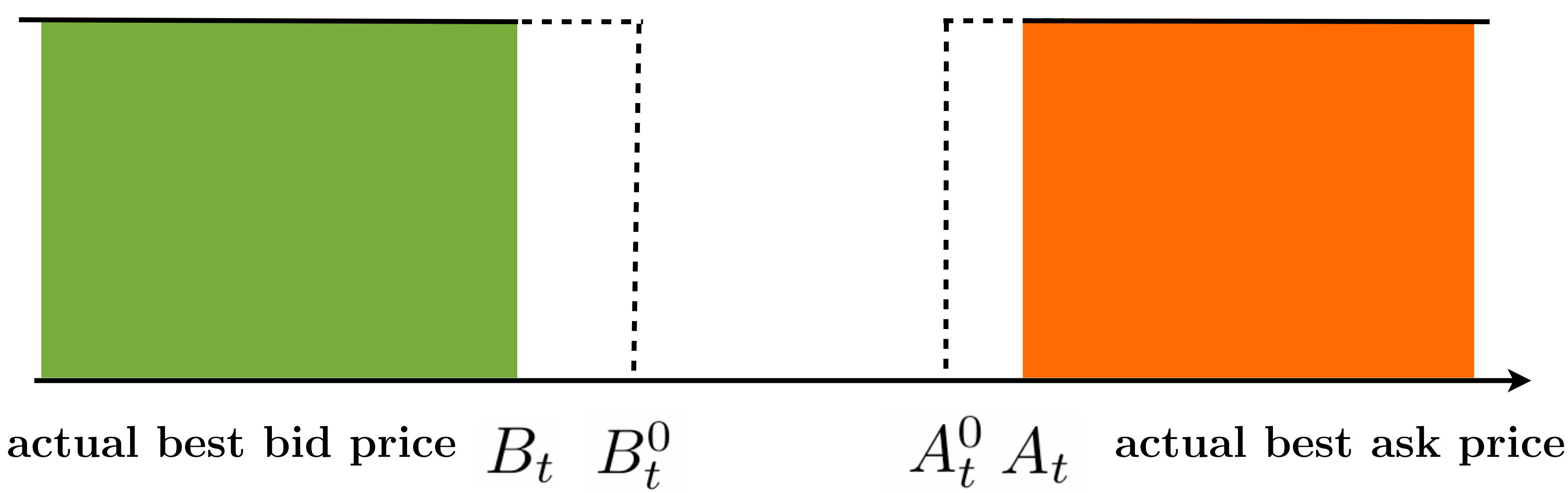
I. Order book models

1. Linear impact, general resilience

Limit order book model without large trader

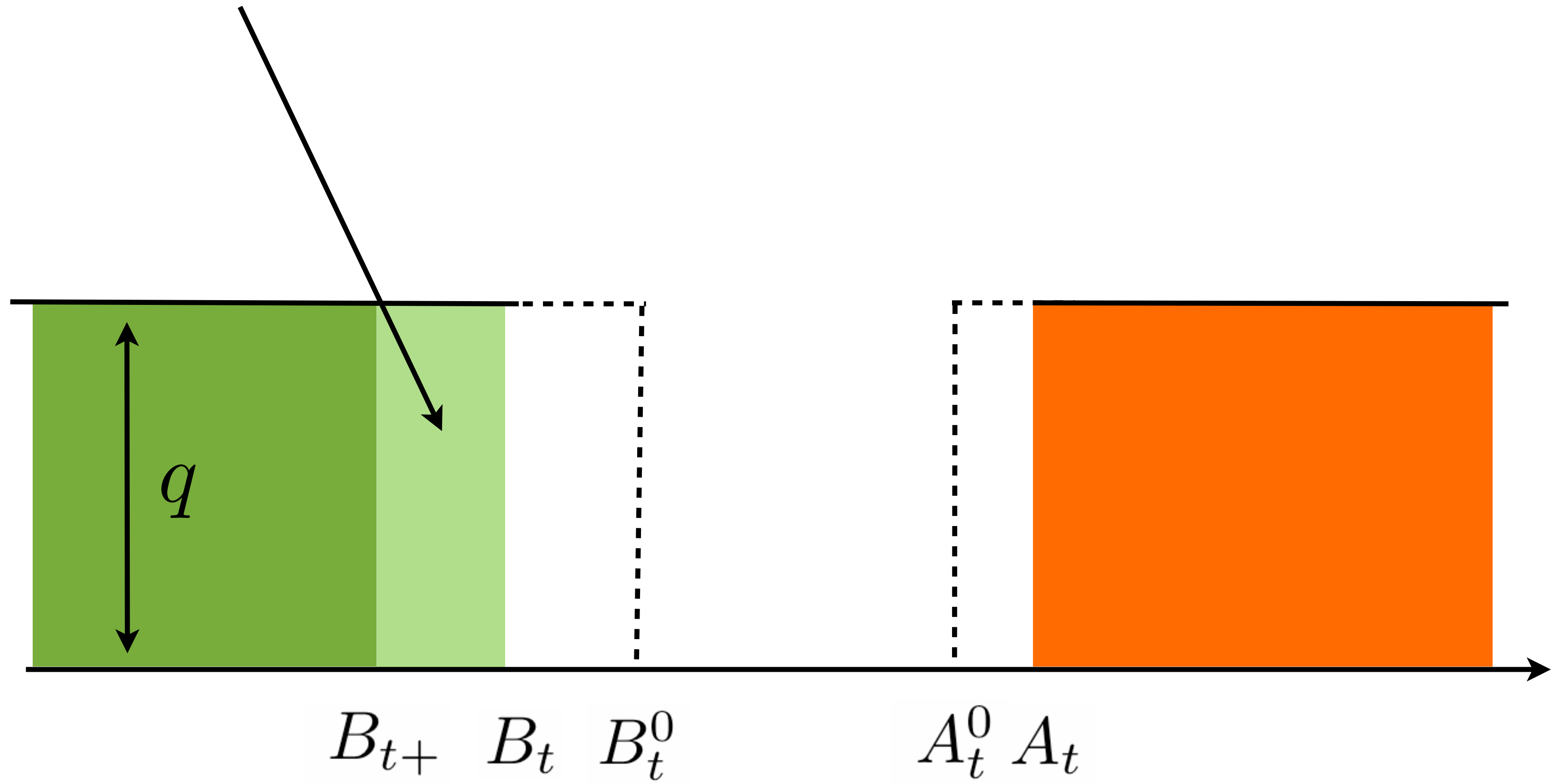


Limit order book model after large trades



Limit order book model at large trade

$$\xi_t = q(B_t - B_{t+})$$

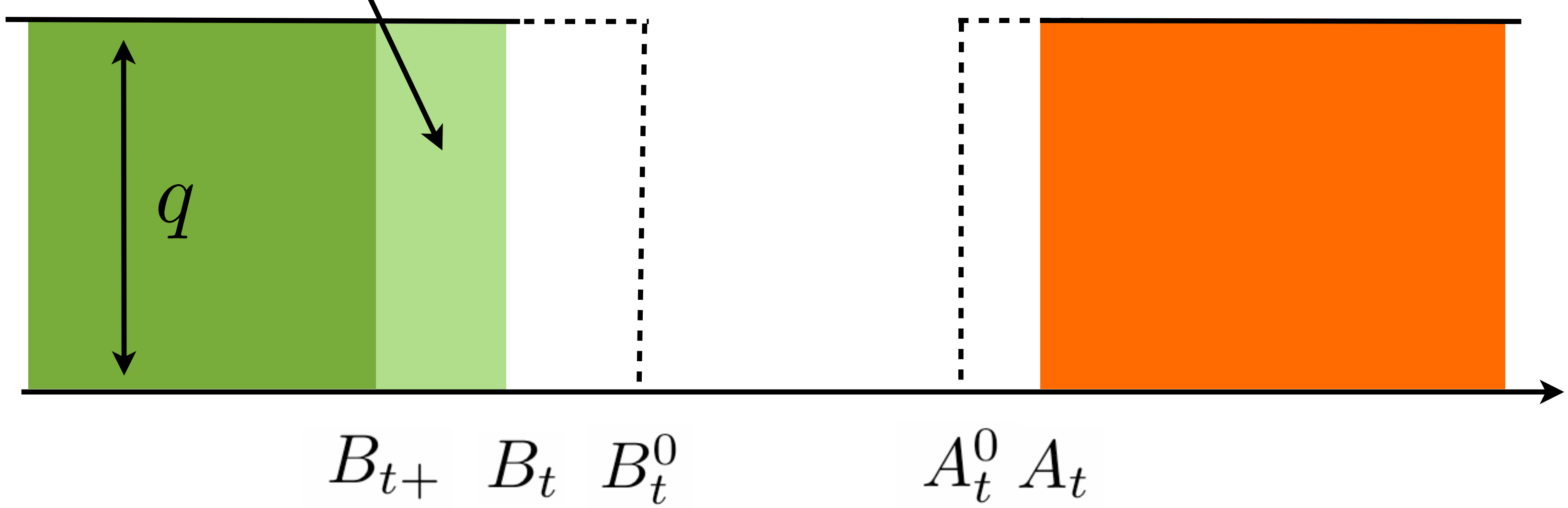


Limit order book model at large trade

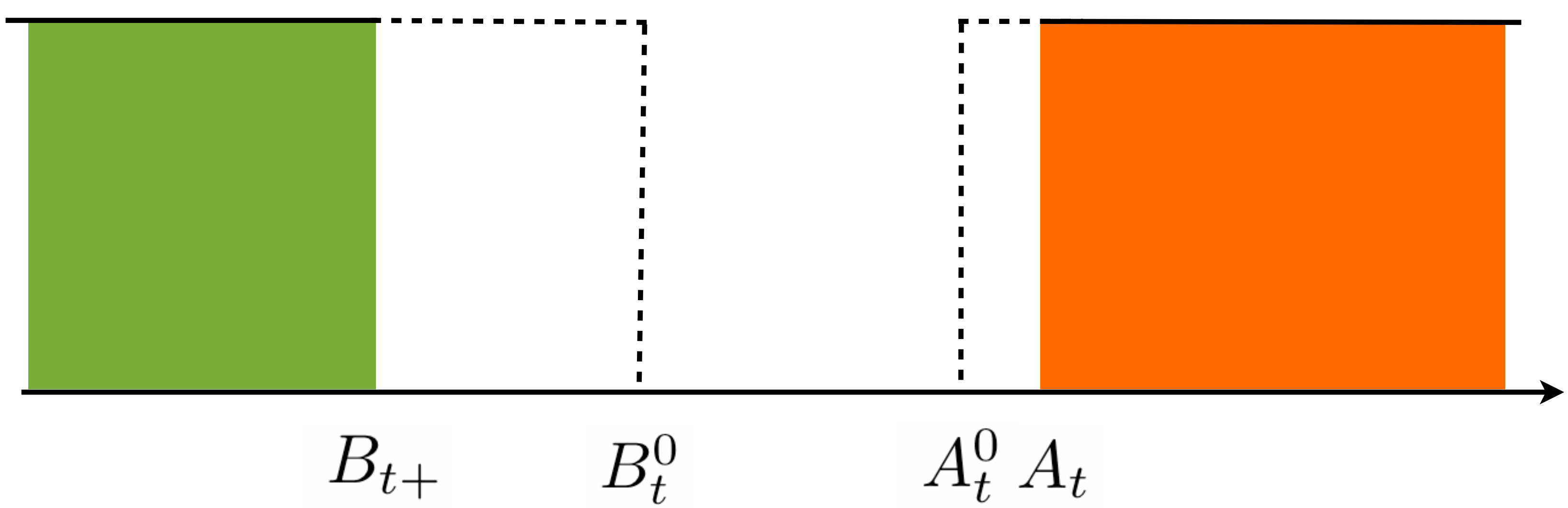
$$\xi_t = q(B_t - B_{t+})$$

sell order executed at average price $\int_{B_{t+}}^{B_t} xq\, dx$

similarly for buy orders



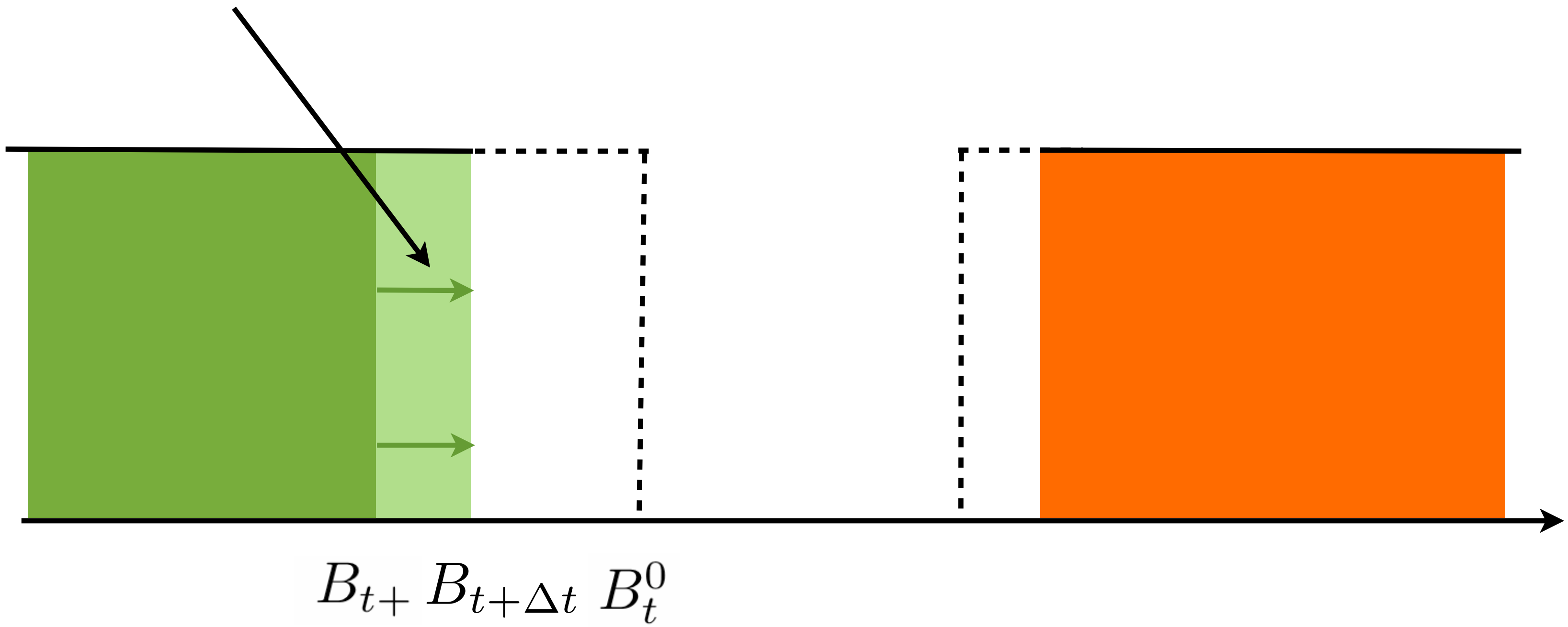
Limit order book model immediately after large trade



Resilience of the limit order book

$\psi : [0, \infty[\to [0,1]$, $\psi(0) = 1$, decreasing

$\frac{\xi_t}{q} \cdot \psi(\Delta t) +$ decay of previous trades



Strategy:

$N+1$ market orders: ξ_n shares placed at time t_n s.th.

a) $0 = t_0 \le t_1 \le \cdots \le t_N = T$
(can also be stopping times)

b) ξ_n is $\mathcal{F}_{t_n}$-measurable and bounded from below,

c) we have $\displaystyle\sum_{n=0}^{N} \xi_n = X_0$

Sell order: $\xi_n > 0$

Buy order: $\xi_n < 0$

Actual best bid and ask prices

$$B_t = B_t^0 - \frac{1}{q} \sum_{\substack{t_n < t \\ \xi_n > 0}} \psi(t - t_n)\xi_n$$

$$A_t = A_t^0 - \frac{1}{q} \sum_{\substack{t_n < t \\ \xi_n < 0}} \psi(t - t_n)\xi_n$$

Cost per trade

$$c_n(\boldsymbol{\xi}) = \begin{cases} \displaystyle\int_{A_{t_n}}^{A_{t_n+}} yq\,dy = \frac{q}{2}(A_{t_n+}^2 - A_{t_n}^2) & \text{for buy order } \xi_n < 0 \\[2em] \displaystyle\int_{B_{t_n}}^{B_{t_n+}} yq\,dy = \frac{q}{2}(B_{t_n+}^2 - B_{t_n}^2) & \text{for sell order } \xi_n > 0 \end{cases}$$

(positive for buy orders, negative for sell orders)

Average cost

$$\mathcal{C}(\boldsymbol{\xi}) = E\Big[\sum_{n=0}^{N} c_n(\boldsymbol{\xi})\Big]$$

A simplified model

No bid-ask spread

S_t^0 = unaffected price, is (continuous) martingale.

$$S_t = S_t^0 - \frac{1}{q} \sum_{t_n < t} \xi_n \psi(t - t_n).$$

Trade ξ_n moves price from S_{t_n} to

$$S_{t_n+} = S_{t_n} - \frac{1}{q} \xi_n.$$

Resulting cost:

$$\overline{c}_n(\boldsymbol{\xi}) := \int_{S_{t_n}}^{S_{t_n+}} yq\,dy = \frac{q}{2}\big[S_{t_n+}^2 - S_{t_n}^2\big] = \frac{1}{2q}\xi_n^2 - \xi_n S_{t_n}$$

(positive for buy orders, negative for sell orders)

Lemma 1. *Suppose that $S^0 = B^0$. Then, for any strategy $\boldsymbol{\xi}$,*

$$\overline{c}_n(\boldsymbol{\xi}) \le c_n(\boldsymbol{\xi}) \qquad \textit{with equality if } \xi_k \ge 0 \textit{ for all } k.$$

Proof: Let

$$\begin{aligned} D_t^B &:= B_t - B_t^0 = -\frac{1}{q}\sum_{\substack{t_n<t\\ \xi_n>0}} \psi(t-t_n)\xi_n \le 0 \\ D_t^A &:= A_t - A_t^0 = -\frac{1}{q}\sum_{\substack{t_n<t\\ \xi_n<0}} \psi(t-t_n)\xi_n \ge 0 \\ D_t &:= D_t^A + D_t^B. \end{aligned}$$

Then

$$S_t = S_t^0 + D_t^A + D_t^B = B_t^0 + D_t$$

and

$$\overline{c}_n(\boldsymbol{\xi}) = \frac{q}{2}\big[S^2_{t_n+} - S^2_{t_n}\big] = \frac{q}{2}\big[(B^0_{t_n} + D_{t_n+})^2 - (B^0_{t_n} + D_{t_n})^2\big].$$

For $\xi_n \geq 0$ we have $D_{t_n+} = D^A_{t_n} + D^B_{t_n+}$ and hence

$$\begin{aligned}
\overline{c}_n(\boldsymbol{\xi}) &= \frac{q}{2}\big[(B^0_{t_n} + D^A_{t_n} + D^B_{t_n+})^2 - (B^0_{t_n} + D^A_{t_n} + D^B_{t_n})^2\big] \\
&= \frac{q}{2}\big[B^2_{t_n+} - B^2_{t_n} + 2D^A_{t_n}(B_{t_n+} - B_{t_n})\big] \\
&\leq \frac{q}{2}\big[B^2_{t_n+} - B^2_{t_n}\big] \\
&= c_n(\boldsymbol{\xi}),
\end{aligned}$$

since $D^A_t \geq 0$ and $B_{t_n+} - B_{t_n} \leq 0$.

For $\xi_n \le 0$, we have $D^A_{t_n+} - D^A_{t_n} \ge 0$ and $B_{t_n} \le B^0_{t_n} \le A^0_{t_n}$. Hence

$$\begin{aligned}
\overline{c}_n(\boldsymbol{\xi}) &= \frac{q}{2}\Big[(B^0_{t_n} + D^A_{t_n+} + D^B_{t_n})^2 - (B^0_{t_n} + D^A_{t_n} + D^B_{t_n})^2\Big] \\
&= \frac{q}{2}\Big[(B_{t_n} + D^A_{t_n+})^2 - (B_{t_n} + D^A_{t_n})^2\Big] \\
&\le \frac{q}{2}\Big[(A^0_{t_n} + D^A_{t_n+})^2 - (A^0_{t_n} + D^A_{t_n})^2\Big] \\
&= c_n(\boldsymbol{\xi}).
\end{aligned}$$

$\square$

Thus: Enough to study the simplified model (as long as all trades ξ_n are positive)

For $\xi_n \leq 0$, we have $D^A_{t_n+} - D^A_{t_n} \geq 0$ and $B_{t_n} \leq B^0_{t_n} \leq A^0_{t_n}$. Hence

$$\begin{aligned}
\overline{c}_n(\boldsymbol{\xi}) &= \frac{q}{2}\big[(B^0_{t_n} + D^A_{t_n+} + D^B_{t_n})^2 - (B^0_{t_n} + D^A_{t_n} + D^B_{t_n})^2\big] \\
&= \frac{q}{2}\big[(B_{t_n} + D^A_{t_n+})^2 - (B_{t_n} + D^A_{t_n})^2\big] \\
&\leq \frac{q}{2}\big[(A^0_{t_n} + D^A_{t_n+})^2 - (A^0_{t_n} + D^A_{t_n})^2\big] \\
&= c_n(\boldsymbol{\xi}).
\end{aligned}$$

$\square$

Thus: Enough to study the simplified model (as long as all trades ξ_n are positive)

Lemma 2. *In the simplified model, the expected cost of a strategy* $\boldsymbol{\xi}$ *is*

$$\overline{\mathcal{C}}(\boldsymbol{\xi}) = E\Big[\sum_{n=0}^{N} \overline{c}_n(\boldsymbol{\xi})\Big] = \frac{1}{2q} E\big[\, C^{\psi}_{\boldsymbol{t}}(\boldsymbol{\xi})\,\big] - X_0 S^0_0,$$

where $C^{\psi}_{\boldsymbol{t}}$ *is the quadratic form*

$$C^{\psi}_{\boldsymbol{t}}(\boldsymbol{x}) = \sum_{m,n=0}^{N} x_n x_m \psi(|t_n - t_m|), \qquad \boldsymbol{x} \in \mathbb{R}^{N+1},\ \boldsymbol{t} = (t_0, \dots, t_N).$$

Proof: We have

$$\begin{aligned}\sum_{n=0}^N \bar{c}_n(\boldsymbol{\xi}) &= \sum_{n=0}^N \Big(\frac{1}{2q}\xi_n^2 - \xi_n S_{t_n}\Big)\\ &= \sum_{n=0}^N \Big(\frac{1}{2q}\xi_n^2 + \xi_n \frac{1}{q}\sum_{t_m<t}\xi_n\psi(t_n - t_m) - \xi_n S^0_{t_n}\Big)\\ &= \frac{1}{2q}\sum_{m,n=0}^N \xi_n\xi_m\psi(|t_n - t_m|) - \sum_{n=0}^N \xi_n S^0_{t_n}.\end{aligned}$$

Letting

$$X_t := X_0 - \sum_{t_n<t}\xi_n \qquad \text{and} \qquad X_{t_{N+1}} := 0,$$

we have

$$\sum_{n=0}^N \xi_n S^0_{t_n} = -\sum_{n=0}^N (X_{t_{n+1}} - X_{t_n})S^0_{t_n} = X_0 S^0_0 + \sum_{n=0}^N X_{t_n}(S^0_{t_n} - S^0_{t_{n-1}}). \quad \square$$

First Question:
What are the conditions on ψ under which the (simplified) model is viable?

Requiring the absence of arbitrage opportunities in the usual sense is not strong enough, as examples will show.

Second Question:
Which strategies minimize the expected cost for given X_0?

This is the optimal execution problem. It is very closely related to the question of model viability.

The usual concept of viability from Hubermann & Stanzl (2004):

Definition

A round trip is a strategy $\boldsymbol{\xi}$ with

$$\sum_{n=0}^{N} \xi_n = X_0 = 0.$$

A market impact model admits

price manipulation strategies

if there is a round trip with negative expected costs.

In the simplified model, the expected costs of a strategy $\boldsymbol{\xi}$ are

$$\overline{\mathcal{C}}(\boldsymbol{\xi}) = \frac{1}{2q} E\big[\, C_{\boldsymbol{t}}^{\psi}(\boldsymbol{\xi})\,\big] - X_0 S_0^0,$$

where

$$C_{\boldsymbol{t}}^{\psi}(\boldsymbol{x}) = \sum_{m,n=0}^{N} x_n x_m \psi(|t_n - t_m|), \qquad \boldsymbol{x} \in \mathbb{R}^{N+1},\ \boldsymbol{t} = (t_0, \dots, t_N).$$

- There are no price manipulation strategies when $C_{\boldsymbol{t}}^{\psi}$ is nonnegative definite for all $\boldsymbol{t} = (t_0, \dots, t_N)$;
- when the minimizer $\boldsymbol{x}^*$ of $C_{\boldsymbol{t}}^{\psi}(\boldsymbol{x})$ with $\sum_i x_i = X_0$ exists, it yields the optimal strategy in the simplified model; in particular, the optimal strategy is then deterministic;
- when the minimizer $\boldsymbol{x}^*$ has only nonnegative components, it yields the optimal strategy in the order book model.

Bochner's theorem (1932):
$C_{\boldsymbol{t}}^{\psi}$ is always nonnegative definite (ψ is "positive definite") if and only if $\psi(|\cdot|)$ is the Fourier transform of a positive Borel measure μ on $\mathbb{R}$.

$C_{\boldsymbol{t}}^{\psi}$ is even strictly positive definite (ψ is "strictly positive definite") when the support of μ is not discrete.

Bochner's theorem (1932):
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$C_{\boldsymbol{t}}^{\psi}$ is even strictly positive definite (ψ is "strictly positive definite") when the support of μ is not discrete.

- Seems to completely settle the question of model viability;
- for strictly positive definite ψ, the optimal strategy is

$$\boldsymbol{\xi}^* = \boldsymbol{x}^* = \frac{X_0}{\mathbf{1}^\top M^{-1}\mathbf{1}} M^{-1}\mathbf{1} \qquad \text{for } M_{ij} = \psi(|t_i - t_j|).$$

Proof of "$\Leftarrow$": Suppose that

$$\psi(|t|) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{itz}\,\mu(dz).$$

Then

$$\begin{aligned} C_{\boldsymbol{t}}^{\psi}(\boldsymbol{x}) &= \sum_{m,n=0}^{N} x_n x_m \psi(|t_n - t_m|) = \frac{1}{\sqrt{2\pi}} \int \sum_{m,n=0}^{N} x_n x_m e^{i(t_n - t_m)z}\,\mu(dz) \\ &= \frac{1}{\sqrt{2\pi}} \int \Big(\sum_{n=0}^{N} x_n e^{it_n z}\Big)\Big(\sum_{n=0}^{N} x_n e^{-it_n z}\Big)\,\mu(dz) \\ &= \frac{1}{\sqrt{2\pi}} \int |g(z)|^2\,\mu(dz) \geq 0, \end{aligned}$$

where

$$g(z) = \sum_{n=0}^{N} x_n e^{it_n z}.$$

Now suppose that $\boldsymbol{x} \neq 0$ but

$$C^{\psi}_{\boldsymbol{t}}(\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}} \int |g(z)|^2 \, \mu(dz) = 0.$$

Then the function g vanishes on the support of μ. But g is analytic and a non-vanishing, so its zero set must be discrete. Hence the support of μ must be discrete. $\square$

Examples

Example 1: Exponential resilience
[Obizhaeva & Wang (2005), Alfonsi, Fruth, S. (2008)]

For the exponential resilience function

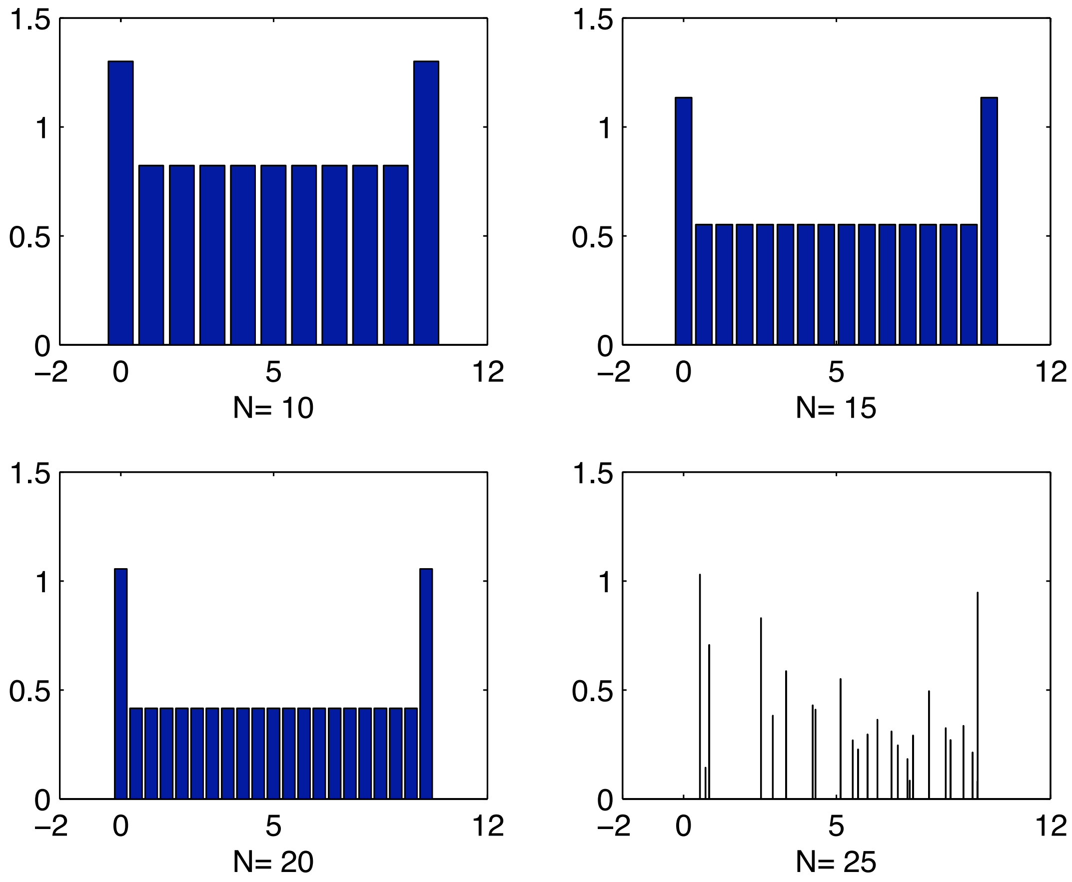
$$\psi(t) = e^{-\rho t},$$

$\psi(|\cdot|)$ is the Fourier transform of the positive measure

$$\mu(dt) = \sqrt{\frac{2}{\pi}} \frac{\rho}{\rho^2 + t^2}\, dt$$

Hence, ψ is strictly positive definite.

Optimal strategies for exponential resilience $\psi(t) = e^{-\rho t}$



The optimal strategy can in fact be computed explicitly for any time grid:

Let $a_n := e^{-\rho(t_n - t_n - 1)}$ for $n = 1, \ldots, N$. Then we can write

$$M = \begin{bmatrix} 1 & a_1 & a_1 a_2 & \cdots & \cdots & a_1 a_2 \cdots a_N \\ a_1 & 1 & a_2 & a_2 a_3 & \cdots & a_2 a_3 \cdots a_N \\ a_1 a_2 & a_2 & 1 & a_3 & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ a_2 \cdots a_N & & & a_{N-1} & 1 & a_N \\ a_1 a_2 \cdots a_N & \cdots & \cdots & a_{N-1} a_N & a_N & 1 \end{bmatrix}.$$

The inverse of M can be computed as the tridiagonal matrix

$$M^{-1} = \begin{bmatrix} \frac{1}{1-a_1^2} & \frac{-a_1}{1-a_1^2} & 0 & \cdots & 0 \\ \frac{-a_1}{1-a_1^2} & \left(\frac{1}{1-a_1^2} + \frac{a_2^2}{1-a_2^2}\right) & \frac{-a_2}{1-a_2^2} & 0 \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \frac{-a_{N-1}}{1-a_{N-1}^2} & \left(\frac{1}{1-a_{N-1}^2} + \frac{a_N^2}{1-a_N^2}\right) & \frac{-a_N}{1-a_N^2} \\ 0 & \cdots & 0 & \frac{-a_N}{1-a_N^2} & \frac{1}{1-a_N^2} \end{bmatrix}$$

From this formula, we get

$$M^{-1}\mathbf{1} = \begin{bmatrix} \frac{1}{1+a_1} \\ \frac{1}{1+a_1} - \frac{a_2}{1+a_2} \\ \vdots \\ \frac{1}{1+a_{N-1}} - \frac{a_N}{1+a_N} \\ \frac{1}{1+a_N} \end{bmatrix}$$

And hence

$$\boldsymbol{x}^* = \lambda_0 M^{-1}\mathbf{1}$$

for

$$\lambda_0 = \frac{X_0}{\mathbf{1}^\top M^{-1}\mathbf{1}} = \frac{X_0}{\frac{2}{1+a_1} + \sum_{n=2}^N \frac{1-a_n}{1+a_n}}.$$

The initial market order of the optimal strategy is hence

$$x_0^* = \frac{\lambda_0}{1 + a_1},$$

the intermediate market orders are given by

$$x_n^* = \lambda_0 \Big(\frac{1}{1 + a_n} - \frac{a_{n+1}}{1 + a_{n+1}} \Big), \qquad n = 1, \ldots, N-1,$$

and the final market order is

$$x_N^* = \frac{\lambda_0}{1 + a_N}.$$

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and the final market order is

$$x_N^* = \frac{\lambda_0}{1 + a_N}.$$

It is clear that x_0^* and x_N^* are strictly positive. For $i = 1, \dots, N - 1$ we have

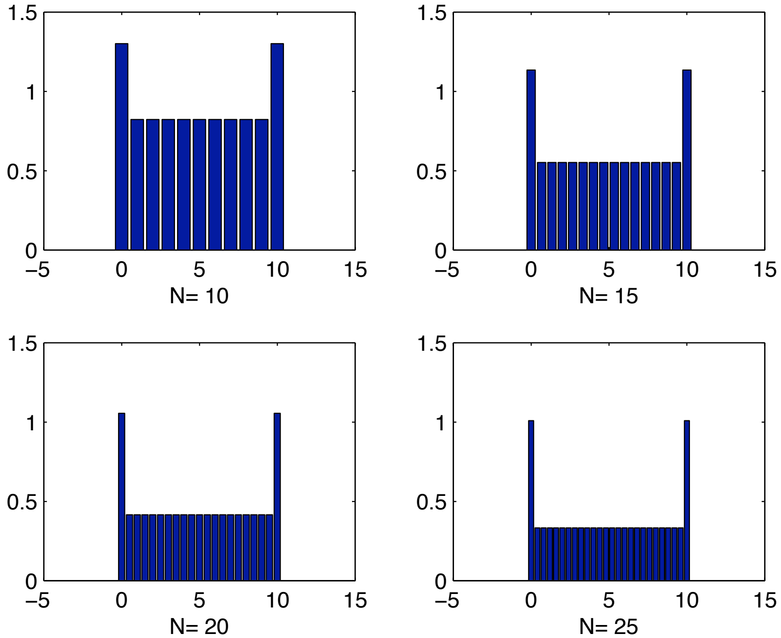
$$x_i^* = \lambda_0 \cdot \frac{(1 - a_i a_{i+1})}{(1 + a_i)(1 + a_{i+1})} > 0.$$

For the equidistant time grid $t_n = nT/N$ the solution simplifies:

$$x_0^* = x_N^* = \frac{X_0}{(N-1)(1-a)+2}$$

and

$$x_1^* = \cdots = x_{N-1}^* = \xi_0^*(1-a).$$



The symmetry of the optimal strategy is a general fact:

Exercise:
Suppose that ψ is strictly positive definite and that the time grid is symmetric, i.e.,

$$t_i = t_N - t_{N-i} \qquad \text{for all } i,$$

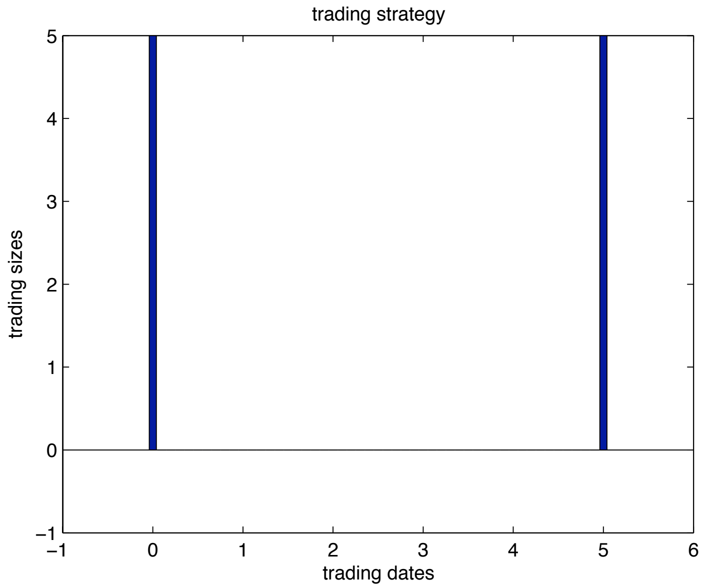
then the optimal strategy is reversible, i.e.,

$$x^*_{t_i} = x^*_{t_{N-i}} \qquad \text{for all } i.$$

Example 2: Linear resilience $\psi(t) = 1 - \rho t$ for some $\rho \leq 1/T$

We will see in a minute that this ψ is strictly positive definite.

The optimal strategy is always of this form:



It is independent of the underlying time grid and consists of two symmetric trades of size $X_0/2$ at $t = 0$ and $t = T$, all other trades are zero.

Proof: Let $\boldsymbol{x}^0$ denote the asserted strategy. It has the cost

$$C_{\boldsymbol{t}}^{\psi}(\boldsymbol{x}^0) = \Big(\frac{X_0}{2}\Big)^2 \big[\psi(0) + 2\psi(T) + \psi(0)\big] = \frac{2-\rho T}{2} X_0^2, \tag{1}$$

regardless of the underlying time grid. We will show that the minimal cost is independent of the time grid and equal to the right-hand side in (1). Since the linear resilience function is strictly positive definite, $\boldsymbol{x}^0$ must then be the unique optimal strategy.

The cost of the optimal

$$\boldsymbol{x}^* = \frac{X_0}{\mathbf{1}^\top M^{-1}\mathbf{1}} M^{-1}\mathbf{1}$$

for an arbitrary time grid is

$$C_{\boldsymbol{t}}^{\psi}(\boldsymbol{x}^*) = (\boldsymbol{x}^*)^\top M \boldsymbol{x}^* = \frac{X_0^2}{\mathbf{1}^\top M^{-1}\mathbf{1}}.$$

Let $\boldsymbol{x} := M^{-1}\mathbf{1} = (x_0, \dots, x_N)$ and $\Delta_i := t_i - t_{i-1}$. Then the first and last lines of the equation $M\boldsymbol{x} = \mathbf{1}$ can be written as follows.

$$x_0 + \big(1 - \rho\Delta_1\big)x_1 + \cdots + \Big(1 - \rho\sum_{i=1}^N \Delta_i\Big)x_N = 1,$$

$$\Big(1 - \rho\sum_{i=1}^N \Delta_i\Big)x_0 + \cdots + \big(1 - \rho\Delta_N\big)x_{N-1} + x_N = 1.$$

Summing both equations yields

$$\sum_{i=0}^N x_i\Big(2 - \rho\sum_{i=1}^N \Delta_i\Big) = 2$$

and thus

$$\mathbf{1}^\top M^{-1}\mathbf{1} = \sum_{i=0}^N x_i = \frac{2}{2 - \rho T}.$$

This proves the assertion. $\square$

More generally: Convex resilience

Theorem 3.
[Carathéodory (1907), Toeplitz (1911), Young (1912)]

ψ is convex, decreasing, nonnegative, and nonconstant $\Longrightarrow$
$\psi(|\cdot|)$ is strictly positive definite.

More generally: Convex resilience

Theorem 3.
[Carathéodory (1907), Toeplitz (1911), Young (1912)]

ψ *is convex, decreasing, nonnegative, and nonconstant* $\Longrightarrow$
$\psi(|\cdot|)$ *is strictly positive definite.*

Proof: W.l.o.g.: ψ is continuous (exercise).
ψ' = right-hand derivative.
$\psi''(dx)$ = second derivative (= Borel measure on $[0, \infty]$).

For $\varepsilon > 0$ let $\psi_\varepsilon(x) := e^{-\varepsilon x}\psi(x)$ (is again convex and decreasing).

The inverse Fourier transform of $\psi_\varepsilon(|\cdot|)$ is proportional to

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi_\varepsilon(|x|)e^{-ixz}\,dx &= 2\int_0^\infty \psi_\varepsilon(x)\cos xz\,dx \\
&= -2\int_0^\infty \psi_\varepsilon'(x)\int_0^x \cos zt\,dt\,dx \\
&= 2\int_0^\infty \int_0^x \int_0^t \cos sz\,ds\,dt\,\psi_\varepsilon''(dx) \\
&= 2\int_0^\infty \frac{1-\cos xz}{z^2}\,\psi_\varepsilon''(dx)
\end{aligned}$$

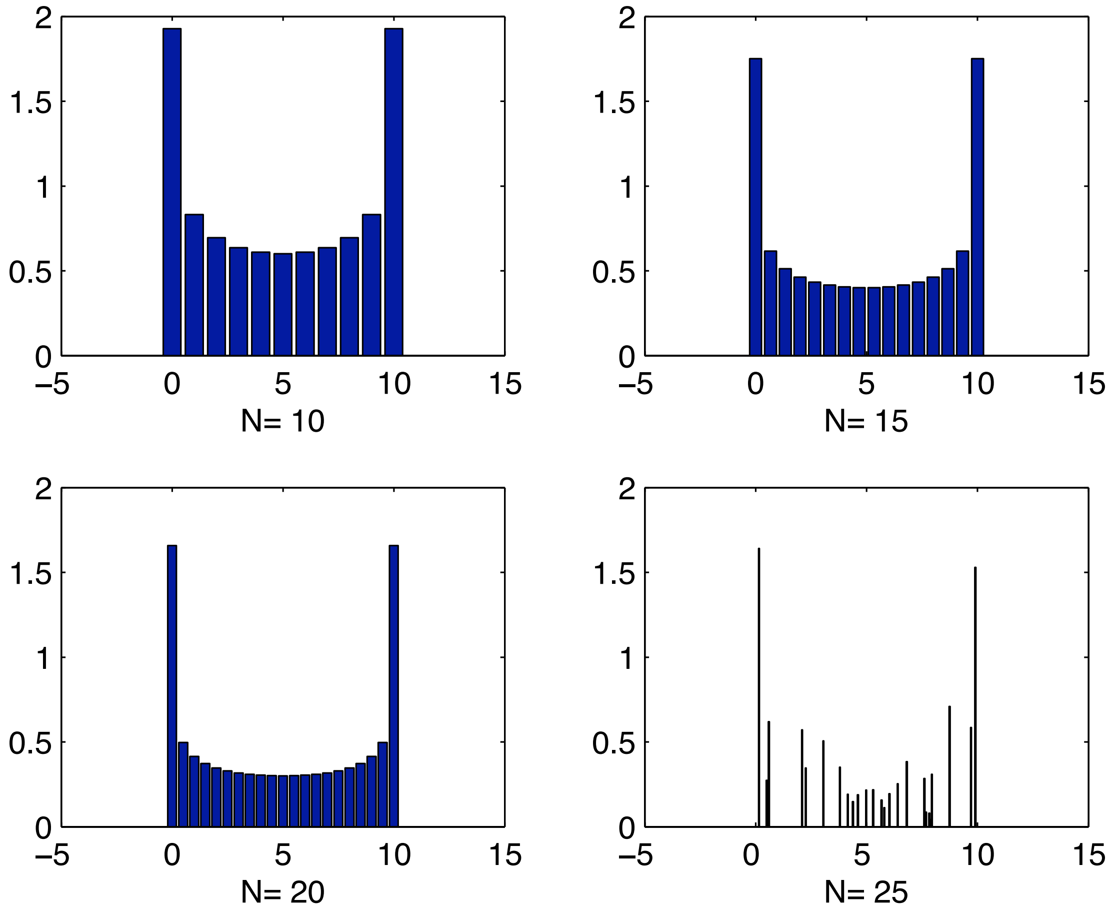
As a function of z, the right-hand side is the density of a positive finite Borel measure μ_ε. It follows that ψ_ε, and hence ψ, are positive definite functions.

Since $\psi_\varepsilon \to \psi$ pointwise, Lévy's theorem entails that μ_ε converges weakly to the measure μ, the inverse Fourier transform of ψ modulo a proportionality factor. Portmanteau theorem:

$$\mu([a,b]) \geq \limsup_{\varepsilon \downarrow 0} \mu_\varepsilon([a,b]) \geq 2 \int_0^\infty \int_a^b \frac{1-\cos xz}{z^2}\, dz\, \psi''(dx) > 0$$

for all $0 < a < b$. Hence, the support of μ is not discrete, and so ψ is strictly positive definite. $\square$

Example 3: Power law resilience $\psi(t) = (1 + \beta t)^{-\alpha}$



Example 4: Trigonometric resilience

The function

$$\cos \rho x$$

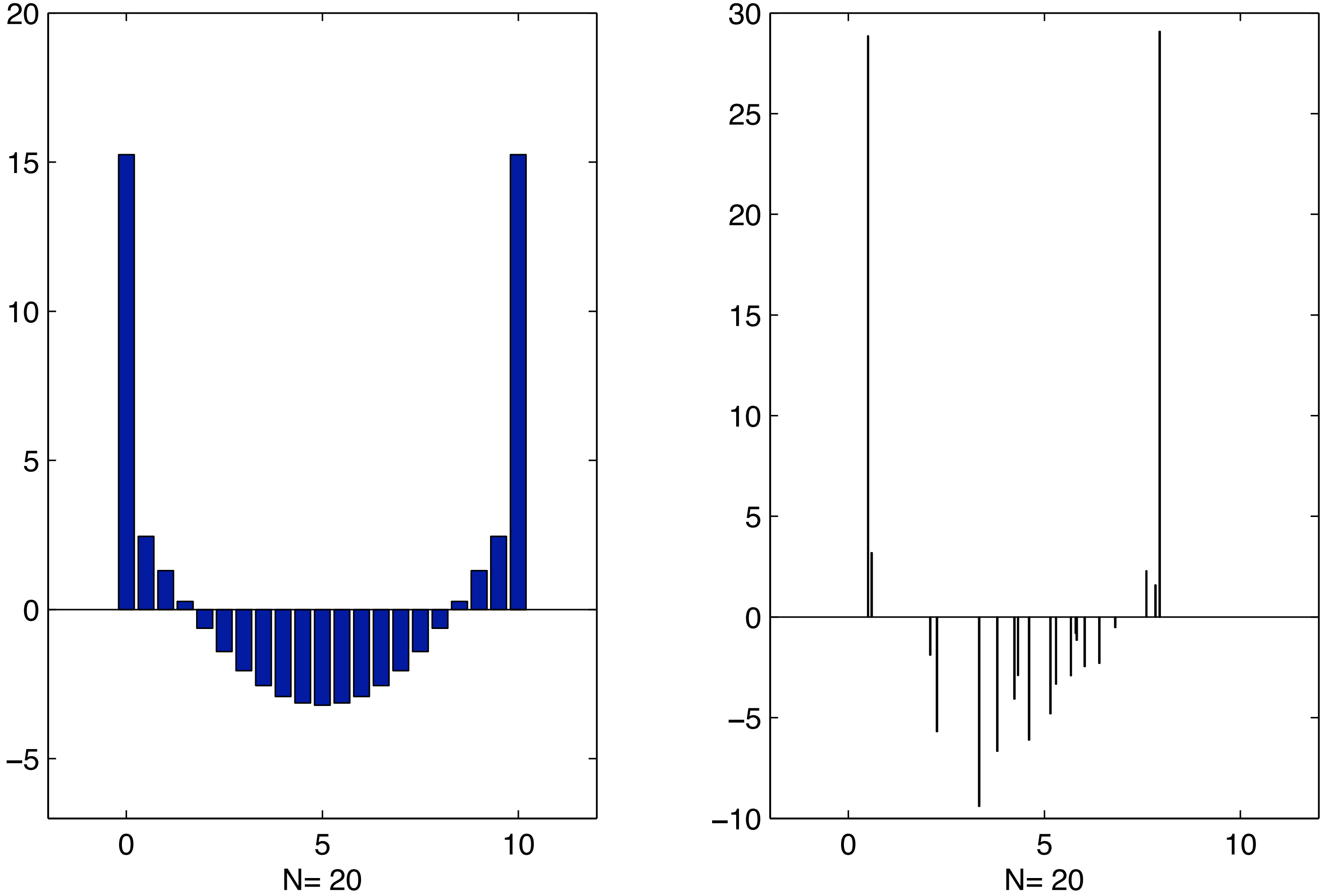
is the Fourier transform of the positive finite measure

$$\mu = \sqrt{\frac{\pi}{2}}(\delta_{-\rho} + \delta_{\rho})$$

Since it is not strictly positive definite, we take

$$\psi(t) = (1-\varepsilon)\cos \rho t + \varepsilon e^{-t} \qquad \text{for some } \rho \leq \frac{\pi}{2T}.$$

Trigonometric resilience $\psi(t) = 0.999\cos(t\pi/2T) + 0.001e^{-t}$



Example 5: Gaussian resilience

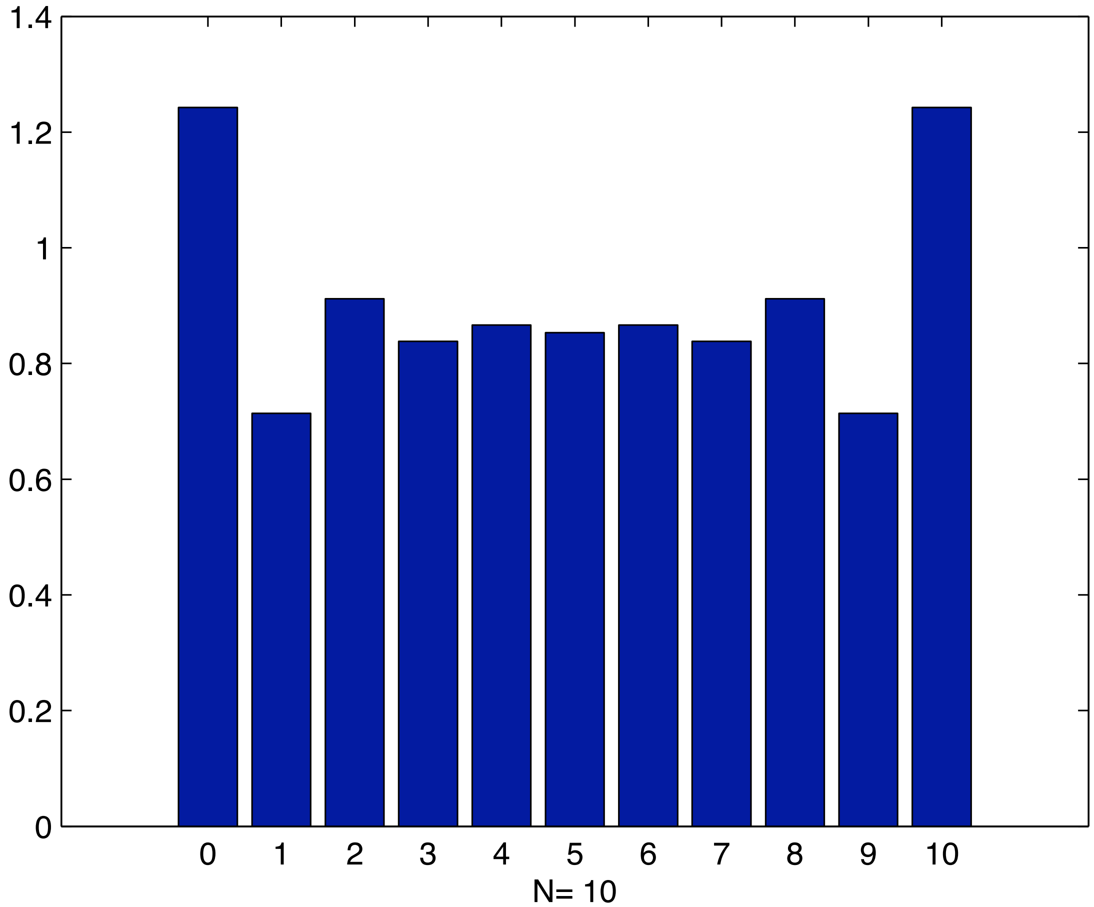
The Gaussian resilience function

$$\psi(t) = e^{-t^2}$$

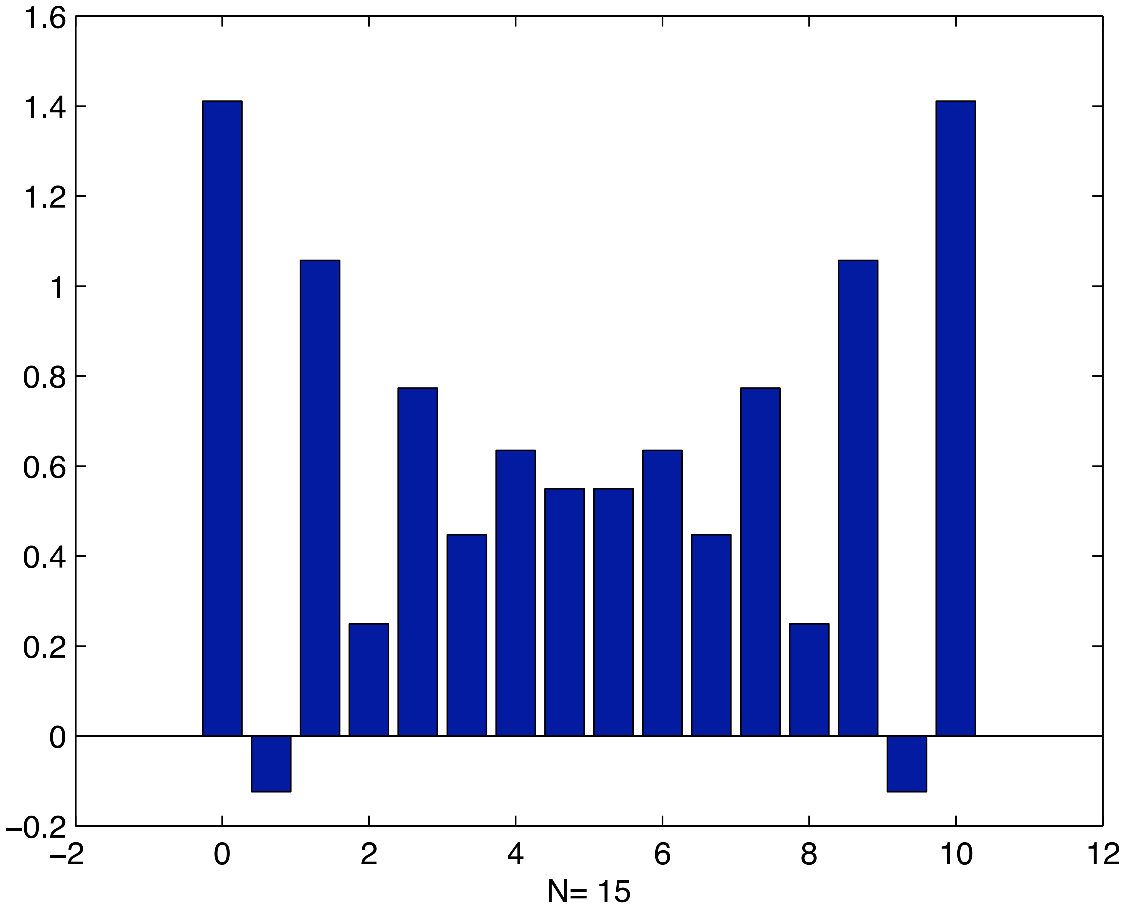
is its own Fourier transform (modulo constants). The corresponding quadratic form is hence positive definite.

Nevertheless.....

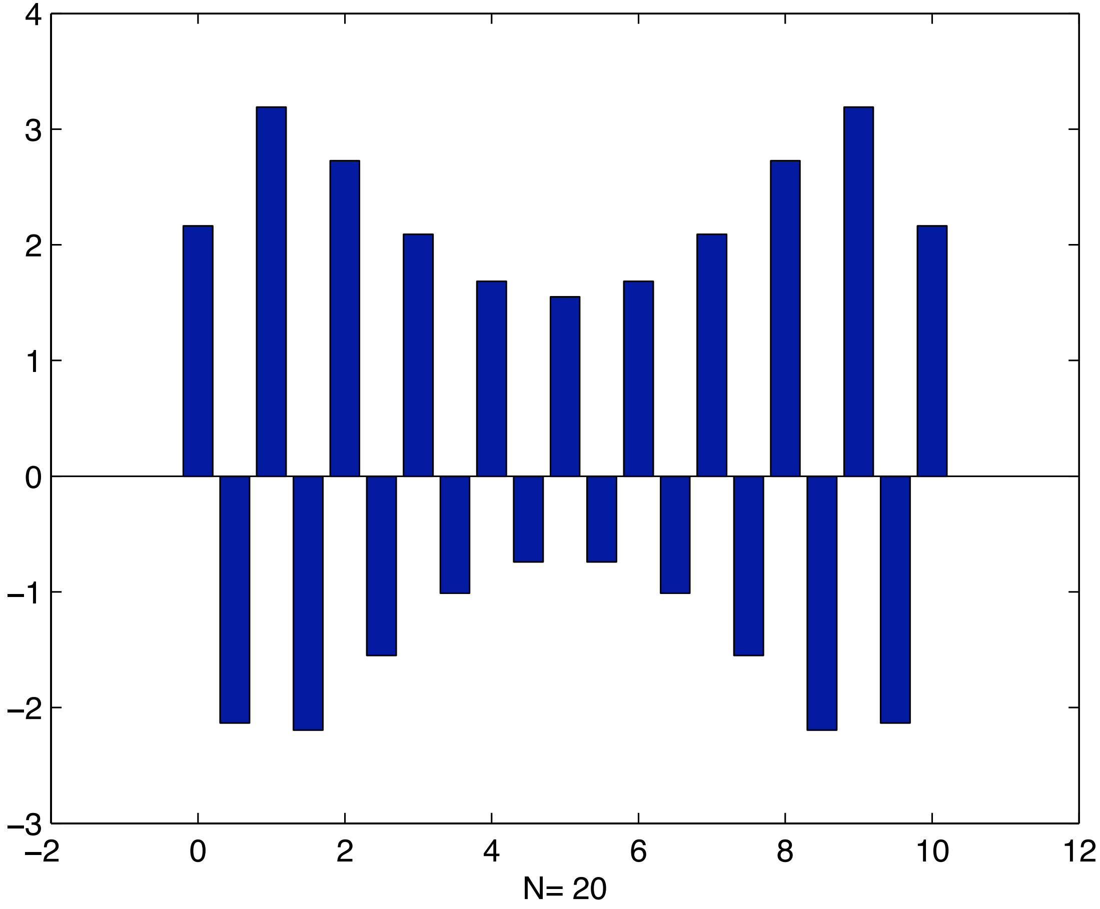
Gaussian resilience $\psi(t) = e^{-t^2}$, $N = 10$



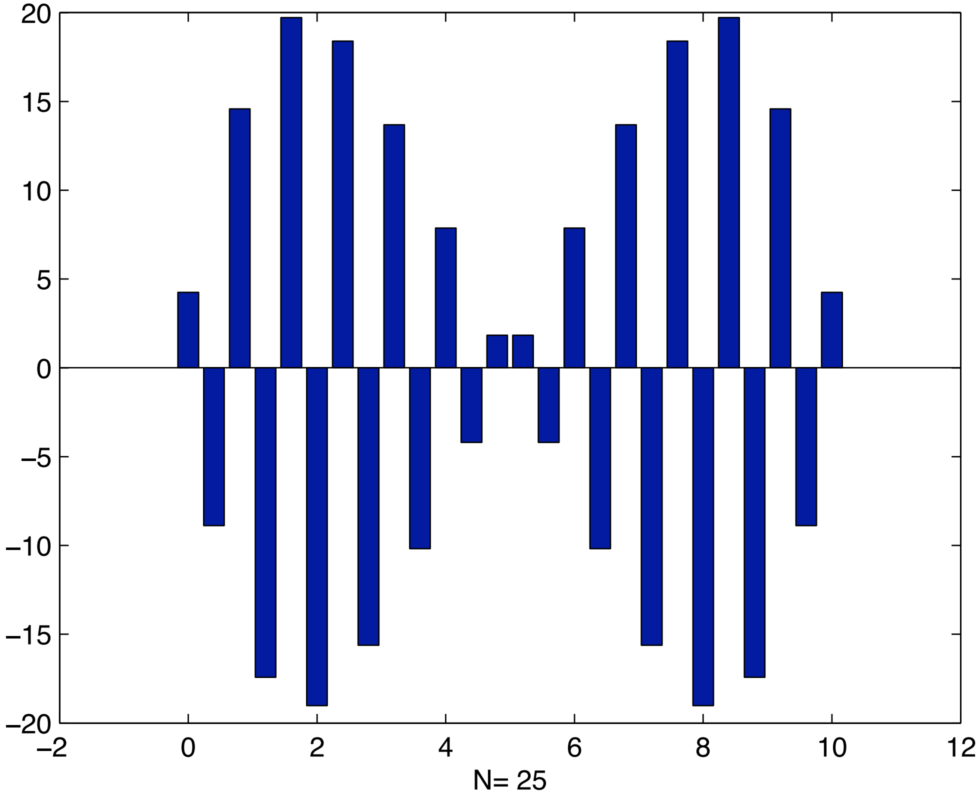
Gaussian resilience $\psi(t) = e^{-t^2}$, $N = 15$



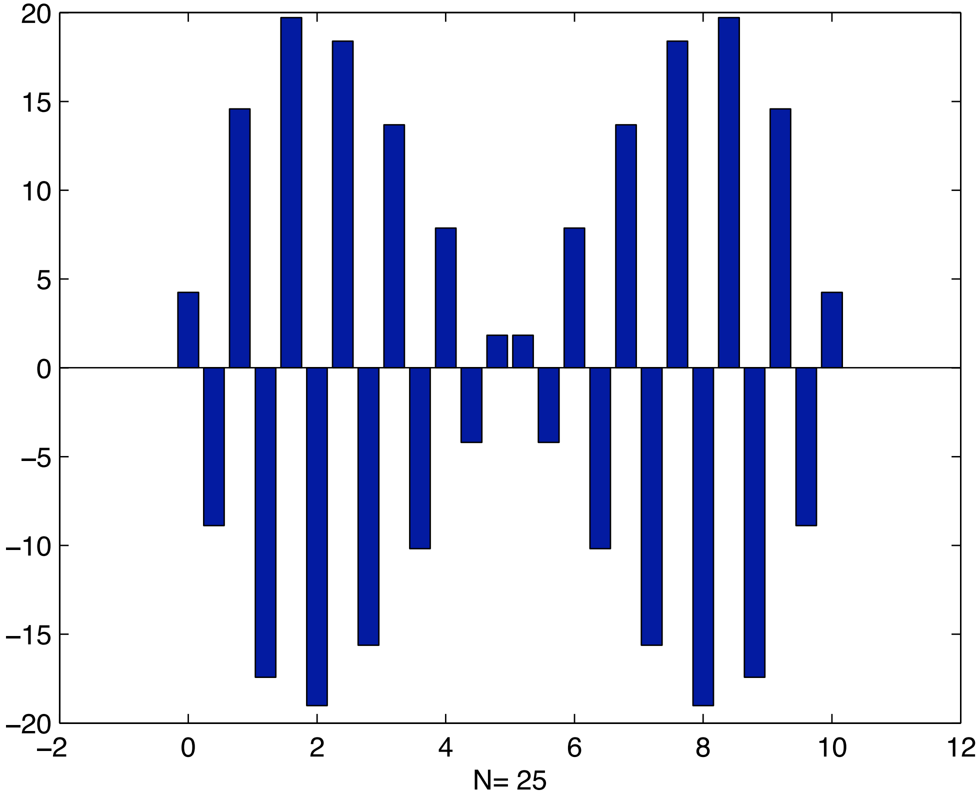
Gaussian resilience $\psi(t) = e^{-t^2}$, $N = 20$



Gaussian resilience $\psi(t) = e^{-t^2}$, $N = 25$



Gaussian resilience $\psi(t) = e^{-t^2}$, $N = 25$



$\Rightarrow$ absence of price manipulation strategies is not enough

Definition [Hubermann & Stanzl (2004)]
A market impact model admits

price manipulation strategies in the strong sense

if there is a round trip with negative expected liquidation costs.

Definition:
A market impact model admits

price manipulation strategies in the weak sense

if the expected liquidation costs of a sell (buy) program can be decreased by intermediate buy (sell) trades.

Question: When does the minimizer $\boldsymbol{x}^*$ of

$$\sum_{i,j} x_i x_j \psi(|t_i - t_j|) \qquad \text{with} \quad \sum_i x_i = X_0$$

have only nonnegative components?

Related to the *positive portfolio problem* in finance:
When are there no short sales in a Markowitz portfolio?

Partial results, e.g., by Gale (1960), Green (1986), Nielsen (1987)

Proposition 4. [Alfonsi, S., Slynko (2009)]
When ψ is strictly positive definite and trading times are equidistant, then

$$x_0^* > 0 \qquad \textit{and} \qquad x_N^* > 0.$$

Proof relies on Trench algorithm for inverting the Toeplitz matrix

$$M_{ij} = \psi(|i-j|/N), \qquad i, j = 0, \dots, N$$

Theorem 5. [Alfonsi, S., Slynko (2009)]

- *If ψ is convex then all components of $\boldsymbol{x}^*$ are nonnegative.*
- *If ψ is strictly convex, then all components are strictly positive.*
- *Conversely, $\boldsymbol{x}^*$ has negative components as soon as, e.g., ψ is strictly concave in a neighborhood of 0.*

Qualitative properties of optimal strategies?

Qualitative properties of optimal strategies?

Proposition 6. [Alfonsi, S., Slynko (2009)]
When ψ is convex and nonconstant, the optimal $\boldsymbol{x}^*$ satisfies

$$x_0^* \geq x_1^* \qquad \textit{and} \qquad x_{N-1}^* \leq x_N^*$$

Proof: Equating the first and second equations in $M\boldsymbol{x}^* = \lambda_0 \mathbf{1}$ gives

$$\sum_{j=0}^{N} x_j^* \psi(t_j) = \sum_{j=0}^{N} x_j^* \psi(|t_j - t_1|).$$

Thus,

$$\begin{aligned} x_0^* - x_1^* &= \sum_{j=0,\ j\neq 1}^{N} x_j^* \psi(|t_j - t_1|) - \sum_{j=1}^{N} x_j^* \psi(t_j) \\ &= x_0^* \psi(t_1) - x_1^* \psi(t_1) + \sum_{j=2}^{N} x_j^* \big[\psi(t_j - t_1) - \psi(t_j)\big] \\ &\geq (x_0^* - x_1^*)\psi(t_1), \end{aligned}$$

by convexity of ψ. Therefore

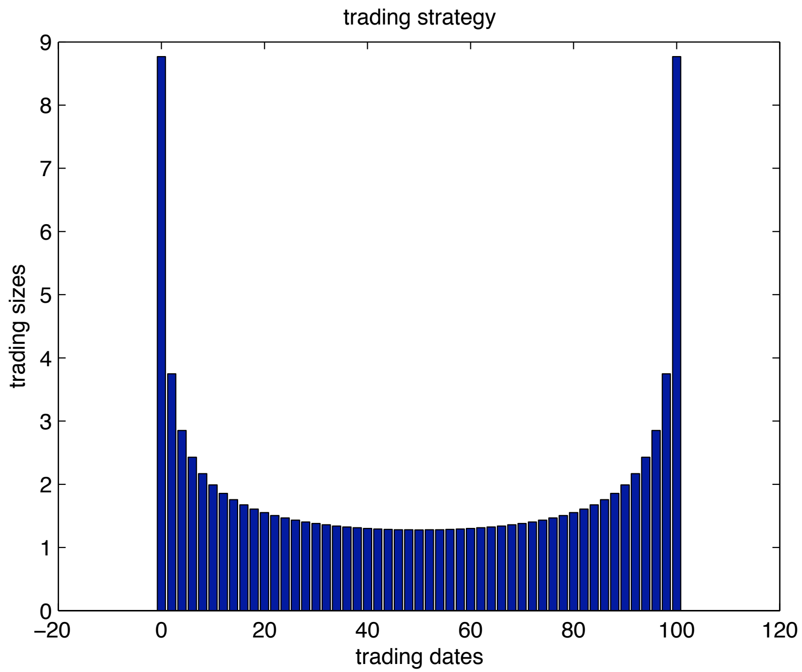
$$(x_0 - x_1)(1 - \psi(t_1)) \geq 0$$

□

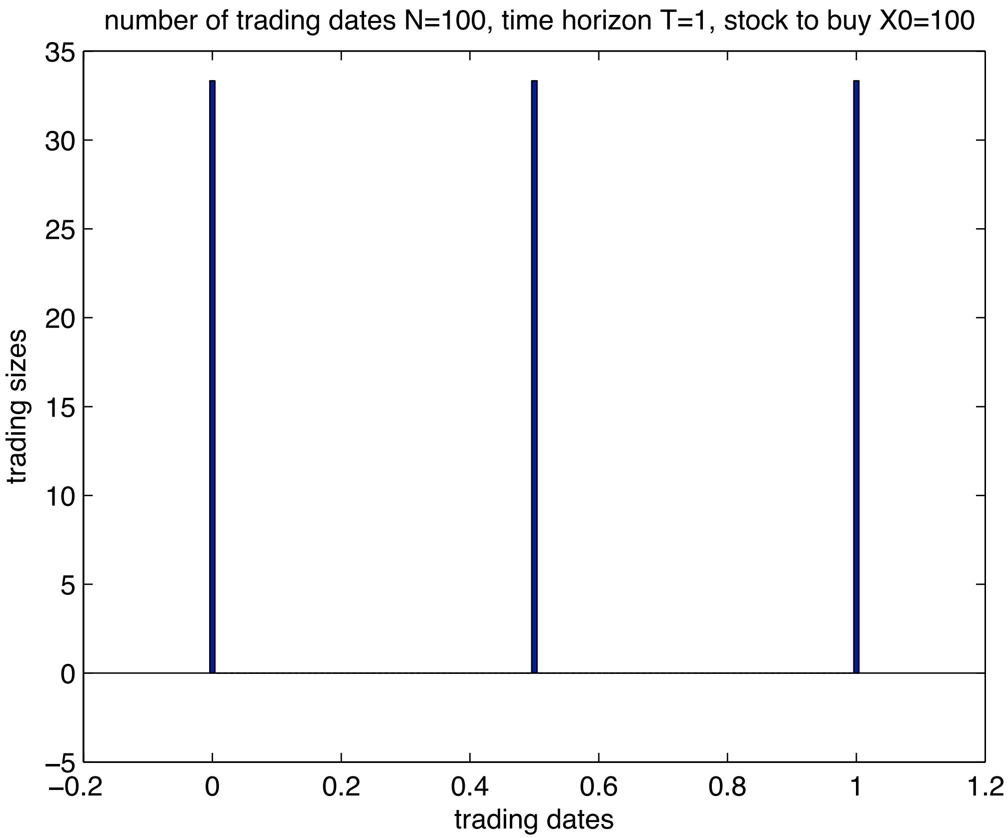
Proposition 6. [Alfonsi, S., Slynko (2009)]
When ψ is convex and nonconstant, the optimal $\boldsymbol{x}^*$ satisfies

$$x_0^* \geq x_1^* \qquad \textit{and} \qquad x_{N-1}^* \leq x_N^*$$

What about other trades? General pattern?



No! Capped linear resilience $\psi(t) = (1 - \rho t)^+, \rho = 2/T$

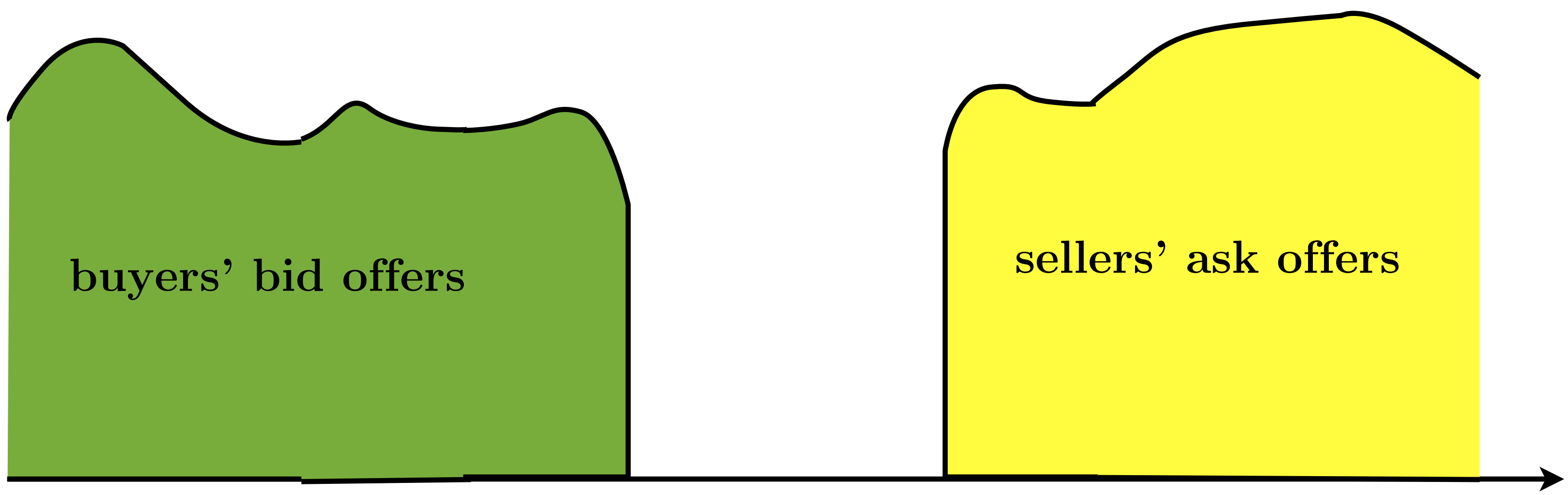


I. Order book models

1. Linear impact, general resilience

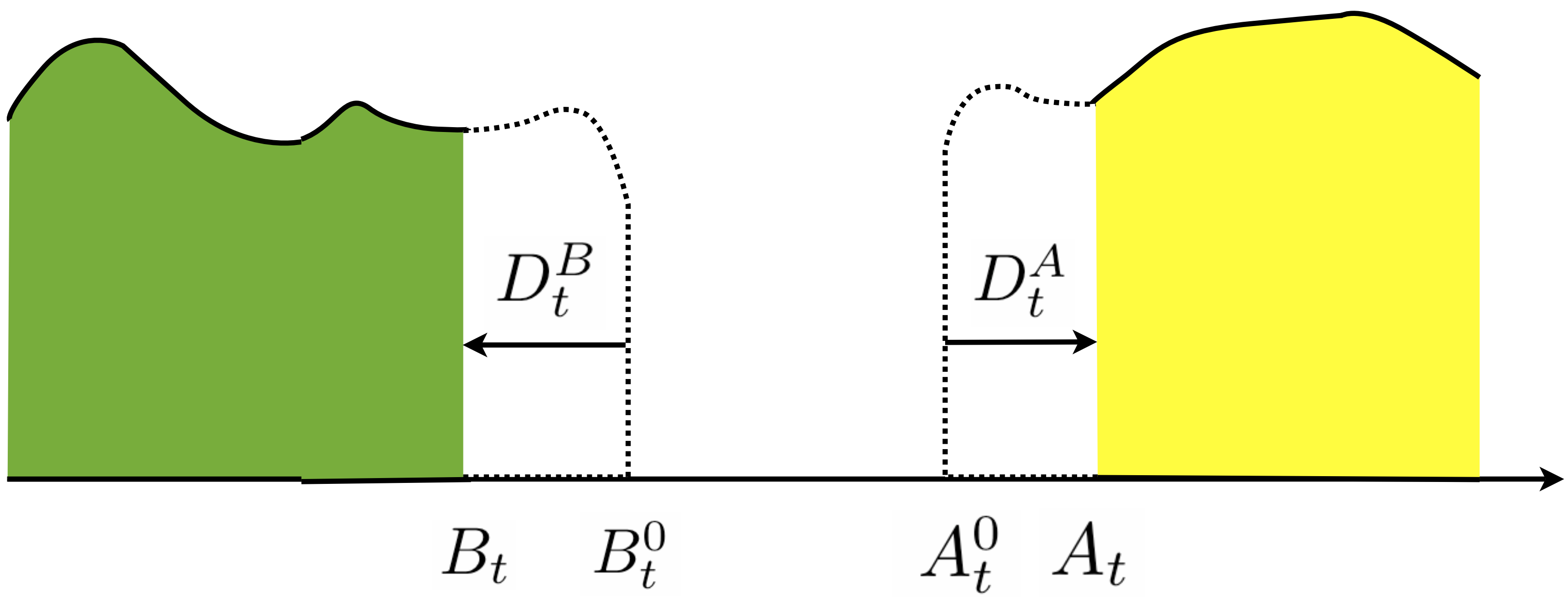
2. Nonlinear impact, exponential resilience

Limit order book model without large trader

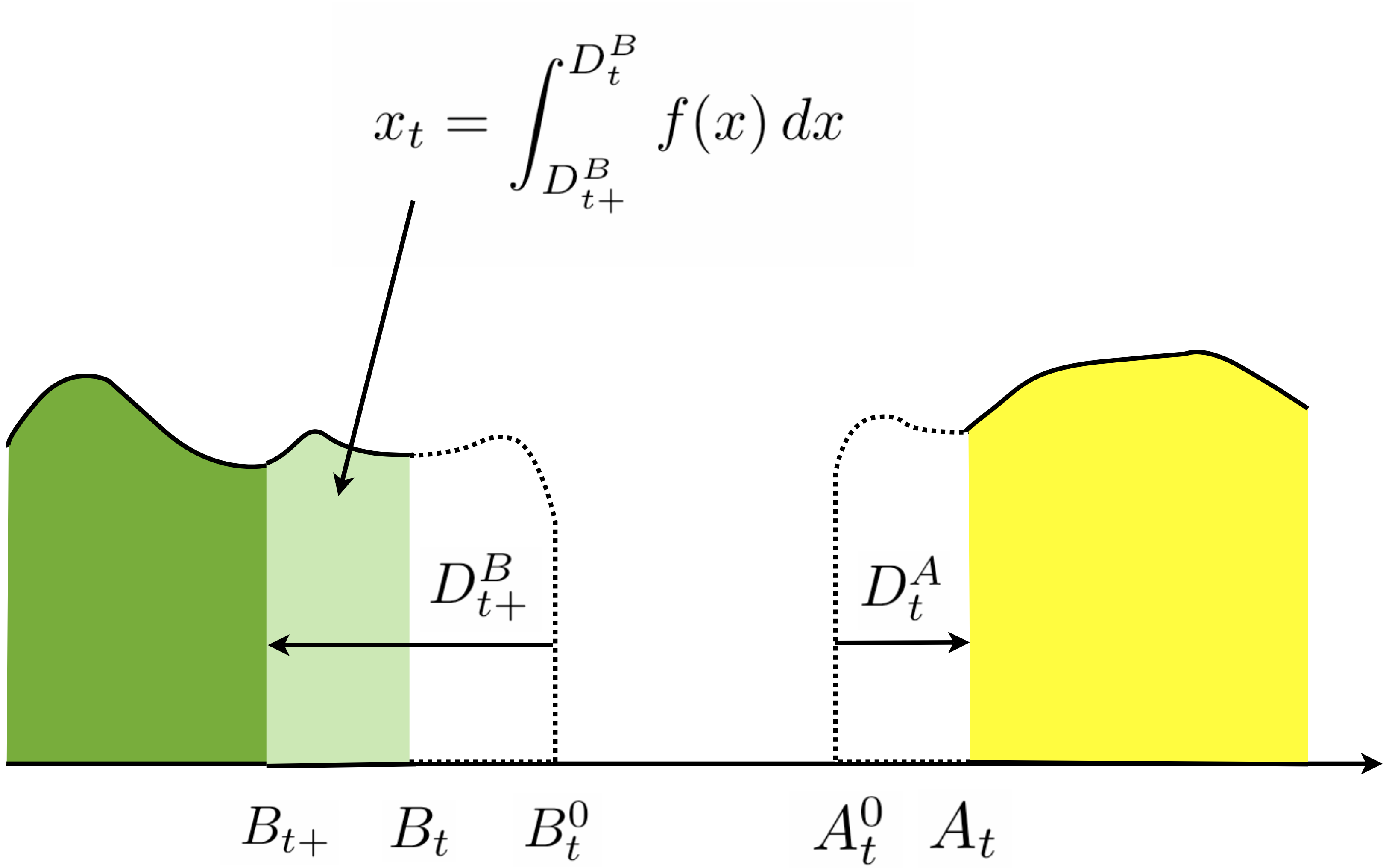


unaffected best bid price, B_t^0 A_t^0 unaffected best ask price
is martingale

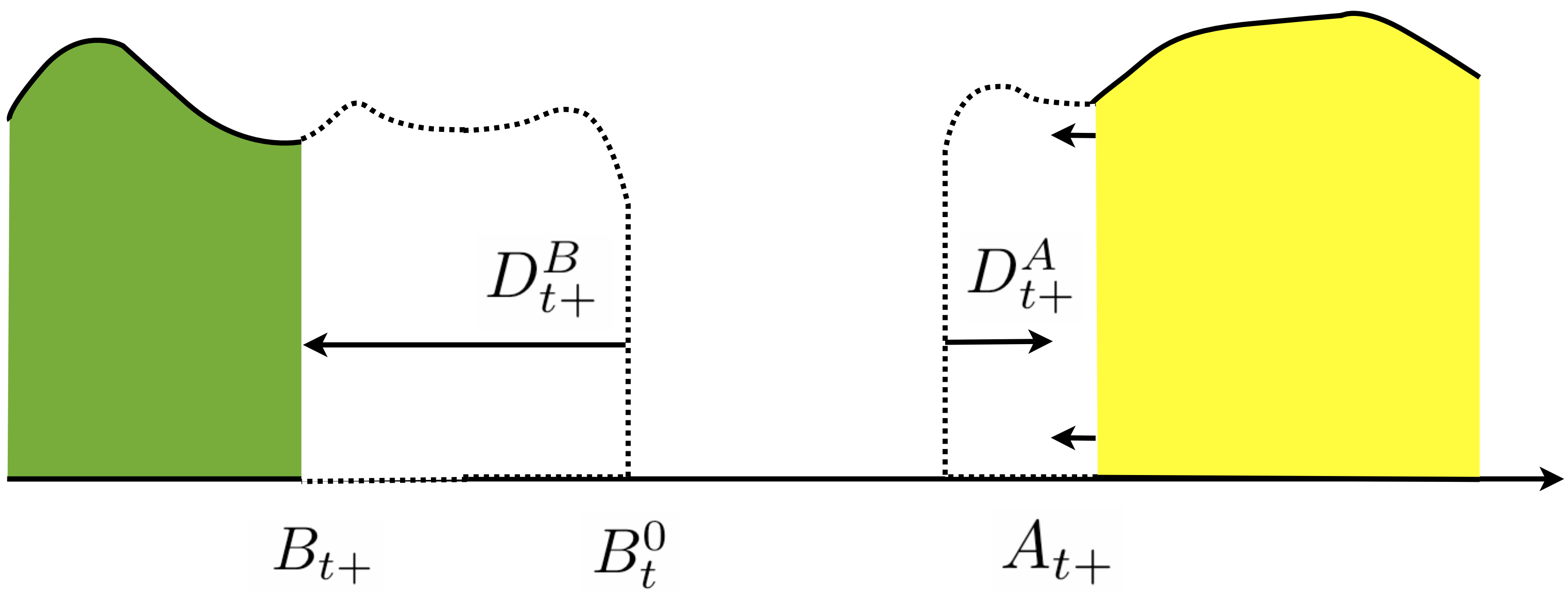
Limit order book model after large trades



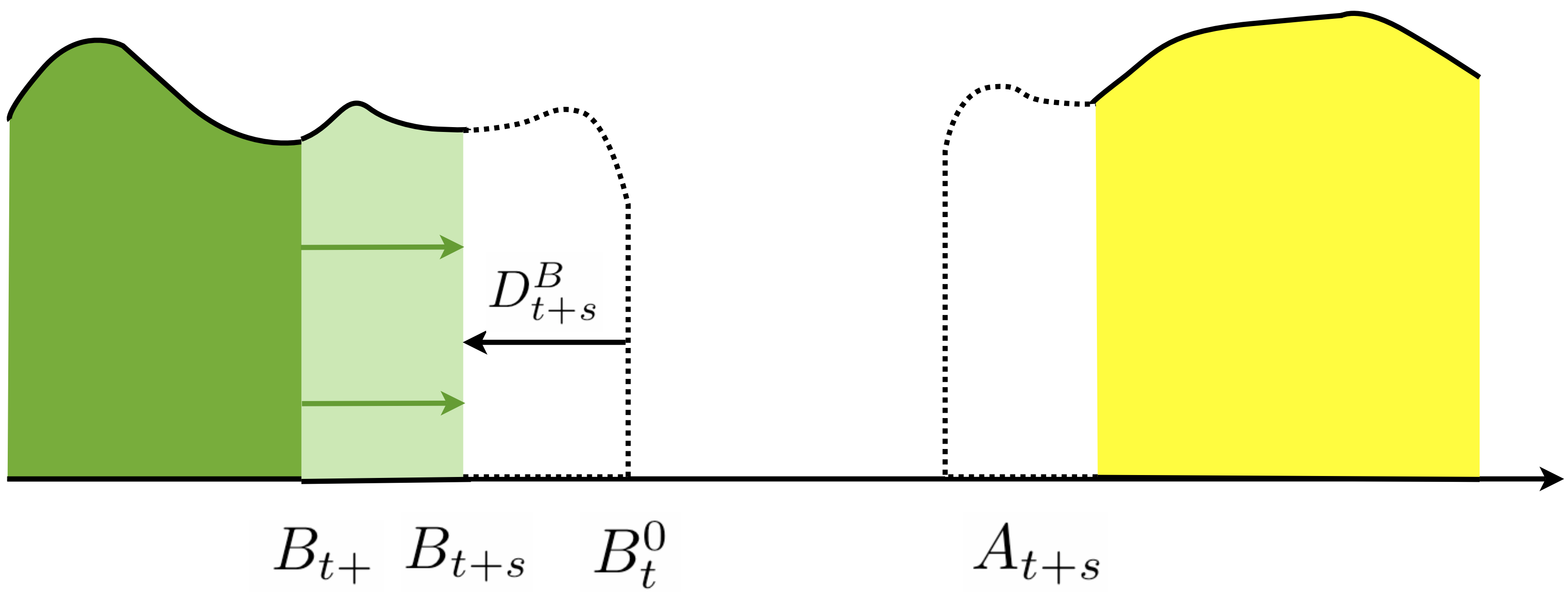
Limit order book model at large trade



Limit order book model immediately after large trade



Limit order book model with resilience



$f(x)$ = shape function = densities of bids for $x < 0$, asks for $x > 0$

B_t^0 = 'unaffected' bid price at time t, is martingale

B_t = bid price after market orders before time t

$D_t^B = B_t - B_t^0$

If sell order of $\xi_t \geq 0$ shares is placed at time t:

D_t^B changes to D_{t+}^B, where

$$\int_{D_t^B}^{D_{t+}^B} f(x)\,dx = -\xi_t$$

and

$$B_{t+} := B_t + D_{t+}^B - D_t^B = B_t^0 + D_{t+}^B,$$

$\Longrightarrow$ nonlinear price impact

A_t^0 = 'unaffected' ask price at time t, satisfies $B_t^0 \leq A_t^0$

A_t = bid price after market orders before time t

$D_t^A = A_t - A_t^0$

If buy order of $\xi_t \leq 0$ shares is placed at time t:

D_t^A changes to D_{t+}^A, where

$$\int_{D_t^A}^{D_{t+}^A} f(x)dx = -\xi_t$$

and

$$A_{t+} := A_t + D_{t+}^A - D_t^A = A_t^0 + D_{t+}^A,$$

For simplicity, we assume that the LOB has infinite depth, i.e., $|F(x)| \to \infty$ as $|x| \to \infty$, where

$$F(x) := \int_0^x f(y)\, dy.$$

If the large investor is inactive during the time interval $[t, t+s[$, there are *two* possibilities:

- **Exponential recovery of the extra spread**

$$D_t^B = e^{-\int_s^t \rho_r \, dr} D_s^B \qquad \text{for } s < t.$$

- **Exponential recovery of the order book volume**

$$E_t^B = e^{-\int_s^t \rho_r \, dr} E_s^B \qquad \text{for } s < t,$$

where

$$E_t^B = \int_{D_t^B}^0 f(x) \, dx =: F(D_t^B).$$

In both cases: analogous dynamics for D^A or E^A

Strategy:

$N+1$ market orders: ξ_n shares placed at time τ_n s.th.

a) the (τ_n) are stopping times s.th. $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_N = T$

b) ξ_n is $\mathcal{F}_{\tau_n}$-measurable and bounded from below,

c) we have $\displaystyle\sum_{n=0}^{N} \xi_n = X_0$

Will write

$$(\boldsymbol{\tau}, \boldsymbol{\xi})$$

and optimize jointly over $\boldsymbol{\tau}$ and $\boldsymbol{\xi}$.

- When selling $\xi_n > 0$ shares, we sell $f(x)\,dx$ shares at price $B^0_{\tau_n} + x$ with x ranging from $D^B_{\tau_n}$ to $D^B_{\tau_n+} < D^B_{\tau_n}$, i.e., the costs are negative:

$$c_n(\boldsymbol{\tau}, \boldsymbol{\xi}) := \int_{D^B_{\tau_n}}^{D^B_{\tau_n+}} (B^0_{\tau_n} + x) f(x)\,dx = -\xi_n B^0_{\tau_n} + \int_{D^B_{\tau_n}}^{D^B_{\tau_n+}} x f(x)\,dx$$

- When buying shares ($\xi_n < 0$), the costs are positive:

$$c_n(\boldsymbol{\tau}, \boldsymbol{\xi}) := -\xi_n A^0_{\tau_n} + \int_{D^A_{\tau_n}}^{D^A_{\tau_n+}} x f(x)\,dx$$

- The expected costs for the strategy $(\boldsymbol{\tau}, \boldsymbol{\xi})$ are

$$\mathcal{C}(\boldsymbol{\tau}, \boldsymbol{\xi}) = \mathbb{E}\Big[\sum_{n=0}^{N} c_n(\boldsymbol{\tau}, \boldsymbol{\xi})\Big]$$

Instead of the τ_k, we will use

$$\alpha_k := \int_{\tau_{k-1}}^{\tau_k} \rho_s ds, \qquad k = 1, \dots, N. \tag{2}$$

The condition $0 = \tau_0 \le \tau_1 \le \cdots \le \tau_N = T$ is equivalent to $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_N)$ belonging to

$$\mathcal{A} := \Big\{ \boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_N) \in \mathbb{R}_+^N \,\Big|\, \sum_{k=1}^N \alpha_k = \int_0^T \rho_s \, ds \Big\}.$$

A simplified model without bid-ask spread

S_t^0 = unaffected price, is (continuous) martingale.

$$S_{t_n} = S_{t_n}^0 + D_n$$

where D and E are defined as follows:

$$E_0 = D_0 = 0, \qquad E_n = F(D_n) \qquad \text{and} \qquad D_n = F^{-1}(E_n).$$

For $n = 0, \dots, N$, regardless of the sign of ξ_n,

$$E_{n+} = E_n - \xi_n \qquad \text{and} \qquad D_{n+} = F^{-1}(E_{n+}) = F^{-1}\left(F\left(D_n\right) - \xi_n\right).$$

For $k = 0, \dots, N-1$,

$$E_{k+1} = e^{-\alpha_{k+1}} E_{k+} = e^{-\alpha_{k+1}}(E_k - \xi_k)$$

The costs are

$$\overline{c}_n(\boldsymbol{\tau}, \boldsymbol{\xi}) = -\xi_n S_{\tau_n}^0 + \int_{D_{\tau_n}}^{D_{\tau_n+}} x f(x)\, dx$$

Lemma 7. *Suppose that $S^0 = B^0$. Then, for any strategy $\boldsymbol{\xi}$,*

$$\overline{c}_n(\boldsymbol{\xi}) \le c_n(\boldsymbol{\xi}) \qquad \textit{with equality if } \xi_k \ge 0 \textit{ for all } k.$$

Moreover,

$$\overline{\mathcal{C}}(\boldsymbol{\tau},\boldsymbol{\xi}) := \mathbb{E}\Big[\sum_{n=0}^{N} \overline{c}_n(\boldsymbol{\tau},\boldsymbol{\xi})\Big] = \mathbb{E}\Big[C(\boldsymbol{\alpha},\boldsymbol{\xi})\Big] - X_0 S_0^0$$

where

$$C(\boldsymbol{\alpha},\boldsymbol{\xi}) := \sum_{n=0}^{N} \int_{D_n}^{D_{n+}} x f(x)\,dx$$

is a deterministic function of $\boldsymbol{\alpha} \in \mathcal{A}$ and $\boldsymbol{\xi} \in \mathbb{R}^{N+1}$.

Implies that is is enough to minimize $C(\boldsymbol{\alpha},\boldsymbol{\xi})$ over $\boldsymbol{\alpha} \in \mathcal{A}$ and

$$\boldsymbol{\xi} \in \Big\{\boldsymbol{x} = (x_0,\dots,x_N) \in \mathbb{R}^{N+1} \,\Big|\, \sum_{n=0}^{N} x_n = X_0\Big\}.$$

Theorem 8. *Suppose f is increasing on $\mathbb{R}_-$ and decreasing on $\mathbb{R}_+$. Then there is a unique optimal strategy $(\boldsymbol{\xi}^*, \boldsymbol{\tau}^*)$ consisting of homogeneously spaced trading times,*

$$\int_{\tau_i^*}^{\tau_{i+1}^*} \rho_r \, dr = \frac{1}{N} \int_0^T \rho_r \, dr =: -\log a,$$

and trades defined via

$$F^{-1}\left(X_0 - N\xi_0^* \left(1 - a\right)\right) = \frac{F^{-1}(\xi_0^*) - aF^{-1}(a\xi_0^*)}{1 - a},$$

and

$$\xi_1^* = \cdots = \xi_{N-1}^* = \xi_0^* \left(1 - a\right),$$

as well as

$$\xi_N^* = X_0 - \xi_0^* - (N-1)\xi_0^* \left(1 - a\right).$$

Moreover, $\xi_i^* > 0$ for all i.

Taking $X_0 \downarrow 0$ yields:

Corollary 9. *Both the original and simplified models admit neither strong nor weak price manipulation strategies.*

$$f(x) = \frac{1}{1 + |x|}$$

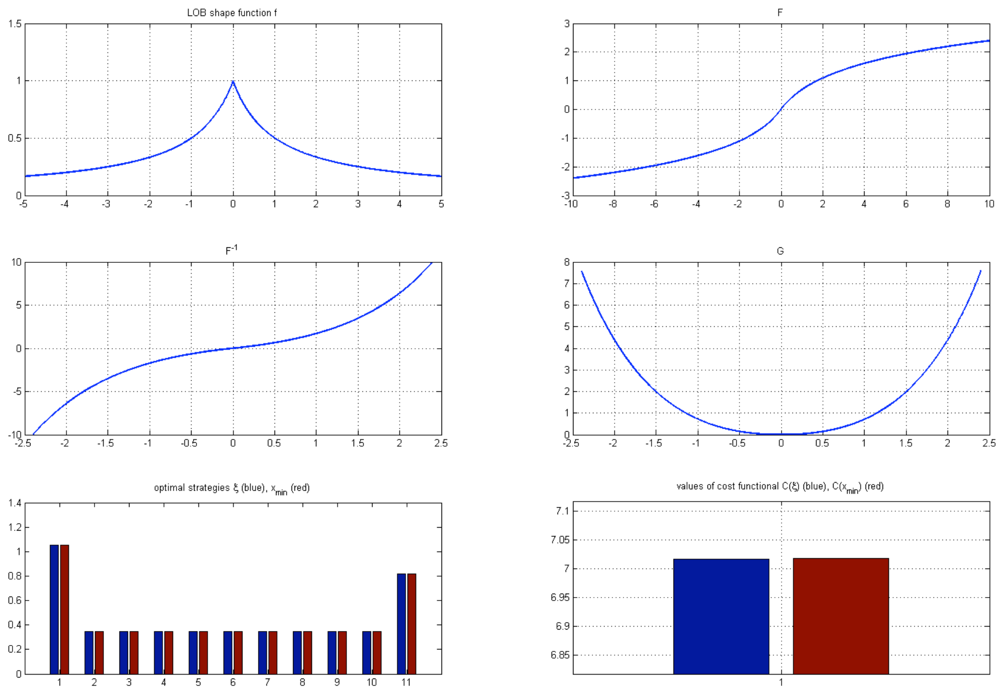


Figure 1: f, F, F^{-1}, G and optimal strategy

Strategy of proving Theorem 8:

(a) Show that there exists a (unique) minimizer $\boldsymbol{x}^*(\boldsymbol{\alpha})$ for each $\boldsymbol{\alpha}$.
(Prove that $C(\boldsymbol{\alpha},\boldsymbol{x}) \to \infty$ for $|\boldsymbol{x}| \to \infty$)

(b) Show that all components of $\boldsymbol{x}^*(\boldsymbol{\alpha})$ are positive
(Use that $\boldsymbol{x}^*(\boldsymbol{\alpha})$ must be a critical point of $\boldsymbol{x} \to C(\boldsymbol{\alpha},\boldsymbol{x}) - \nu \boldsymbol{x}^\top \mathbf{1}$ for some Lagrange multiplier ν. Then compute gradient of $C(\boldsymbol{\alpha},\cdot)$ and use explicit estimates....)

(c) By (a) and (b) we can restrict the optimization of $C(\boldsymbol{\alpha},\boldsymbol{x})$ to $(\boldsymbol{\alpha},\boldsymbol{x})$ belonging to the compact simplex

$$\mathcal{A} \times \Big\{ \boldsymbol{x} \in \mathbb{R}^{N+1} \,\Big|\, x_i \geq 0 \text{ and } \sum_{n=0}^{N} x_n = X_0 \Big\}.$$

Hence a minimizer $(\boldsymbol{\alpha}^*,\boldsymbol{x}^*)$ exists.

(d) Use again Lagrange multipliers to identify $(\boldsymbol{\alpha}^*,\boldsymbol{x}^*)$:

Let us introduce the functions

$$\tilde{F}(x) := \int_0^x z f(z)\,dz \qquad \text{and} \qquad G = \tilde{F} \circ F^{-1}.$$

Then, since $D_n = F^{-1}(E_n)$ and $D_{n+} = F^{-1}(E_{n+})$

$$\begin{aligned} C(\boldsymbol{\alpha}, \boldsymbol{x}) &= \sum_{n=0}^{N} \int_{D_n}^{D_{n+}} x f(x)\,dx = \sum_{n=0}^{N} \Big[\widetilde{F}(D_{n+}) - \widetilde{F}(D_n)\Big] \\ &= \sum_{n=0}^{N} \Big[G(E_{n+}) - G(E_n)\Big] = \sum_{n=0}^{N} \Big[G(E_n - x_n) - G(E_n)\Big] \end{aligned}$$

where

$$E_0 = 0 \qquad \text{and} \qquad E_n = -\sum_{i=0}^{n-1} x_i e^{-\sum_{k=i+1}^{n} \alpha_k}, \qquad 1 \le n \le N.$$

Lemma 10. *For $i = 0, \dots, N-1$, we have the following recursive formula,*

$$(3)\qquad \frac{\partial C}{\partial x_i} = e^{-\alpha_{i+1}} F^{-1}(E_{i+1}) - F^{-1}(E_i - x_i) + e^{-\alpha_{i+1}} \frac{\partial C}{\partial x_{i+1}}.$$

Moreover, for $i = 1, \dots, N$,

$$(4)\qquad \frac{\partial C}{\partial \alpha_i} = E_i \sum_{n=i}^{N} \big[F^{-1}(E_n - x_n) - F^{-1}(E_n) \big] e^{-\sum_{k=i+1}^{n} \alpha_k}.$$

When $(\boldsymbol{\alpha}, \boldsymbol{x})$ is a minimizer, then it is a critical point of

$$(\boldsymbol{\beta}, \boldsymbol{y}) \longmapsto C(\boldsymbol{\beta}, \boldsymbol{y}) - \nu \boldsymbol{y}^\top \mathbf{1} - \lambda \boldsymbol{\beta}^\top \mathbf{1}.$$

Hence

$$\frac{\partial C}{\partial x_i} = \nu \qquad \text{and} \qquad \frac{\partial C}{\partial \alpha_j} = \lambda \qquad \text{for all } i, j$$

Plugging this into (3) yields $\nu = -F^{-1}(E_N - x_N)$ and

$$\nu = e^{-\alpha_{i+1}} F^{-1}(E_{i+1}) - F^{-1}(E_i - x_i) + e^{-\alpha_{i+1}} \nu$$

or, since $E_{i+1} = e^{-\alpha_{i+1}}(E_i - x_i)$,

$$\nu = -\frac{F^{-1}(E_i - x_i) - a_{i+1} F^{-1}(a_{i+1}(E_i - x_i))}{1 - a_{i+1}}$$

where $a_{i+1} = e^{-\alpha_{i+1}}$.

Plugging this into (3) yields $\nu = -F^{-1}(E_N - x_N)$ and

$$\nu = e^{-\alpha_{i+1}} F^{-1}(E_{i+1}) - F^{-1}(E_i - x_i) + e^{-\alpha_{i+1}} \nu$$

or, since $E_{i+1} = e^{-\alpha_{i+1}}(E_i - x_i)$,

$$\nu = -\frac{F^{-1}(E_i - x_i) - a_{i+1} F^{-1}(a_{i+1}(E_i - x_i))}{1 - a_{i+1}}$$

where $a_{i+1} = e^{-\alpha_{i+1}}$.

Similarly,

$$\begin{aligned}
\frac{\lambda}{E_j} &= \sum_{n=j}^{N} \big[F^{-1}(E_n - x_n) - F^{-1}(E_n)\big] e^{-\sum_{k=j+1}^{n} \alpha_k} \\
&= -F^{-1}(E_j) + \big[F^{-1}(E_j - x_j) - F^{-1}(E_{j+1}) e^{-\alpha_{j+1}}\big] + \dots \\
&\quad + \big[F^{-1}(E_{N-1} - x_{N-1}) - F^{-1}(E_N) e^{-\alpha_N}\big] e^{-\sum_{k=j+1}^{N-1} \alpha_k} \\
&\quad + F^{-1}(E_N - x_N) e^{-\sum_{k=j+1}^{N} \alpha_k}
\end{aligned}$$

Plugging this into (3) yields $\nu = -F^{-1}(E_N - x_N)$ and

$$\nu = e^{-\alpha_{i+1}} F^{-1}(E_{i+1}) - F^{-1}(E_i - x_i) + e^{-\alpha_{i+1}}\nu$$

or, since $E_{i+1} = e^{-\alpha_{i+1}}(E_i - x_i)$,

$$\nu = -\frac{F^{-1}(E_i - x_i) - a_{i+1}F^{-1}(a_{i+1}(E_i - x_i))}{1 - a_{i+1}}$$

where $a_{i+1} = e^{-\alpha_{i+1}}$.

Similarly,

$$\begin{aligned}
\frac{\lambda}{E_j} &= \sum_{n=j}^{N} \Big[F^{-1}(E_n - x_n) - F^{-1}(E_n)\Big] e^{-\sum_{k=j+1}^{n}\alpha_k} \\
&= -F^{-1}(E_j) + \Big[F^{-1}(E_j - x_j) - F^{-1}(E_{j+1})e^{-\alpha_{j+1}}\Big] + \dots \\
&\quad + \Big[F^{-1}(E_{N-1} - x_{N-1}) - F^{-1}(E_N)e^{-\alpha_N}\Big] e^{-\sum_{k=j+1}^{N-1}\alpha_k} \\
&\quad + F^{-1}(E_N - x_N) e^{-\sum_{k=j+1}^{N}\alpha_k}
\end{aligned}$$

$$
\begin{aligned}
&= -F^{-1}(E_j) - (1-e^{-\alpha_{j+1}})\nu - \cdots - (1-e^{-\alpha_N})\nu e^{-\sum_{k=j+1}^{N-1}\alpha_k} \\
&\qquad -\nu e^{-\sum_{k=j+1}^{N}\alpha_k} \\
&= -F^{-1}(E_j) - \nu
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda &= -E_j(F^{-1}(E_j)+\nu) \\
&= E_j\left[\frac{F^{-1}(E_j-x_j) - a_{j+1}F^{-1}(a_{j+1}(E_j-x_j))}{1-a_{j+1}} - F^{-1}(E_j)\right]
\end{aligned}
$$

Altogether:

$$
\begin{aligned}
\nu &= -\frac{F^{-1}(E_{i-1}-x_{i-1}) - e^{-\alpha_i}F^{-1}(e^{-\alpha_i}(E_{i-1}-x_{i-1}))}{1-e^{-\alpha_i}}, \\
\lambda &= e^{-\alpha_i}(E_{i-1}-x_{i-1})\frac{F^{-1}(E_{i-1}-x_{i-1}) - F^{-1}(e^{-\alpha_i}(E_{i-1}-x_{i-1}))}{1-e^{-\alpha_i}},
\end{aligned}
$$

for $i = 1, \ldots, N$.

$$\begin{aligned}
\nu &= -\frac{F^{-1}(E_{i-1}-x_{i-1}) - e^{-\alpha_i}F^{-1}(e^{-\alpha_i}(E_{i-1}-x_{i-1}))}{1-e^{-\alpha_i}}, \\
\lambda &= e^{-\alpha_i}(E_{i-1}-x_{i-1})\frac{F^{-1}(E_{i-1}-x_{i-1}) - F^{-1}(e^{-\alpha_i}(E_{i-1}-x_{i-1}))}{1-e^{-\alpha_i}},
\end{aligned}$$

for $i = 1, \dots, N$.

Lemma 11. *Given ν and λ, these equations uniquely determine α_i and $E_{i-1} - x_{i-1}$*

It follows that

$$\alpha_1 = \cdots = \alpha_N \qquad \text{and} \qquad -x_0 = E_1 - x_1 = \cdots = E_{N-1} - x_{N-1}.$$

The theorem now follows easily. $\square$

Robustness of the optimal strategy
[Plots by C. Lorenz (2009)]

First figure:

$$f(x) = \frac{1}{1+|x|}$$

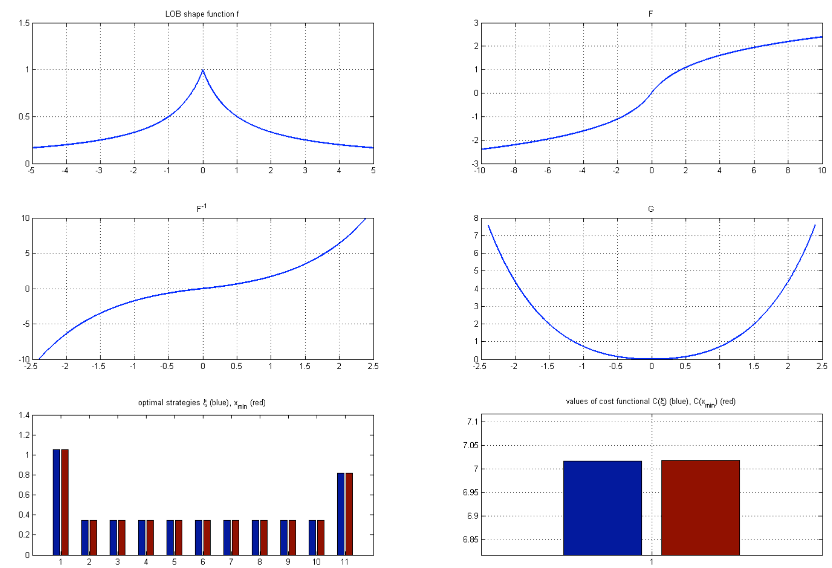


Figure 2: f, F, F^{-1}, G and optimal strategy

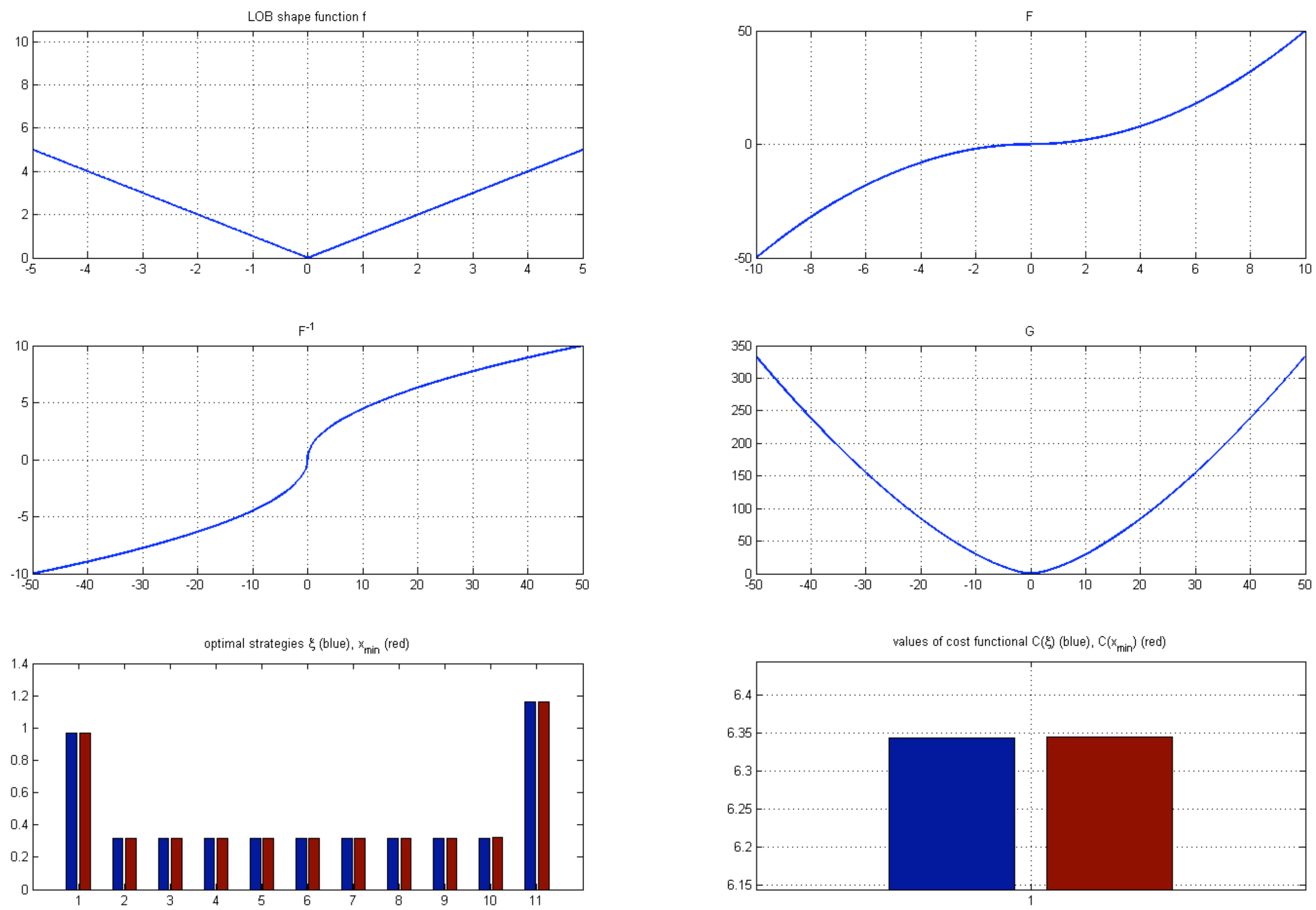

LOB shape function f
F
F^{-1}
G
optimal strategies ξ (blue), x_{min} (red)
values of cost functional C(ξ) (blue), C(x_{min}) (red)

Figure 3: $f(x) = |x|$

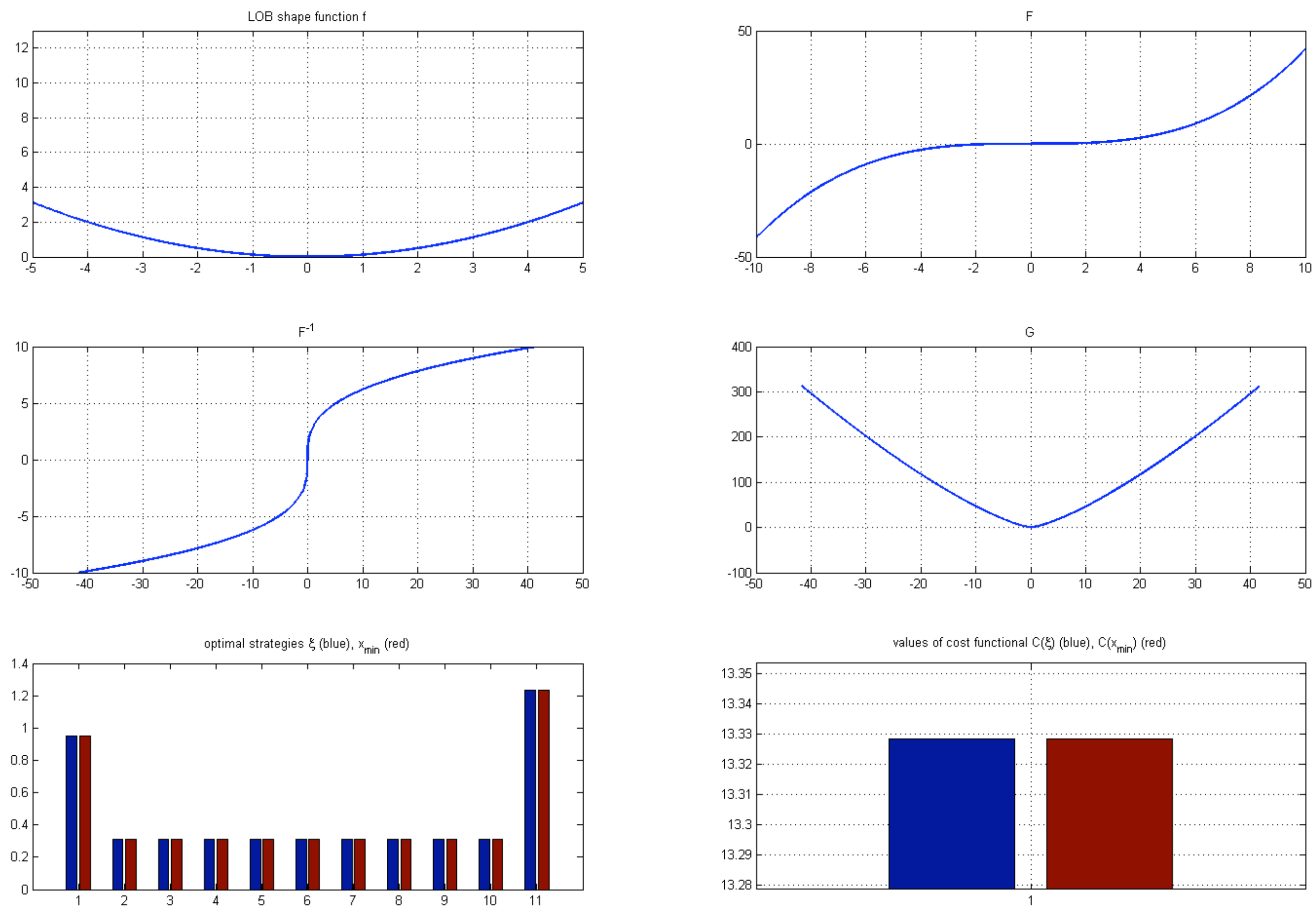


Figure 4: $f(x) = \frac{1}{8}x^2$

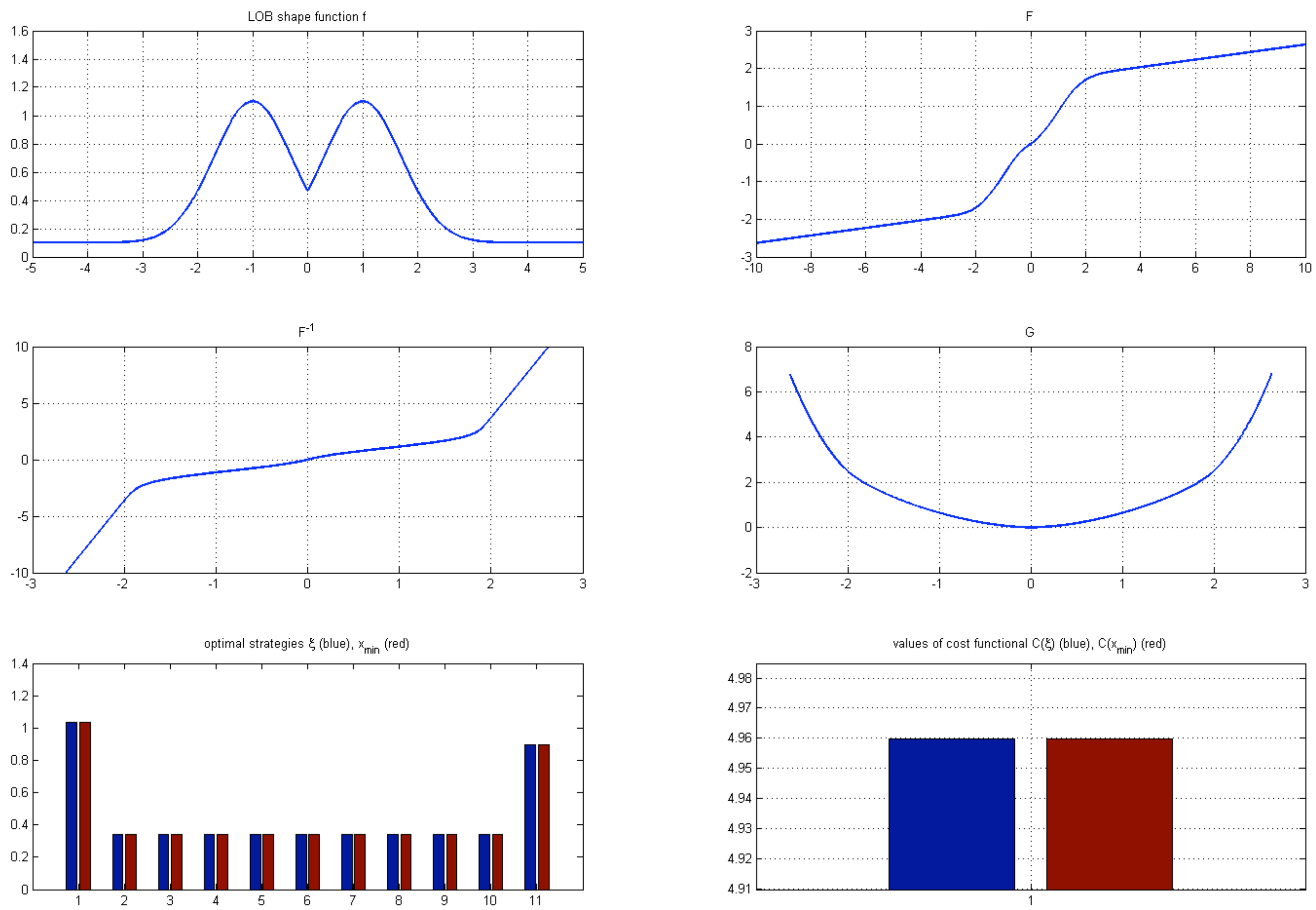


Figure 5: $f(x) = \exp(-(|x|-1)^2) + 0.1$

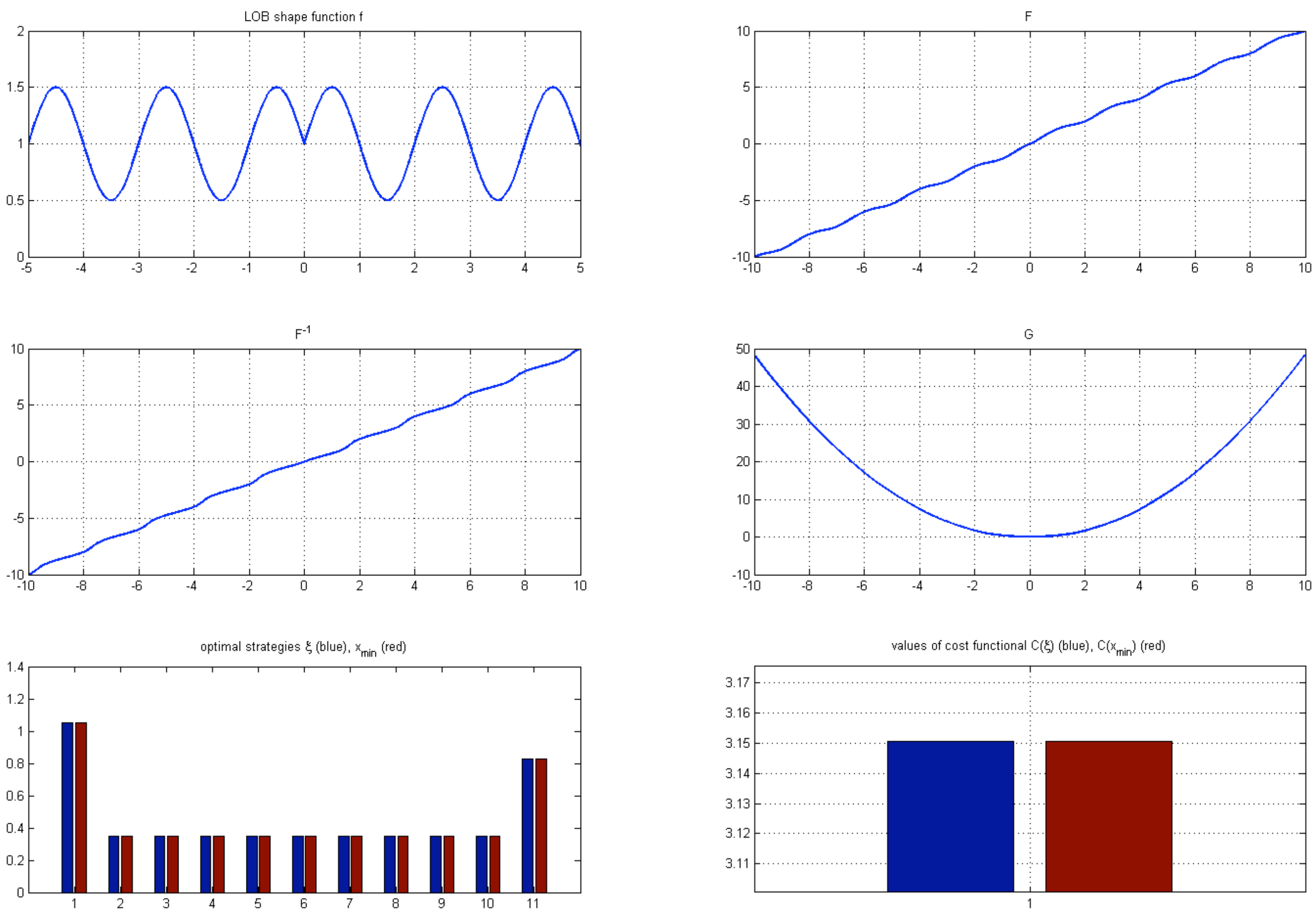


Figure 6: $f(x) = \frac{1}{2}\sin(\pi|x|) + 1$

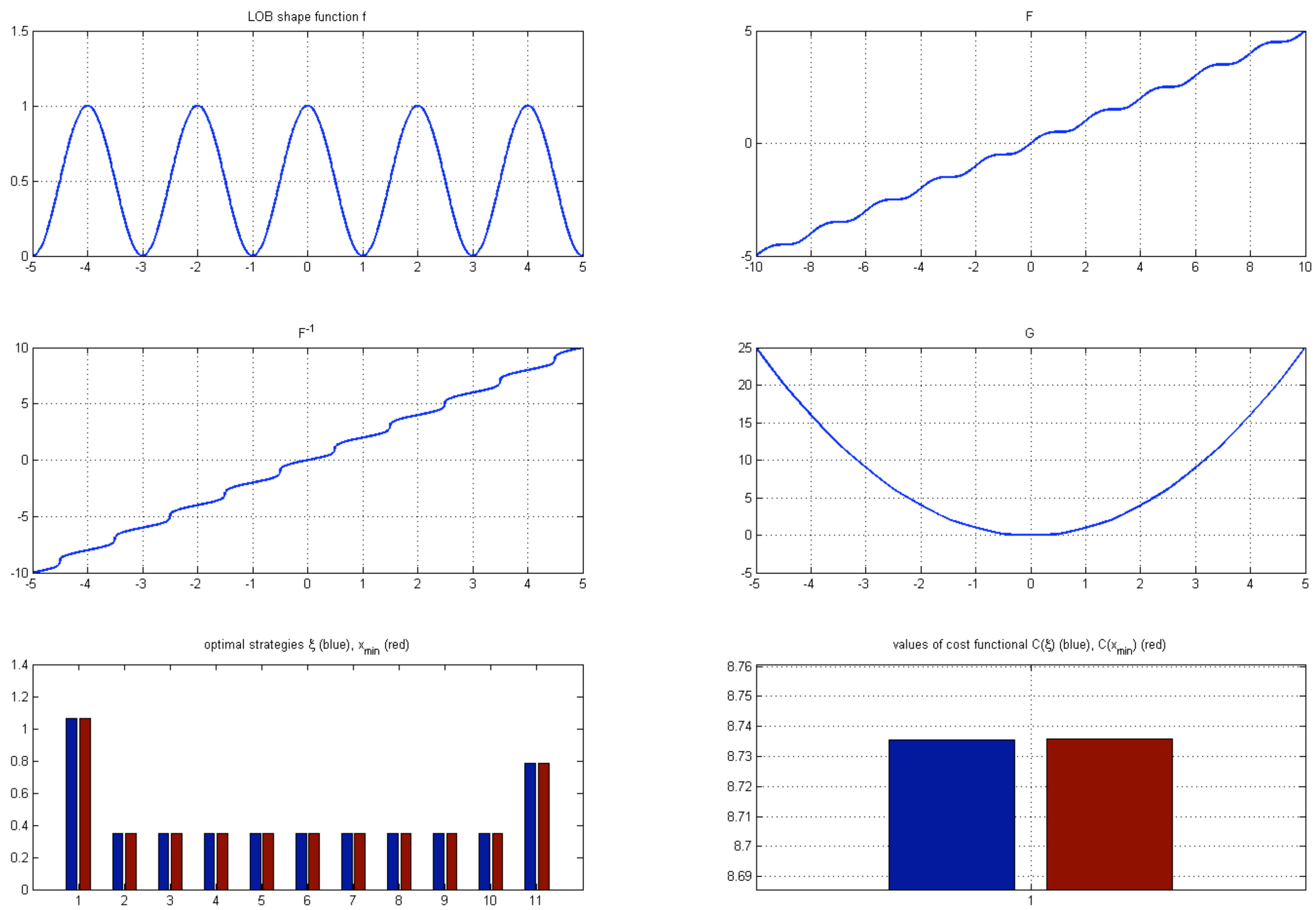


Figure 7: $f(x) = \frac{1}{2}\cos(\pi|x| + \frac{1}{2})$

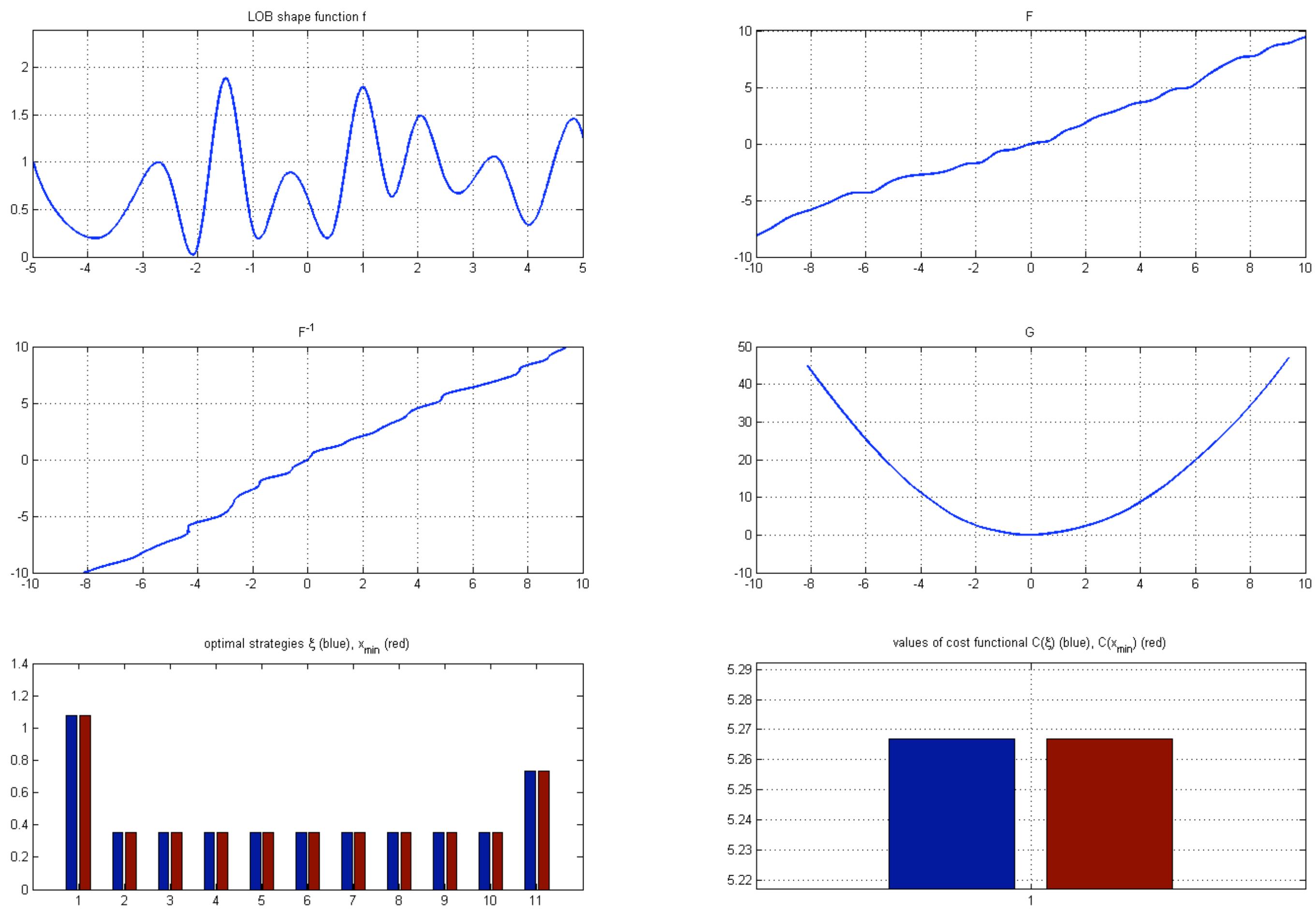


Figure 8: f random

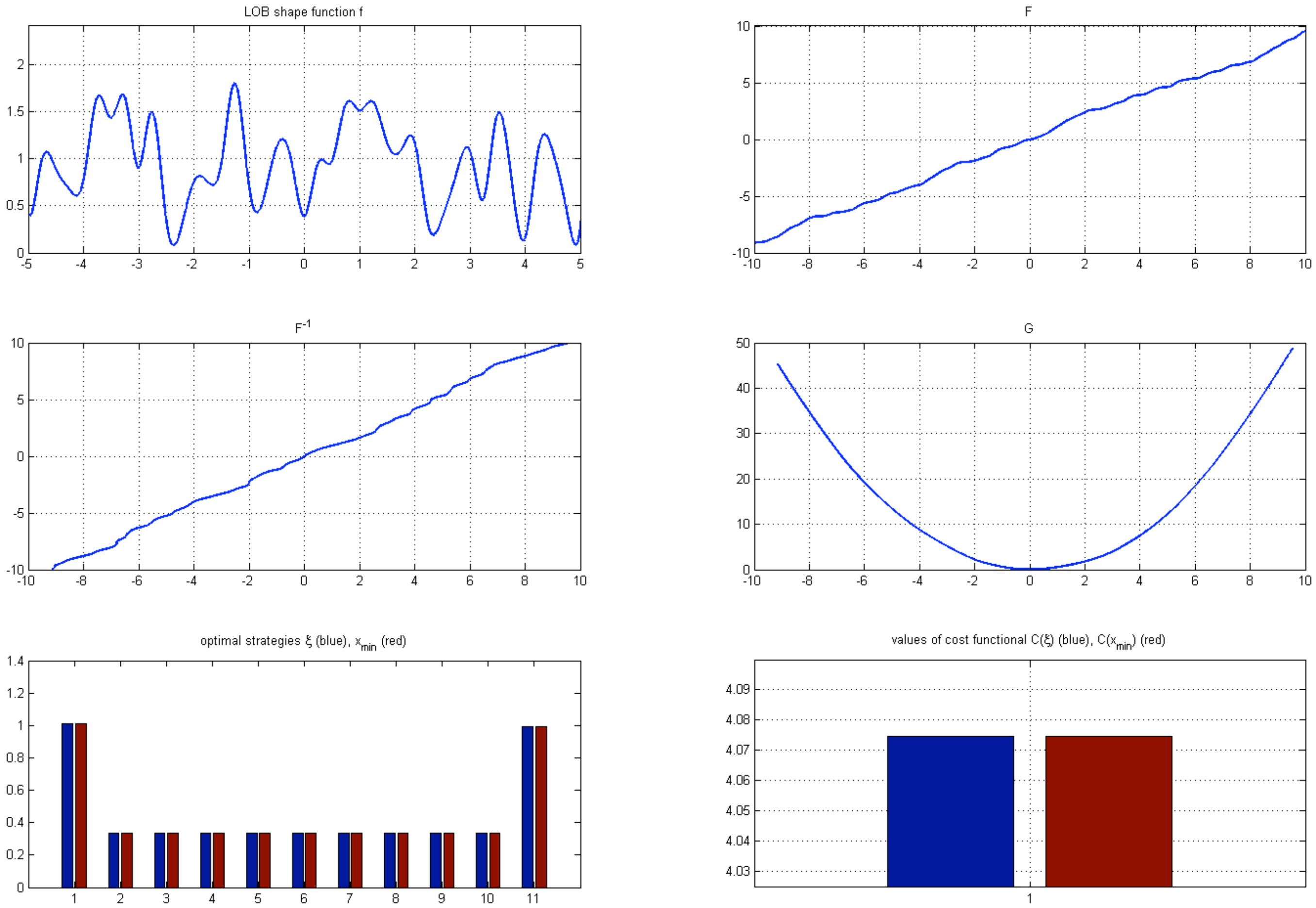


Figure 9: f random

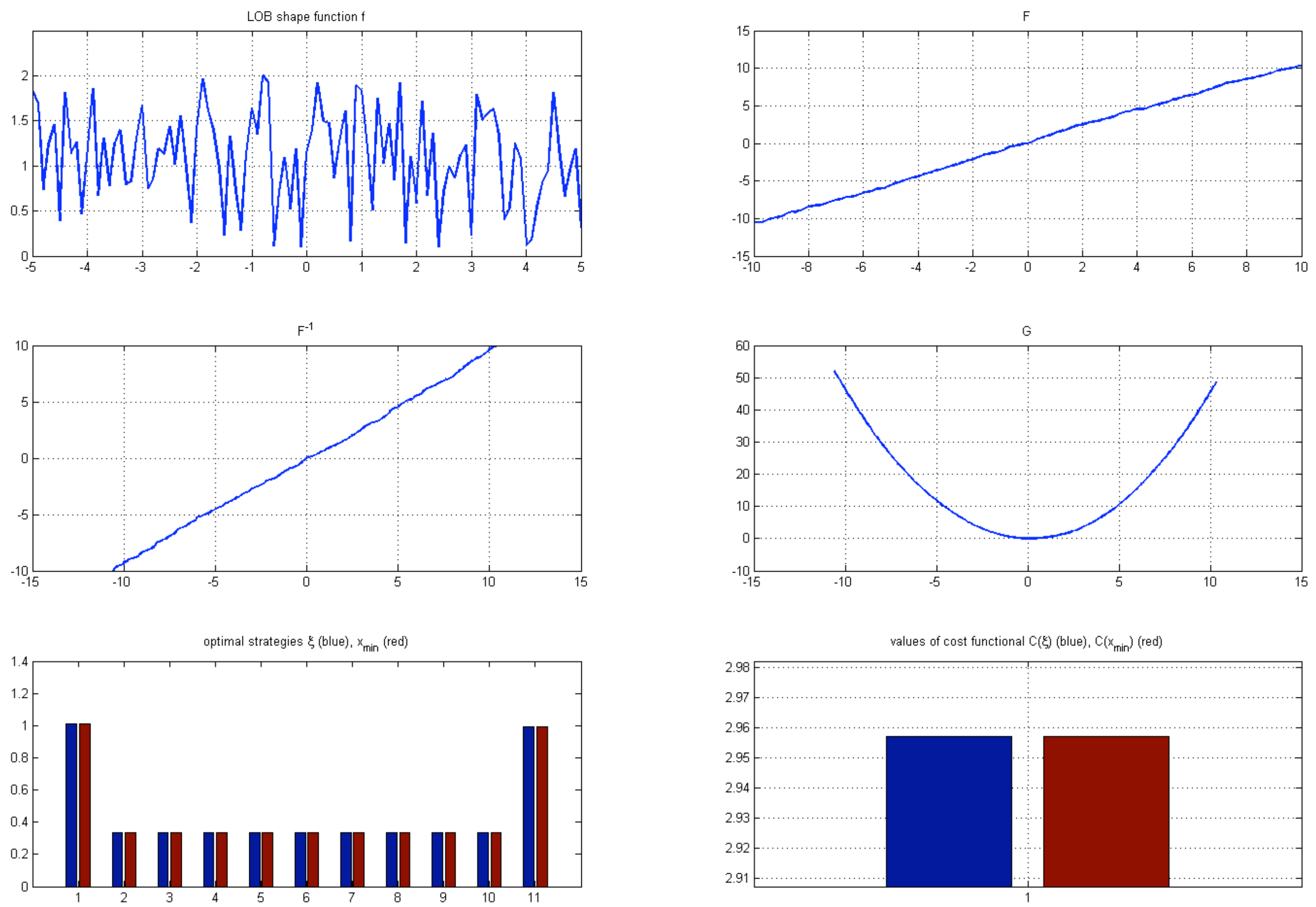


Figure 10: f random

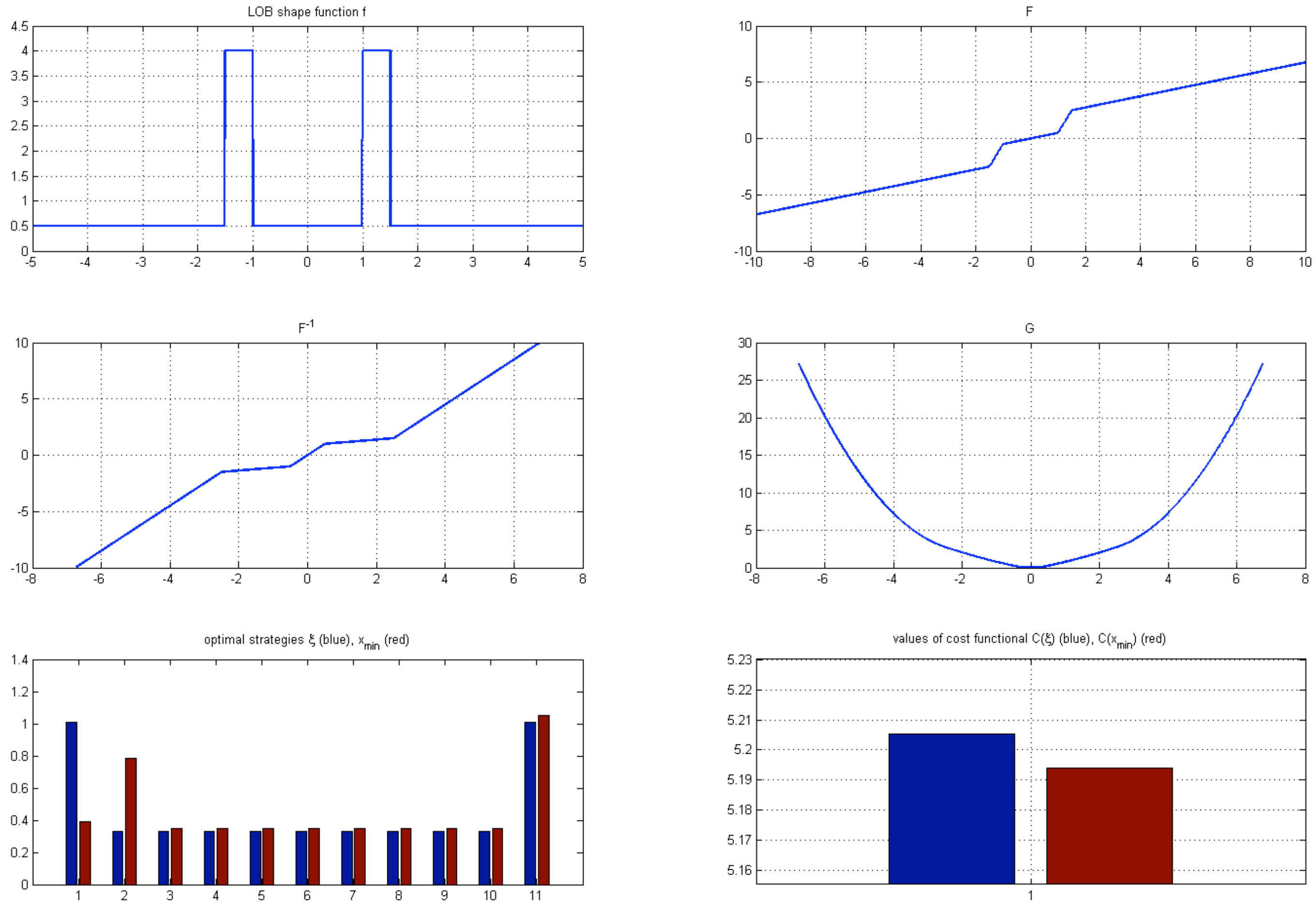


Figure 11: f piecewise constant

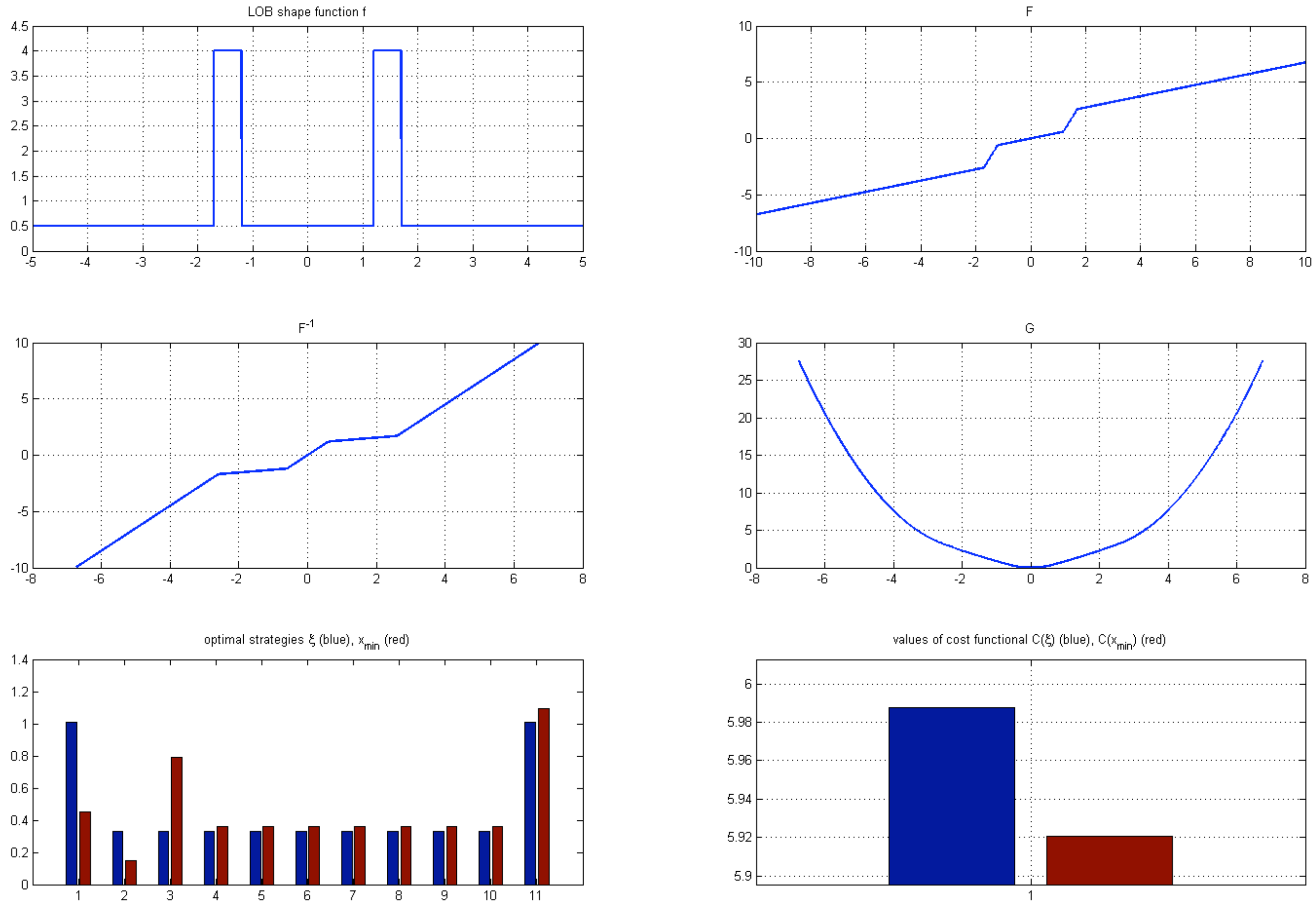


Figure 12: f piecewise constant

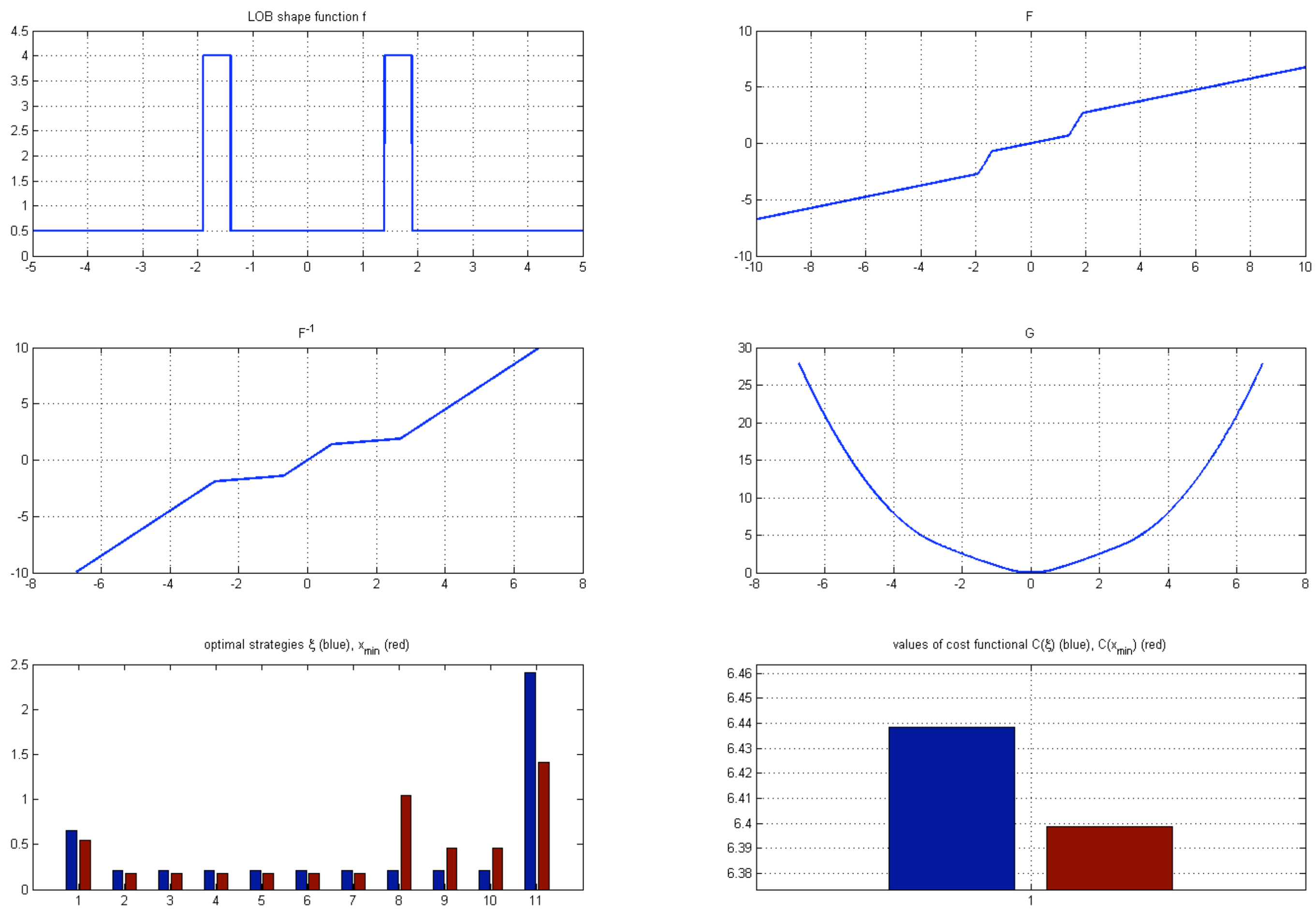


Figure 13: f piecewise constant

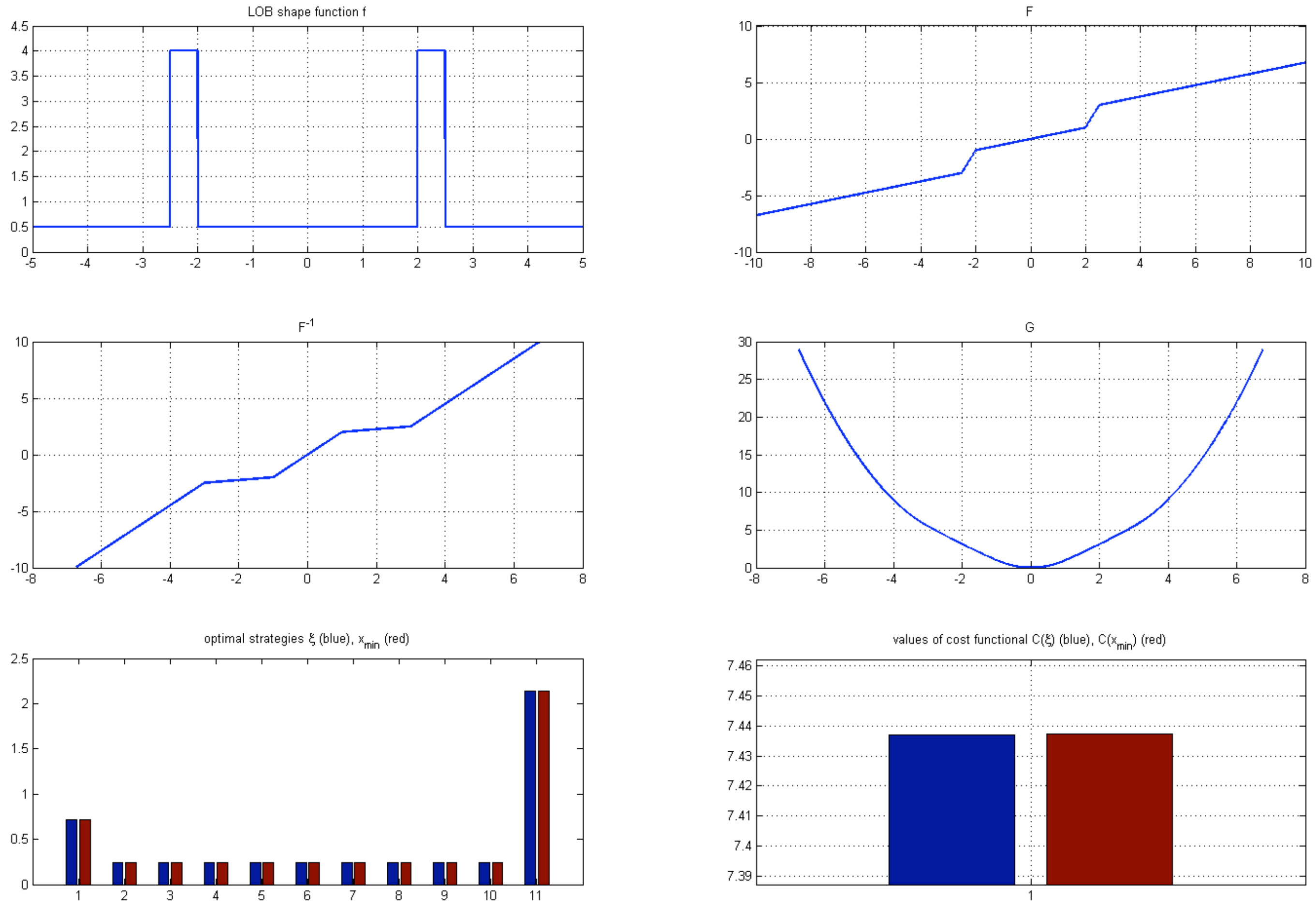


Figure 14: f piecewise constant

Continuous-time limit of the optimal strategy

- Initial block trade of size ξ_0^*, where

$$F^{-1}\Big(X_0 - \xi_0^* \int_0^T \rho_s\,ds\Big) = F^{-1}(\xi_0^*) + \frac{\xi_0^*}{f(F^{-1}(\xi_0^*))}$$

- Continuous trading in $]0, T[$ at rate

$$\xi_t^* = \rho_t \xi_0^*$$

- Terminal block trade of size

$$\xi_T^* = X_0 - \xi_0^* - \xi_0^* \int_0^T \rho_t\,dt$$

I. Order book models

1. Linear impact, general resilience

2. Nonlinear impact, exponential resilience

3. Gatheral's model

Liquidation time: $T \geq 0$.

Strategy: X adapted with $X_0 > 0$ fixed and $X_T = 0$.
Admissible: X_t bounded, absolutely continuous in t.

Liquidation time: $T \geq 0$.

Strategy: X adapted with $X_0 > 0$ fixed and $X_T = 0$.
Admissible: X_t bounded, absolutely continuous in t.

Note: These strategies are of bounded variation.
So there will be no liquidation costs in many of the models from Peter Bank's course

Liquidation time: $T \geq 0$.

Strategy: X adapted with $X_0 > 0$ fixed and $X_T = 0$.
Admissible: X_t bounded, absolutely continuous in t.

Market impact model: S^0 unaffected price, = martingale

$$S_t = S_t^0 + \int_0^t h(-\dot{X}_t)\psi(t-s)\,ds$$

- For $h(x) = \lambda x$ continuous-time version of simplified model in I.1.
- For nonlinear h close to continuous-time version of simplified model in I.2.
- $\psi \equiv const$ corresponds to purely permanent impact
- $\psi(t-s) = \delta(t-s)$ corresponds to purely temporary impact
- **Almgren-Chriss model:** (studied in next lectures)

$$\psi(t-s) = \lambda\delta(t-s) + \gamma$$

Costs:

$\dot{X}_t\,dt$ shares are sold at price $S_t \Rightarrow$ infinitesimal costs $= -\dot{X}_t S_t\,dt$

$$\begin{aligned}\text{Total costs} &= -\int_0^T \dot{X}_t S_t\,dt \\ &= -\int_0^T \dot{X}_t S_t^0\,dt + \int_0^T \int_0^t (-\dot{X}_t) h(-\dot{X}_s)\psi(t-s)\,ds\,dt\end{aligned}$$

Letting $\xi_t := -\dot{X}_t$, we get

$$\text{Expected costs} = -X_0 S_0^0 + \mathbb{E}\Big[\int_0^T \int_0^t \xi_t h(\xi_s)\psi(t-s)\,ds\,dt\Big]$$

Remark: Model formulation is not complete since optimal strategies typically will not be absolutely continous (see continous-time limit in preceding section)

Are there price manipulation strategies?

Find $\xi \in L^2[0,T]$ such that

$$\int_0^T \int_0^t \xi_t h(\xi_s)\psi(t-s)\,ds\,dt < 0.$$

For linear impact $h(x) = x$: **Bochner-Schwartz theorem**

Theorem 12. [Gatheral (2008)]
Suppose that

$$\psi(t) = e^{-\rho t}$$

and market impact is not linear. Then the model admits price manipulation strategies in the strong sense.

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Very puzzling result in view of Corollary 9!

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Suppose that

$$\psi(t) = e^{-\rho t}$$

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Very puzzling result in view of Corollary 9!

The resolution of this paradox is surprising ... stay tuned.

Theorem 12. [Gatheral (2008)]
Suppose that

$$\psi(t) = e^{-\rho t}$$

and market impact is not linear. Then the model admits price manipulation strategies in the strong sense.

Taking $\rho \downarrow 0$ yields:

Corollary 13. [Huberman & Stanzl (2004)]
Suppose that market impact is permanent and nonlinear. Then the model admits price manipulation strategies in the strong sense.

Sketch of proof of Theorem 12: For simplicity assume

$$h(-x) = -h(x)$$

Consider a strategy of the form

$$\xi_t = v_1 \text{ for } 0 \le t \le T_0 \text{ and } \xi_t = -v_2 \text{ for } T_0 < t \le T.$$

'Round trip' requires that

$$v_1 T_0 = v_2 (T - T_0)$$

A calculation yields that for this specific strategy

$$\int_0^T \int_0^t \xi_t h(\xi_s) \psi(t-s) \, ds \, dt = \cdots$$

$$\begin{aligned}\cdots \quad = \quad & v_1 h(v_1)\Big(e^{-\frac{v_2\rho T}{v_1+v_2}} - 1 + \frac{v_2\rho T}{v_1+v_2}\Big) + v_2 h(v_2)\Big(e^{-\frac{v_1\rho T}{v_1+v_2}} - 1 + \frac{v_1\rho T}{v_1+v_2}\Big)\\ & -v_2 h(v_1)\Big(1 + e^{-\rho T} - e^{-\frac{v_2\rho T}{v_1+v_2}} - e^{-\frac{v_1\rho T}{v_1+v_2}}\Big)\end{aligned}$$

$$
\begin{aligned}
\cdots \quad &= \quad v_1 h(v_1)\Big(e^{-\frac{v_2\rho T}{v_1+v_2}} - 1 + \frac{v_2\rho T}{v_1+v_2}\Big) + v_2 h(v_2)\Big(e^{-\frac{v_1\rho T}{v_1+v_2}} - 1 + \frac{v_1\rho T}{v_1+v_2}\Big) \\
&\qquad\qquad -v_2 h(v_1)\Big(1 + e^{-\rho T} - e^{-\frac{v_2\rho T}{v_1+v_2}} - e^{-\frac{v_1\rho T}{v_1+v_2}}\Big) \\
&\approx \quad \frac{v_1 v_2\big[v_1 h(v_2) - v_2 h(v_1)\big](\rho T)^2}{2(v_1+v_2)^2} + O((\rho T)^3) \qquad \text{for } \rho T \to 0
\end{aligned}
$$

$$\begin{aligned}\cdots &= v_1 h(v_1)\Big(e^{-\frac{v_2\rho T}{v_1+v_2}} - 1 + \frac{v_2\rho T}{v_1+v_2}\Big) + v_2 h(v_2)\Big(e^{-\frac{v_1\rho T}{v_1+v_2}} - 1 + \frac{v_1\rho T}{v_1+v_2}\Big)\\ &\qquad - v_2 h(v_1)\Big(1 + e^{-\rho T} - e^{-\frac{v_2\rho T}{v_1+v_2}} - e^{-\frac{v_1\rho T}{v_1+v_2}}\Big)\\ &\approx \frac{v_1 v_2\big[v_1 h(v_2) - v_2 h(v_1)\big](\rho T)^2}{2(v_1+v_2)^2} + O((\rho T)^3) \qquad \text{for } \rho T \to 0\end{aligned}$$

Can always choose v_1, v_2 such that $[\dots] < 0$, then take T such that ρT small enough. $\square$

More econo-physics:

$\psi(t) = t^{-\gamma}$, $h(v) = v^{\delta}$

Gatheral finds that

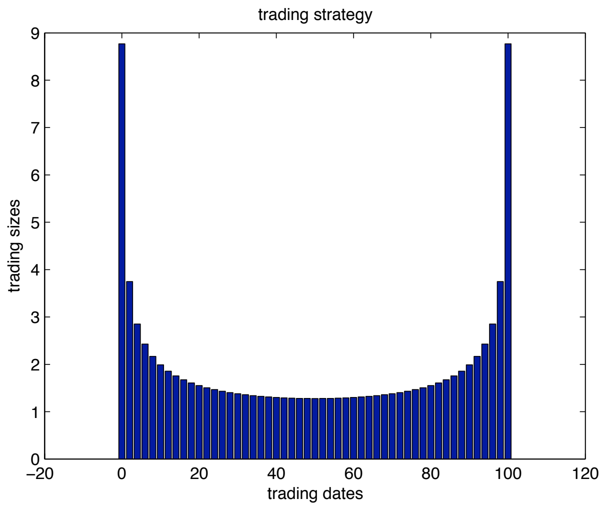
$$\gamma \text{ must be such that } \gamma \geq \gamma^* := 2 - \frac{\log 3}{\log 2} \approx 0.415$$

$$\delta + \gamma \approx 1$$

Consistent with (some) empirical studies.

Conclusion for Part I:

- Market impact should decay as a convex function of time
- Exponential or power law resilience leads to "bathtub solutions"



  which are extremely robust

- Many open problems
- Minimizing *expected* costs does not take into account volatility risk. Must introduce risk aversion — see next part.

II. The qualitative effects of risk aversion

1. Exponential utility and mean-variance

2. General utility functions

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II. The qualitative effects of risk aversion

1. Exponential utility and mean-variance

Liquidation time: $T \in [0, \infty]$.

Strategy: X adapted with $X_0 > 0$ fixed and $X_T = 0$.

Admissible: X_t bounded, absolutely continuous in t. Take

$$\xi_t := -\dot{X}_t$$

as controll. Then

$$X_t^\xi := X_0 - \int_0^t \xi_s \, ds$$

Market impact model: Following Almgren (2003),

$$\begin{array}{ccccccccc} S_t^\xi & = & S_0 & + & \sigma B_t & + & \gamma(X_t^\xi - X_0) & + & h(\xi_t) \\ & & \text{initial} & & \text{Brownian} & & \text{permanent} & & \text{temporary} \\ & & \text{price} & & \text{motion} & & \text{impact} & & \text{impact} \end{array}$$

Most common model in practice; *drift, multiple assets, general Lévy process, Gatheral-type impact* possible.

Assumption:

$$f(x) := xh(x)$$

is convex, C^1, and satisfies $f(x) = f(-x)$ and $f(x)/x \to \infty$ for $|x| \to \infty$.
E.g., $h(x) = \alpha\,\mathrm{sign}(x)\sqrt{|x|} + \beta x$.

Sales revenues:

$$\begin{aligned}\mathcal{R}_T(\xi) &= \int_0^T (-\dot{X}_t) S_t^\xi \, dt = \dots \\ &= S_0 X_0 - \frac{\gamma}{2} X_0^2 + \sigma \int_0^T X_t^\xi \, dB_t - \int_0^T f(\xi_t) \, dt.\end{aligned}$$

Goal: maximize expected utility

$$\mathbb{E}[\, u(\mathcal{R}_T(\xi)) \,]$$

over admissible strategies for $u(x) = -e^{-\alpha x}$

Setup as control problem

- controlled diffusion:

$$R_t^\xi = R_0 + \sigma \int_0^t X_s^\xi \, dB_s - \int_0^t f(\xi_s)\, ds$$

- value function

$$v(T, X_0, R_0) = \sup_{\xi \in \mathcal{X}(T,X_0)} \mathbb{E}\big[\, u(R_T^\xi)\,\big],$$

where

$$\mathcal{X}(T, X_0) = \Big\{ \xi \,\big|\, X^\xi \text{ is bounded and } \int_0^T \xi_t \, dt = X_0 \Big\}$$

Heuristic derivation of HJB equation

$$dv(T-t, X_t^\xi, R_t^\xi) = \sigma v_R X_t^\xi \, dB_t + \Big(- v_t - \xi_t v_X + v_R f(\xi_t) + \frac{\sigma^2}{2} (X_t^\xi)^2 v_{RR} \Big) \, dt$$

Hence

$$v_t = \frac{\sigma^2}{2} X^2 v_{RR} - \inf_\xi \big(\xi v_X + v_R f(\xi) \big)$$

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What about the constraint $\int_0^T \xi_t \, dt = X_0$?

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$$v(0, X, R) = \lim_{T \downarrow 0} v(T, X, R) = \begin{cases} u(R) & \text{if } X = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

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Theorem 14. [A.S. & Schöneborn (2008), A.S., Schöneborn & Tehranchi (2009)]
If $u(x) = -e^{-\alpha x}$ for some $\alpha > 0$, then the unique optimal strategy ξ^* is a deterministic function of t. Moreover, v is a classical solution of the singular HJB equation.

The fact that optimal strategies for CARA investors are deterministic is very robust. Is also true

- if Brownian motion is replaced by a Lévy process;
- for Gatheral-type impact
- other models with functionally dependent impact

Sketch of proof: For simplicity: $\sigma = 1$. We have

$$\begin{aligned}\mathbb{E}\big[\,u(R_T^\xi)\,\big] &= -e^{-\alpha R_0}\mathbb{E}\Big[\,e^{-\alpha\int_0^T X_t^\xi\,dB_t+\alpha\int_0^T f(\xi_t)\,dt}\,\Big]\\ &= -e^{-\alpha R_0}\mathbb{E}^\xi\Big[\,e^{\frac{\alpha^2}{2}\int_0^T (X_t^\xi)^2\,dt+\alpha\int_0^T f(\xi_t)\,dt}\,\Big]\end{aligned}$$

where

$$\frac{d\mathbb{P}^\xi}{d\mathbb{P}} = e^{-\alpha\int_0^T X_t^\xi\,dB_t-\frac{\alpha^2}{2}\int_0^T (X_t^\xi)^2\,dt}$$

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where

$$\frac{d\mathbb{P}^\xi}{d\mathbb{P}} = e^{-\alpha\int_0^T X_t^\xi\,dB_t-\frac{\alpha^2}{2}\int_0^T (X_t^\xi)^2\,dt}$$

Now, by Jensen's inequality,

$$\mathbb{E}^\xi\Big[\,e^{\frac{\alpha^2}{2}\int_0^T (X_t^\xi)^2\,dt+\alpha\int_0^T f(\xi_t)\,dt}\,\Big] \geq \exp\Big(\mathbb{E}^\xi\Big[\,\frac{\alpha^2}{2}\int_0^T (X_t^\xi)^2\,dt+\alpha\int_0^T f(\xi_t)\,dt\,\Big]\Big)$$

with equality if and only if ξ is deterministic.

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with equality if and only if ξ is deterministic. Moreover

$$\mathbb{E}^\xi\Big[\frac{\alpha^2}{2}\int_0^T (X_t^\xi)^2\,dt+\alpha\int_0^T f(\xi_t)\,dt\Big] \geq \frac{\alpha^2}{2}\int_0^T (X_t^{\overline{\xi}})^2\,dt+\alpha\int_0^T f(\overline{\xi}_t)\,dt$$

where $\overline{\xi}_t := \mathbb{E}^\xi[\,\xi_t\,]$.

Hence, the value function is

$$v(T, X_0, R_0) = \sup_{\xi \in \mathcal{X}(T,X_0)} \mathbb{E}\big[\, u(R_T^\xi) \,\big] = \sup_{\xi \in \mathcal{X}_{\det}(T,X_0)} \mathbb{E}\big[\, u(R_T^\xi) \,\big]$$

$$= -\exp\Big(-\alpha R_0 + \alpha \inf_{\xi \in \mathcal{X}_{\det}(T,X_0)} \int_0^T L(X_t^\xi, \xi_t)\, dt \Big)$$

where $\mathcal{X}_{\det}(T, X_0)$ are the deterministic strategies in $\mathcal{X}(T, X_0)$ and L is the Lagrangian

$$L(q,p) = \frac{\alpha}{2} q^2 + f(-p) = \frac{\alpha}{2} q^2 + f(p)$$

Classical mechanics: the action function

$$S(T,X) := \inf_{\xi \in \mathcal{X}_{\det}(T,X)} \int_0^T L(X_t^\xi, \xi_t)\,dt = \inf_{\xi \in \mathcal{X}_{\det}(T,X)} \int_0^T L(X_t^\xi, \dot{X}_t^\xi)\,dt$$

is a classical solution of the Hamilton-Jacobi equation

$$S_T(T,X) + H(X, S_X(T,X)) = 0 \qquad T > 0,\; X \in \mathbb{R}$$

where H is the Hamiltonian

$$H(q,p) = -\frac{\alpha}{2}q^2 + f^*(p)$$

Boundary conditions:

$$S(0,0) = 0 \qquad \text{and} \qquad S(0,X) = \infty \text{ for } X \neq 0.$$

[Side remark: this fact is classical when $f \in C^2$ but more subtle when $f \in C^1$ as for $h(x) = \sqrt{|x|}$]

Plugging the Hamilton-Jacobi equation into

$$\begin{aligned} v(T, X_0, R_0) &= -\exp\Big(-\alpha R_0 + \alpha \inf_{\xi \in \mathcal{X}_{\mathrm{det}}(T, X_0)} \int_0^T L(X_t^\xi, \xi_t)\, dt \Big) \\ &= -\exp\Big(-\alpha R_0 + \alpha S(T, X_0) \Big) \end{aligned}$$

yields the singular HJB-equation for v. $\square$

Alternative proof: Define the function

$$w(T,X_0,R_0) := -\exp\Big(-\alpha R_0 + \alpha S(T,X_0)\Big)$$

so that it's a classical solution of the singular HJB-equation. Then use a verification argument to show that $w = v$ (subtle).

Then there is $\xi^* \in \mathcal{X}_{\text{det}}(T,X_0)$ such that

$$S(T,X_0) = \int_0^T L(X_t^{\xi^*}, \xi_t^*)\,dt$$

and this ξ^* must hence be optimal. $\square$

The relation with mean-variance optimization

For $\xi \in \mathcal{X}_{\text{det}}(T, X_0)$,

$$R_t^\xi = R_0 + \sigma \int_0^t X_s^\xi \, dB_s - \int_0^t f(\xi_s)\, ds$$

is Gaussian, and so

$$\mathbb{E}[\, u(R_T^\xi)\,] = -\exp\Big(-\alpha \mathbb{E}[\, R_T^\xi\,] + \frac{\alpha^2}{2} \mathrm{var}(R_T^\xi)\Big)$$

Hence, exponential-utility maximization is equivalent to the maximization of the mean-variance functional

$$\mathbb{E}[\, R_T^\xi\,] - \frac{\alpha}{2} \mathrm{var}(R_T^\xi)$$

for deterministic strategies [Markowitz,. . . , Almgren & Chriss (2000)].
Different for adaptive strategies [Almgren & Lorenz (2008)].

Computation of the optimal strategy

Classical mechanics: X^{ξ^*} is solution of the Euler-Lagrange equation

$$\alpha X = f''(\dot{X}_t)\ddot{X}_t \qquad \text{with } X_0 = \textit{ initial portfolio } \text{ and } X_T = 0$$

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Not clear when $f \notin C^2$ as for $h(x) = \sqrt{|x|}$

Theorem 15. [A.S. & Schöneborn (2008)]
The optimal X^{ξ^*} is C^1 and uniquely solves the Hamilton equations

$$\begin{aligned} \dot{X}_t &= H_p(X_t, p(t)) = -(f^*)'(-p(t)) \\ \dot{p}(t) &= -H_q(X_t, p(t)) = \alpha X_t \end{aligned}$$

with initial conditions $X_0^{\xi^*} = X_0$ *and* $p(0) = -(f^*)'(\xi_0^*)$.

Example: For linear temporary impact, $f(x) = \lambda x^2$, the optimal strategy is

$$\xi_t^* = X_0 \sqrt{\frac{\alpha\sigma^2}{2\lambda}} \cdot \frac{\cosh\left((T-t)\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right)}{\sinh\left(T\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right)}$$

$$X_t^{\xi^*} = X_0 \cdot \frac{\cosh\left(t\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right)\sinh\left(T\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right) - \cosh\left(T\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right)\sinh\left(t\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right)}{\sinh\left(T\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right)}$$

The value function is

$$v(T, R_0, X_0) = -\exp\left[-\alpha(R_0 + S_0 X_0 - \frac{\gamma}{2} X_0^2) + X_0^2 \sqrt{\frac{\lambda\alpha^3\sigma^2}{2}} \coth\left(T\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right)\right]$$

II. The qualitative effects of risk aversion

1. Exponential utility and mean-variance

2. General utility functions

Problem with $T < \infty$ difficult because of singular initial condition of HJB equation.

$\Longrightarrow$ Consider infinite time horizon instead

- Assume also linear temporary impact (for simplicity only)

$$f(x) = \lambda x^2$$

- Utility function $u \in C^6(\mathbb{R})$ such that the absolute risk aversion,

$$A(R) := -\frac{u''(R)}{u'(R)} \qquad (= \text{constant for exponential utility}),$$

satisfies

$$0 < A_{min} \leq A(R) \leq A_{max} < \infty.$$

Entire section based on A.S. & Schöneborn (2009)

Recall

$$R_t^\xi = R_0 + \sigma \int_0^t X_s^\xi \, dB_s - \lambda \int_0^t \xi_s^2 \, ds.$$

- Optimal liquidation:

$$\text{maximize} \quad \mathbb{E}[\, u(R_\infty^\xi)\,]$$

- Maximization of asymptotic portfolio value:

$$\text{maximize} \quad \lim_{t \uparrow \infty} \mathbb{E}[\, u(R_t^\xi)\,]$$

Note: Liquidation enforced by the fact that a risk-averse investor does not want to hold a stock whose price process is a martingale.

HJB equation for finite time horizon:

$$v_t = \frac{\sigma^2}{2} X^2 v_{RR} - \inf_c \left(c v_X + \lambda v_R c^2 \right)$$

Guess for infinite time horizon:

$$0 = \frac{\sigma^2}{2} X^2 v_{RR} - \inf_c \left(c v_X + \lambda v_R c^2 \right)$$

Initial condition:

$$v(0, R) = u(R).$$

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$$0 = \frac{\sigma^2}{2} X^2 v_{RR} - \inf_c \left(c v_X + \lambda v_R c^2 \right)$$

Initial condition:

$$v(0, R) = u(R).$$

Corresponding reduced-form equation:

$$v_X^2 = -2\lambda \sigma^2 X^2 v_R \cdot v_{RR}$$

Not a straightforward PDE either......

Way out: consider optimal Markov control in HJB equation

$$\widehat{c}(X,R) = -\frac{v_X(X,R)}{2\lambda v_R(X,R)}$$

and let

$$\widetilde{c}(Y,R) = \frac{\widehat{c}(\sqrt{Y},R)}{\sqrt{Y}}.$$

If v solves the HJB equation, then $\widetilde{c}$ solves

$$(*) \qquad \begin{cases} \widetilde{c}_Y = \dfrac{\sigma^2}{4\widetilde{c}}\widetilde{c}_{RR} - \dfrac{3}{2}\lambda\widetilde{c}\widetilde{c}_R \\[2ex] \widetilde{c}(0,R) = \sqrt{\frac{\sigma^2 A(R)}{2\lambda}} \end{cases}$$

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$$(*) \qquad \begin{cases} \widetilde{c}_Y = \dfrac{\sigma^2}{4\widetilde{c}}\widetilde{c}_{RR} - \dfrac{3}{2}\lambda\widetilde{c}\widetilde{c}_R \\ \\ \widetilde{c}(0,R) = \sqrt{\frac{\sigma^2 A(R)}{2\lambda}} \end{cases}$$

Theorem 16. $(*)$ *admits a unique classical solution* $\widetilde{c} \in C^{2,4}$ *s.th.*

$$\sqrt{\frac{\sigma^2 A_{min}}{2\lambda}} \le \widetilde{c}(Y,R) \le \sqrt{\frac{\sigma^2 A_{max}}{2\lambda}}$$

Follows from:

Theorem 17. [Ladyzhenskaya, Solonnikov & Uraltseva (1968)] *There is a classical $C^{2,4}$-solution for the parabolic partial differential equation*

$$f_t - \frac{\partial}{\partial x}\big[a(x,t,f,f_x)\big] + b(x,t,f,f_x) = 0$$

with initial value condition $f(0,x) = \psi_0(x)$ if all of the following conditions are satisfied:

- *$\psi_0(x)$ is smooth (C^4) and bounded*
- *a and b are smooth (C^3 respectively C^2)*
- *There are constants b_1 and $b_2 \geq 0$ such that for all x and u:*

$$\left(b(x,t,u,0) - \frac{\partial a}{\partial x}(x,t,u,0)\right) u \geq -b_1 u^2 - b_2.$$

- *For all $M > 0$, there are constants $\mu_M \geq \nu_M > 0$ such that for all x, t, u and p that are bounded in modulus by M:*

$$\nu_M \leq \frac{\partial a}{\partial p}(x,t,u,p) \leq \mu_M \tag{5}$$

and

$$\left(|a| + \left|\frac{\partial a}{\partial u}\right|\right)(1 + |p|) + \left|\frac{\partial a}{\partial x}\right| + |b| \leq \mu_M (1 + |p|)^2. \tag{6}$$

Proof: Obtained from original existence theorem by cutting off the coefficients of the PDE. □

Next, consider the transport equation

$$\begin{cases} \widetilde{w}_Y = -\lambda \widetilde{c}\widetilde{w}_R \\ \widetilde{w}(0,R) = u(R). \end{cases}$$

Proposition 18. *The transport equation admits a $C^{2,4}$-solution $\widetilde{w}$. Moreover, $w(X,R) := \widetilde{w}(X^2,R)$ is a classical solution of the HJB equation*

$$0 = \frac{\sigma^2}{2} X^2 w_{RR} - \inf_c \left(c w_X + w_R c^2 \right), \qquad w(0,R) = u(R)$$

The unique minimum above is attained at

$$c(X,R) := \widetilde{c}(X^2,R)X.$$

Sketch of proof: Existence and uniqueness of solutions follws by method of characteristics. Assume for the moment that

$$\widetilde{c}^2 = -\frac{\sigma^2 \widetilde{w}_{RR}}{2\lambda \widetilde{w}_R}.$$

Then with $Y = X^2$:

$$\begin{aligned}
0 &= -\lambda X^2 \widetilde{w}_R \left(\frac{\sigma^2 \widetilde{w}_{RR}}{2\lambda \widetilde{w}_R} + \widetilde{c}^2 \right) \\
&= -\lambda X^2 \widetilde{w}_R \left(\frac{\sigma^2 \widetilde{w}_{RR}}{2\lambda \widetilde{w}_R} + \frac{\widetilde{w}_Y^2}{\lambda^2 \widetilde{w}_R^2} \right) \\
&= -\frac{1}{2}\sigma^2 X^2 w_{RR} - \frac{w_X^2}{4\lambda w_R} \\
&= \inf_c \left[-\frac{1}{2}\sigma^2 X^2 w_{RR} + \lambda w_R c^2 + w_X c \right]
\end{aligned}$$

We now show that

$$\widetilde{c}^2 = -\frac{\sigma^2 \widetilde{w}_{RR}}{2\lambda \widetilde{w}_R}.$$

First, observe that it holds for $Y = 0$. For general Y, consider

$$\frac{d}{dY}\widetilde{c}^2 = -3\lambda \widetilde{c}^2 \widetilde{c}_R + \frac{\sigma^2}{2}\widetilde{c}_{RR}$$

$$-\frac{d}{dY}\frac{\sigma^2 \widetilde{w}_{RR}}{2\lambda \widetilde{w}_R} = \sigma^2 \widetilde{c}\frac{d}{dR}\frac{\widetilde{w}_{RR}}{2\widetilde{w}_R} + \sigma^2 \widetilde{c}_R \frac{\widetilde{w}_{RR}}{2\widetilde{w}_R} + \frac{\sigma^2}{2}\widetilde{c}_{RR}$$

The first holds by PDE for $\widetilde{c}$, the second by transport eqn. for $\widetilde{w}$.

Next,

$$\begin{aligned}\frac{d}{dY}\left(\widetilde{c}^2 + \frac{\sigma^2 \widetilde{w}_{RR}}{2\lambda \widetilde{w}_R}\right) &= -3\lambda \widetilde{c}^2 \widetilde{c}_R + \frac{\sigma^2}{2}\widetilde{c}_{RR} - \sigma^2 \widetilde{c}\frac{d}{dR}\frac{\widetilde{w}_{RR}}{2\widetilde{w}_R} - \sigma^2 \widetilde{c}_R \frac{\widetilde{w}_{RR}}{2\widetilde{w}_R} - \frac{\sigma^2}{2}\widetilde{c}_{RR} \\ &= -\lambda \widetilde{c}\frac{d}{dR}\left(\widetilde{c}^2 + \frac{\sigma^2 \widetilde{w}_{RR}}{2\lambda \widetilde{w}_R}\right) - \lambda \widetilde{c}_R\left(\widetilde{c}^2 + \frac{\sigma^2 \widetilde{w}_{RR}}{2\lambda \widetilde{w}_R}\right).\end{aligned}$$

We now show that

$$\widetilde{c}^2 = -\frac{\sigma^2 \widetilde{w}_{RR}}{2\lambda \widetilde{w}_R}.$$

First, observe that it holds for $Y = 0$. For general Y, consider

$$\frac{d}{dY}\widetilde{c}^2 = -3\lambda\widetilde{c}^2\widetilde{c}_R + \frac{\sigma^2}{2}\widetilde{c}_{RR}$$

$$-\frac{d}{dY}\frac{\sigma^2 \widetilde{w}_{RR}}{2\lambda \widetilde{w}_R} = \sigma^2\widetilde{c}\frac{d}{dR}\frac{\widetilde{w}_{RR}}{2\widetilde{w}_R} + \sigma^2\widetilde{c}_R\frac{\widetilde{w}_{RR}}{2\widetilde{w}_R} + \frac{\sigma^2}{2}\widetilde{c}_{RR}$$

The first holds by PDE for $\widetilde{c}$, the second by transport eqn. for $\widetilde{w}$.

Next,

$$\frac{d}{dY}\left(\widetilde{c}^2 + \frac{\sigma^2 \widetilde{w}_{RR}}{2\lambda \widetilde{w}_R}\right) = -3\lambda\widetilde{c}^2\widetilde{c}_R + \frac{\sigma^2}{2}\widetilde{c}_{RR} - \sigma^2\widetilde{c}\frac{d}{dR}\frac{\widetilde{w}_{RR}}{2\widetilde{w}_R} - \sigma^2\widetilde{c}_R\frac{\widetilde{w}_{RR}}{2\widetilde{w}_R} - \frac{\sigma^2}{2}\widetilde{c}_{RR}$$

$$= -\lambda\widetilde{c}\frac{d}{dR}\left(\widetilde{c}^2 + \frac{\sigma^2 \widetilde{w}_{RR}}{2\lambda \widetilde{w}_R}\right) - \lambda\widetilde{c}_R\left(\widetilde{c}^2 + \frac{\sigma^2 \widetilde{w}_{RR}}{2\lambda \widetilde{w}_R}\right).$$

Therefore need $u \in C^6$!

Hence,

$$f(Y,R) := \widetilde{c}^2 + \frac{\sigma^2 \widetilde{w}_{RR}}{2\lambda \widetilde{w}_R}$$

satisfies the linear PDE

$$f_Y = -\lambda \widetilde{c} f_R - \lambda \widetilde{c}_R f$$

with initial value condition $f(0,R) = 0$. One obvious solution to this PDE is $f(Y,R) \equiv 0$. By the method of characteristics this is the unique solution to the PDE, since $\widetilde{c}$ and $\widetilde{c}_R$ are smooth and hence locally Lipschitz. $\square$

A (rather technical) verification argument yields:

Theorem 19. *The value functions for optimal liquidation and for maximization of asymptotic portfolio value are equal and are classical solutions of the HJB equation*

$$-\frac{1}{2}\sigma^2 X^2 v_{RR} + \inf_c \left[\lambda v_R c^2 + v_X c\right] = 0$$

with boundary condition $v(0, R) = u(R)$. The a.s. unique optimal control $\hat{\xi}_t$ is Markovian and given in feedback form by

$$\hat{\xi}_t = c(X_t^{\hat{\xi}}, R_t^{\hat{\xi}}) = -\frac{v_X}{2\lambda v_R}(X_t^{\hat{\xi}}, R_t^{\hat{\xi}}). \tag{7}$$

For the value functions, we have convergence:

$$v(X_0, R_0) = \lim_{t\to\infty} \mathbb{E}[u(R_t^{\hat{\xi}})] = \mathbb{E}[u(R_\infty^{\hat{\xi}})] \tag{8}$$

Corollary 20. *If* $u(R) = -e^{-AR}$, *then*

$$X_t^{\xi^*} = X_0 \exp\Big(-t\sqrt{\frac{\sigma^2 A}{2\lambda}}\Big).$$

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$$X_t^{\xi^*} = X_0 \exp\Big(-t\sqrt{\frac{\sigma^2 A}{2\lambda}}\Big).$$

General result:

Theorem 21. *The optimal strategy $c(X,R)$ is increasing (decreasing) in R iff $A(R)$ is increasing (decreasing). I.e.,*

Utility function		*Optimal trading strategy*
DARA	$\Longleftrightarrow$	*Passive in the money*
CARA	$\Longleftrightarrow$	*Neutral in the money*
IARA	$\Longleftrightarrow$	*Aggresive in the money*

Theorem 22. *If u^1 and u^0 are such that $A^1 \geq A^0$ then $c^1 \geq c^0$.*

Idea of Proof: $g := \widetilde{c}^1 - \widetilde{c}^0$ solves

$$g_Y = \frac{1}{2} a g_{RR} + b g_R + V g,$$

where

$$a = \frac{\sigma^2}{2\widetilde{c}^0}, \qquad b = -\frac{3}{2}\lambda \widetilde{c}^1, \qquad \text{and} \qquad V = -\frac{\sigma^2 \widetilde{c}^1_{RR}}{4\widetilde{c}^0 \widetilde{c}^1} - \frac{3}{2}\lambda \widetilde{c}^0_R.$$

The boundary condition of g is

$$g(0,R) = \sqrt{\frac{\sigma^2 A^1(R)}{2\lambda}} - \sqrt{\frac{\sigma^2 A^0(R)}{2\lambda}} \geq 0$$

Now maximum principle or Feynman-Kac argument....
(plus localization) $\square$

Relation to forward utilities

Theorem 23.
For every $X > 0$, the value function $v(X,R)$ is again a utility function in R. Moreover,

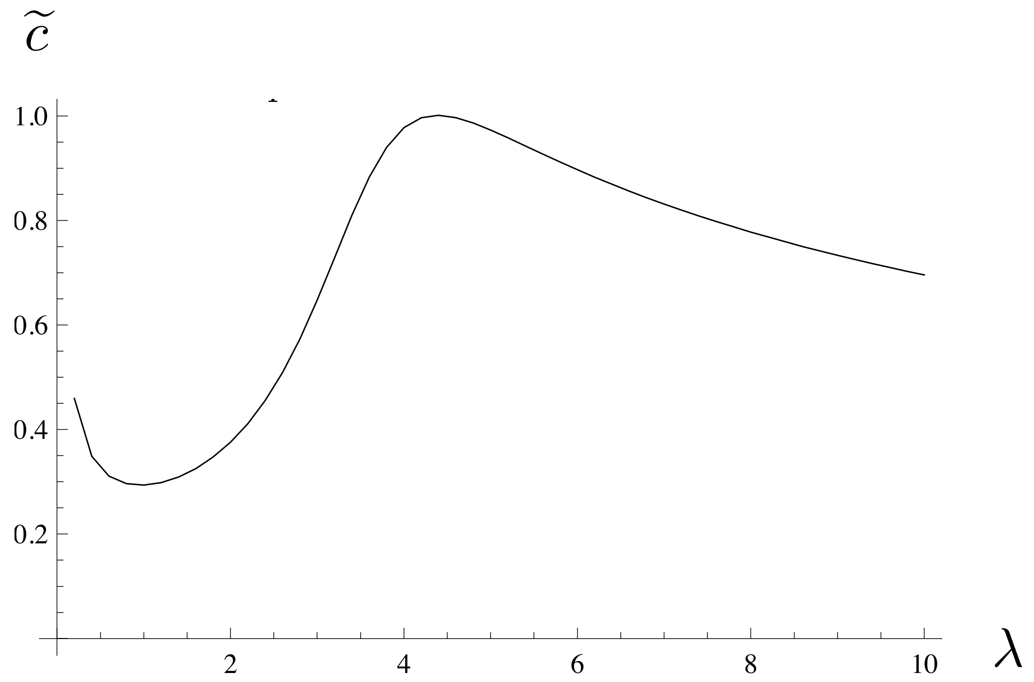
$$\widetilde{c}(Y,R) = \sqrt{\frac{\sigma^2 A(\sqrt{Y},R)}{2\lambda}}. \tag{9}$$

where

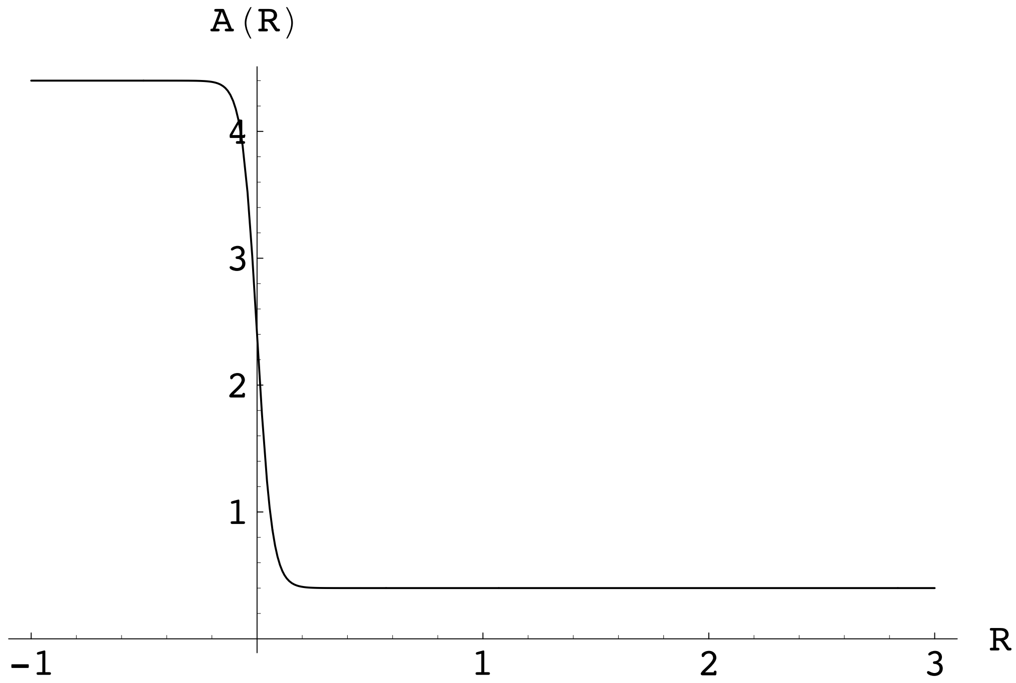
$$A(X,R) := -\frac{v_{RR}(X,R)}{v_R(X,R)}$$

What about other monotonicity relations?

• Monotonicity in λ: intuitively, an increase in liquidaton costs should lead to a decrease of liquidation speed.

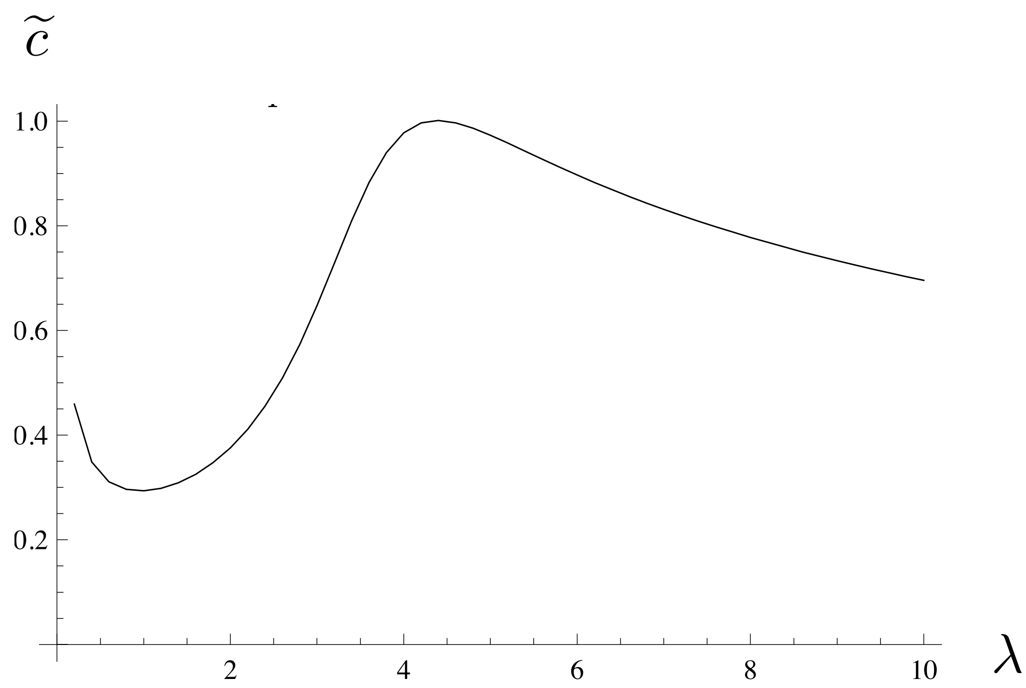


Dependence of the transformed optimal strategy $\widetilde{c}$ on λ for the DARA utility function with $A(R) = 2(1.2 - \tanh(15R))^2$.



The shape of the absolute risk aversion

$$A(R) = 2(1.2 - \tanh(15R))^2$$



Dependence of the transformed optimal strategy $\widetilde{c}$ on λ for the DARA utility function with $A(R) = 2(1.2 - \tanh(15R))^2$.

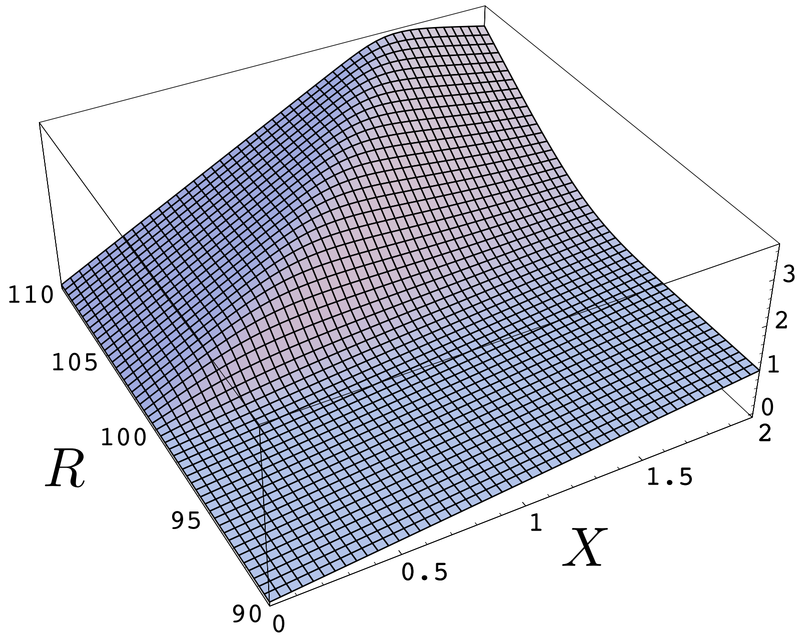
Theorem 24. *IARA* $\Longrightarrow$ *c is decreasing in λ.*

Proof similiar to Theorem 22. $\square$

What about other monotonicity relations?

- Monotonicity in λ: intuitively, an increase in liquidaton costs should lead to a decrease of liquidation speed.

- Monotonicity in X: intuitively, larger asset position should lead to an *increased* liquidation speed.



$$\hat{\xi}(X, R)$$

IARA utility function with $A(R) = 2(1.5 + \tanh(R - 100))^2$ and parameter $\lambda = \sigma = 1$.

What about other monotonicity relations?

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• Monotonicity in σ: intuitively, an increase in volatility should lead to an increase in the liquidation speed.

?

The multi-asset case

Initial portfolio of d assets

$$\boldsymbol{X}_0 = (X_0^1, \dots, X_0^d)$$

Strategy

$$\boldsymbol{X}_t^{\boldsymbol{\xi}} = \boldsymbol{X}_0 - \int_0^t \boldsymbol{\xi}_s \, ds$$

Price process:

$$\boldsymbol{S}_t = \boldsymbol{S}_0^0 + \sigma \boldsymbol{B}_t + \boldsymbol{\gamma}^\top (\boldsymbol{X}_t^{\boldsymbol{\xi}} - \boldsymbol{X}_0) - \boldsymbol{h}(\boldsymbol{\xi}_t)$$

for d-dim Brownian motion $\boldsymbol{B}$ and covariance matrix $\Sigma := \sigma\sigma^\top$.
Letting

$$f(\boldsymbol{\xi}) := \xi^\top \boldsymbol{h}(\boldsymbol{\xi}),$$

The revenues are

$$R_t^{\boldsymbol{\xi}} = R_0 + \int_0^t (\boldsymbol{X}_2^{\boldsymbol{\xi}})^\top \sigma \, d\boldsymbol{B}_s - \int_0^t f(\boldsymbol{\xi}_s)\, ds.$$

Guess for HJB equation

$$0 = \frac{1}{2}\boldsymbol{X}^\top \Sigma \boldsymbol{X} v_{RR} - \inf_{\boldsymbol{c}} \big(\boldsymbol{c}^\top \nabla_X v + v_R f(\boldsymbol{c})\big)$$

with initial condition

$$v(0,R) = u(R).$$

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Formally: Nonlinear PDE of "parabolic" type with d time parameters

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Guess for HJB equation

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with initial condition

$$v(0, R) = u(R).$$

Formally: Nonlinear PDE of "parabolic" type with d time parameters

Solvability completely unclear, a priori:

$$\nabla_{\boldsymbol{X}} v = g$$

typically not solvable (Poincaré lemma)

Theorem 25. [Schöneborn (2008)]
Under analogous conditions as in the onedimensional case and f having the scaling property

$$f(a\boldsymbol{\xi}) = a^{\alpha+1} f(\boldsymbol{\xi}), \qquad a \geq 0,$$

the value function is a classical solution of the HJB equation

$$0 = \frac{1}{2}\boldsymbol{X}^\top \Sigma \boldsymbol{X} v_{RR} - \inf_{\boldsymbol{c}} \big(\boldsymbol{c}^\top \nabla_{\boldsymbol{X}} v + v_R f(\boldsymbol{c}) \big)$$

with initial condition

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The minimizer $\widehat{c}$ determines the optimal strategy....

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with initial condition

$$v(0, R) = u(R).$$

The minimizer $\widehat{c}$ determines the optimal strategy....

How can this be proved??

Theorem 26. [Schöneborn (2008)]
The optimal control is

$$\widehat{c}(\boldsymbol{X}, R) = \widetilde{c}\big(\overline{v}(\boldsymbol{X}), R)\overline{c}(\boldsymbol{X}),$$

where $\overline{v}(\boldsymbol{X})$ *is the cost and* $\overline{c}(\boldsymbol{X})$ *is the vector field (optimal strategy) for mean-variance optimal liquidation of* $\boldsymbol{X}$, *and* $\widetilde{c}(Y, R)$ *is the unique solution of the nonlinear PDE*

$$\widetilde{c}_Y = -\frac{2\alpha+1}{\alpha+1}\widetilde{c}^{\alpha}\widetilde{c}_R + \frac{\alpha(\alpha-1)}{\alpha+1}\Big(\frac{\widetilde{c}_R}{\widetilde{c}}\Big)^2 + \frac{\alpha}{\alpha+1}\frac{\widetilde{c}_{RR}}{\widetilde{c}}$$

with initial condition

$$\widetilde{c}(0, R) = A(R)^{\frac{1}{\alpha+1}}$$

III. Multi-agent equilibrium

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Entire section based on Schöneborn and A.S. (2007)

Information leakage creates multi-player situations

- One trader ('the seller') must liquidate large portfolio by T_1
- Informed traders ('the predators') can exploit the resulting drift:
  - first short the asset
  - buy back shortly before T_1 at lower price

  "predatory trading"
- Suggests 'stealth trading strategy' for seller

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  "predatory trading"
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- **But why, then, do some sellers practice 'sunshine trading'?**

- $n+1$ traders with positions $X_0(t)$, $X_1(t)$, ..., $X_n(t)$
- Trades at time t are executed at the price

$$S(t) = S(0) + \sigma B(t) + \gamma \sum_{i=0}^{n} (X_i(t) - X_i(0)) + \lambda \sum_{i=0}^{n} \dot{X}_i(t)$$

- Player 0 (the seller) has $X_0(0) > 0$, $X_0(t) = 0$ for $t \geq T_1$
- Players $1, \ldots, n$ have $X_i(0) = 0$, $X_i(T_1) = \text{arbitrary}$, $X_i(T_2) = 0$
- Strategies are deterministic
- Players are risk-neutral and aim to maximize expected return

Goal: Find Nash equilibrium

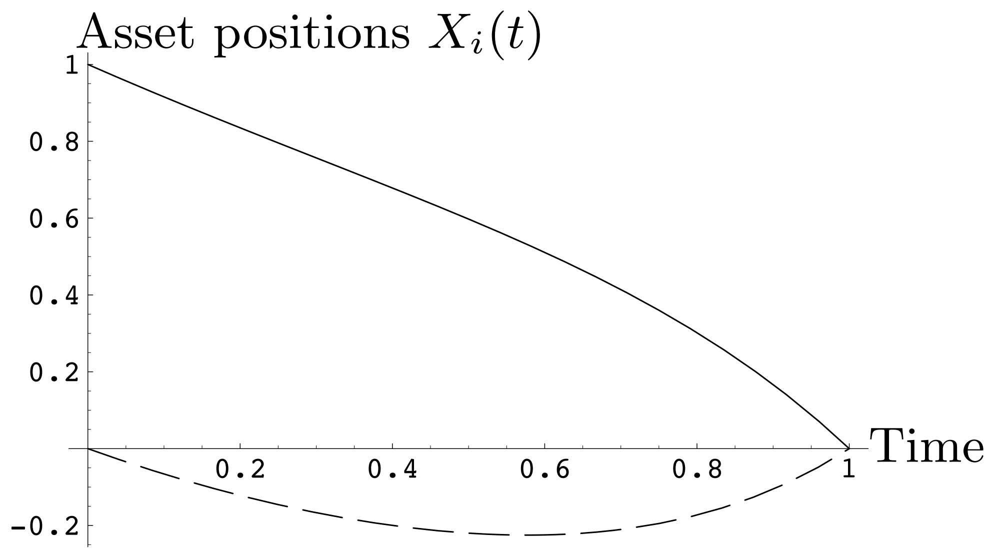
Situation in a one-stage framework

Theorem 1. [Carlin, Lobo, Viswanathan]
If $T_1 = T_2$, then the unique optimal strategies for these $n+1$ players are given by:

$$\dot{X}_i(t) = ae^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t} + b_i e^{\frac{\gamma}{\lambda}t}$$

with

$$\begin{aligned} a &= \frac{n}{n+2}\frac{\gamma}{\lambda}\left(1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}\right)^{-1}\frac{\sum_{i=0}^{n}(X_i(T_1) - X_i(0))}{n+1} \\ b_i &= \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T_1} - 1\right)^{-1}\left(X_i(T_1) - X_i(0) - \frac{\sum_{j=0}^{n}(X_j(T_1) - X_j(0))}{n+1}\right). \end{aligned}$$



Solid line ~ seller, dashed line ~ predator

- Predation occurs irrespective of the market parameters
- Predators always decrease the seller's return
- Predation becomes fiercer when the number of predators increases

$\Longrightarrow$ Model cannot explain sunshine trading or liquidity provision

Theorem 2.
In the two-stage framework, $T_2 > T_1$, there is a unique Nash equilibrium, in which all predators acquire the same asset positions, and these are determined by their value at T_1:

$$X_i(T_1) = \frac{A_2 n^2 + A_1 n + A_0}{B_3 n^3 + B_2 n^2 + B_1 n + B_0} X_0.$$

The coefficients A_i and B_i are functions of n that converge in the limit $n \uparrow \infty$.

Idea of Proof: Use result from Carlin et al., optimize over $X_i(T_1)$.

Coefficients in theorem can be computed exlicitly, e.g.,

$$A_0 = 2\Bigg(-e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(n(3+2n)T_1+(2+n)T_2)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{\gamma\left(\left(2+2n+n^2\right)T_1+n(2+n)T_2\right)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} - e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{n\gamma T_1+\gamma T_2}{\lambda+n\lambda}} - e^{\frac{\gamma T_1+n\gamma T_2}{\lambda+n\lambda}}\Bigg).$$

Are there new effects in the two-stage model?

- **Plastic market:**

$$\text{temporary impact } \lambda \ll \text{permanent impact } \gamma$$

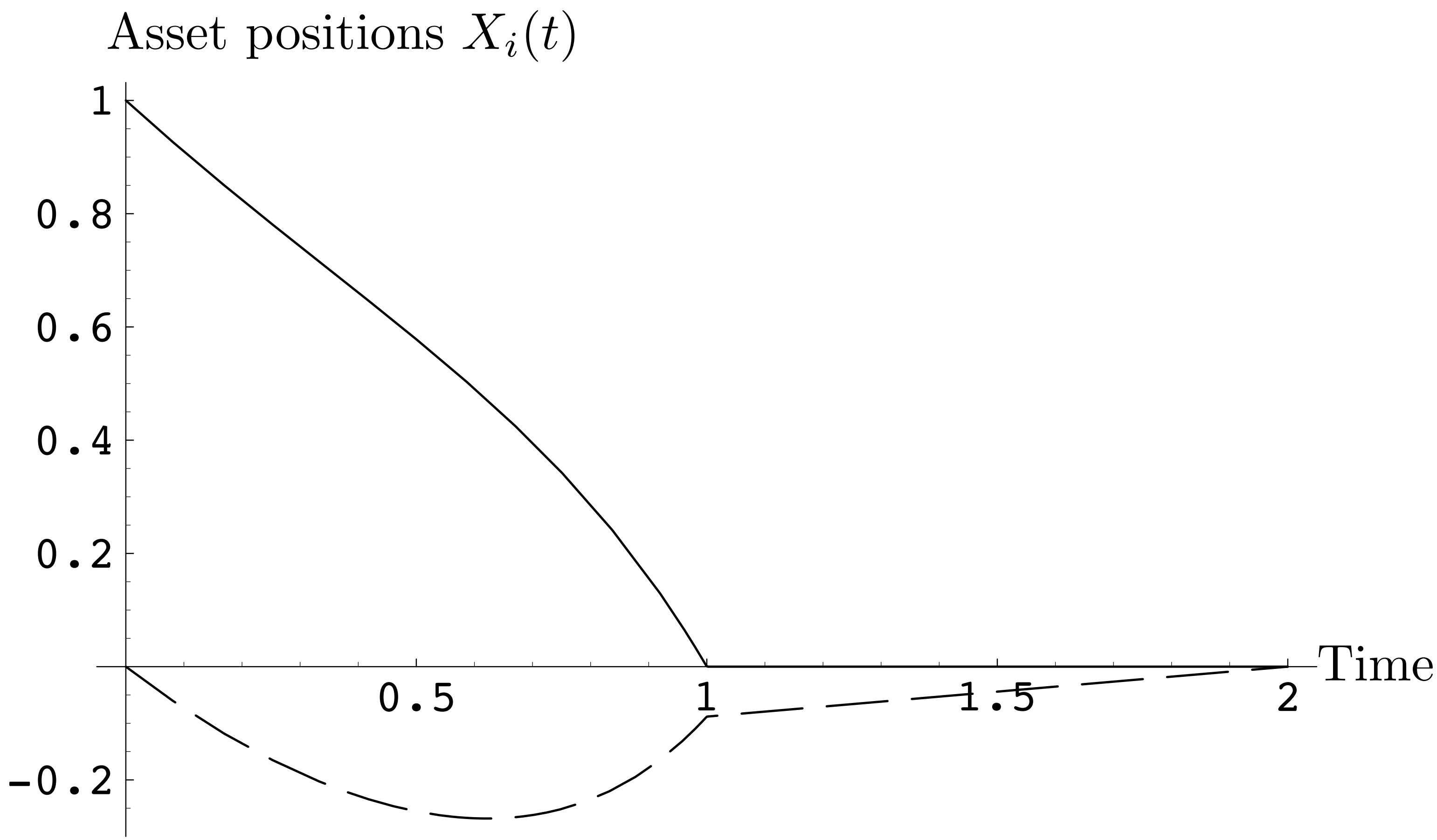
- **Elastic market:**

$$\text{temporary impact } \lambda \gg \text{permanent impact } \gamma$$

- **Intermediate market:**

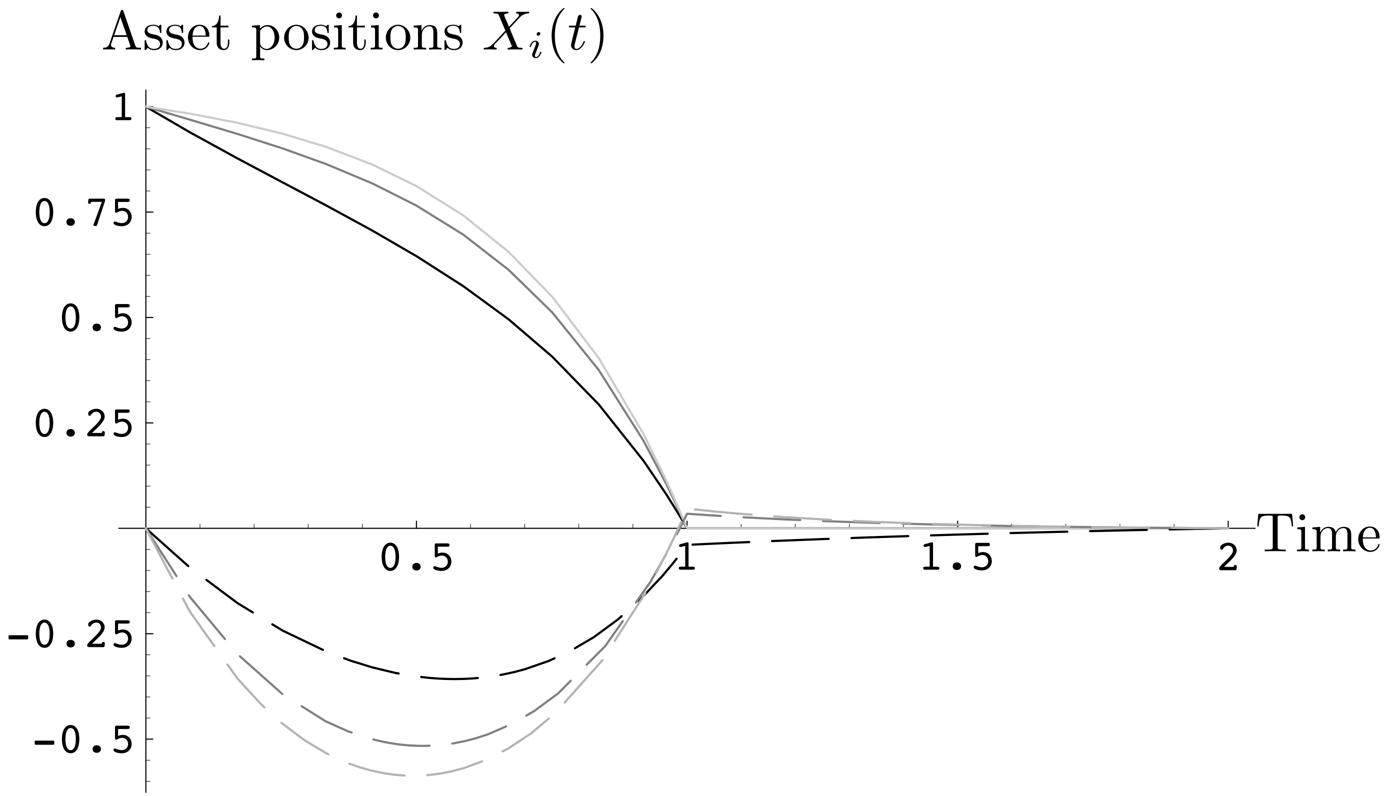
$$\text{temporary impact } \lambda \sim \text{permanent impact } \gamma$$

Plastic market (large perm. impact) one predator



Solid line $\sim$ seller, dashed line $\sim$ predator

Plastic market (large perm. impact)

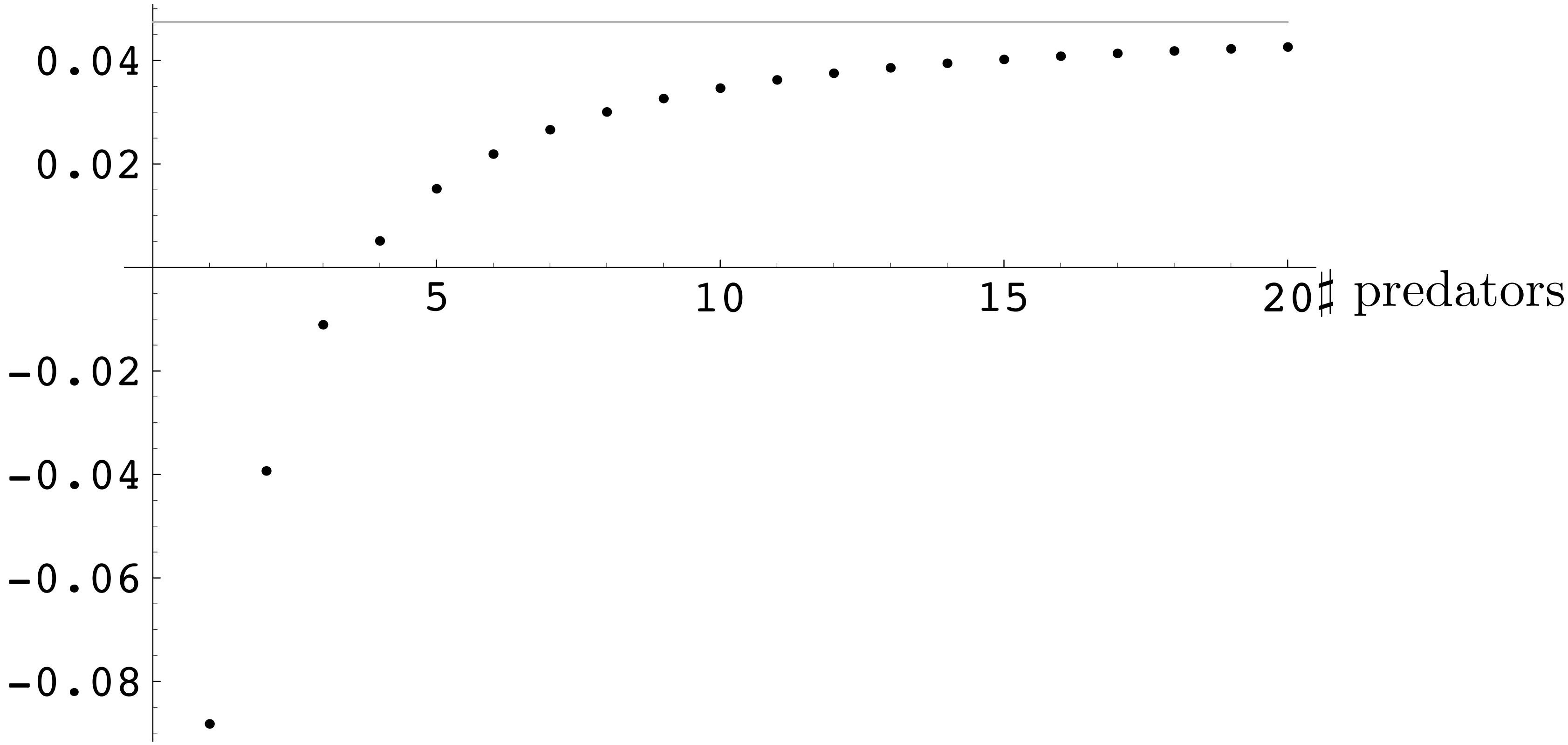


Solid lines $\sim$ seller, dashed lines $\sim$ n predators

Black $\sim n = 2$, dark grey $\sim n = 10$, light grey $\sim n = 100$

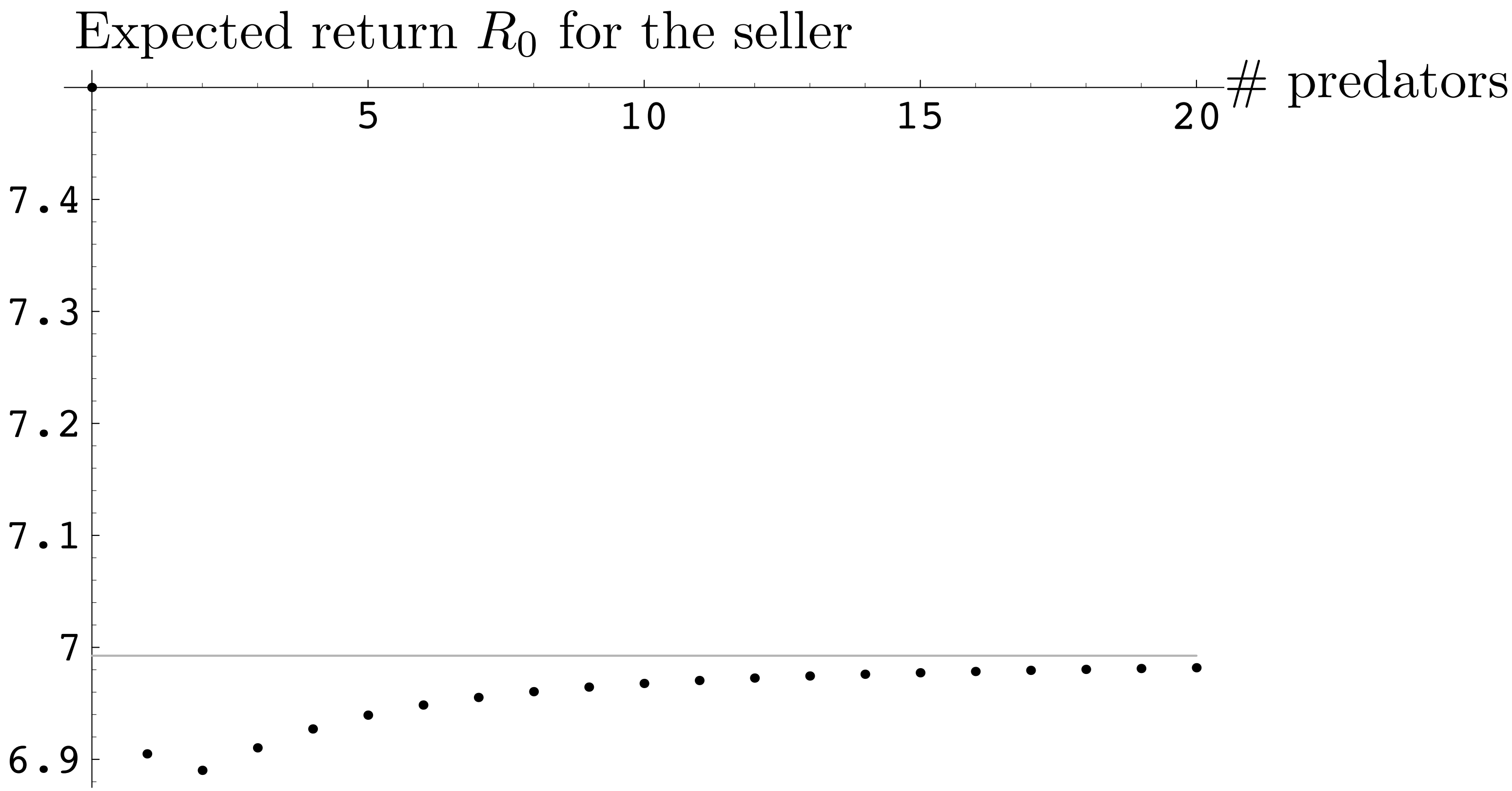
Plastic market (large perm. impact)

Joint asset position $\sum_{i=1}^{n} X_i(T_1)$ of all predators



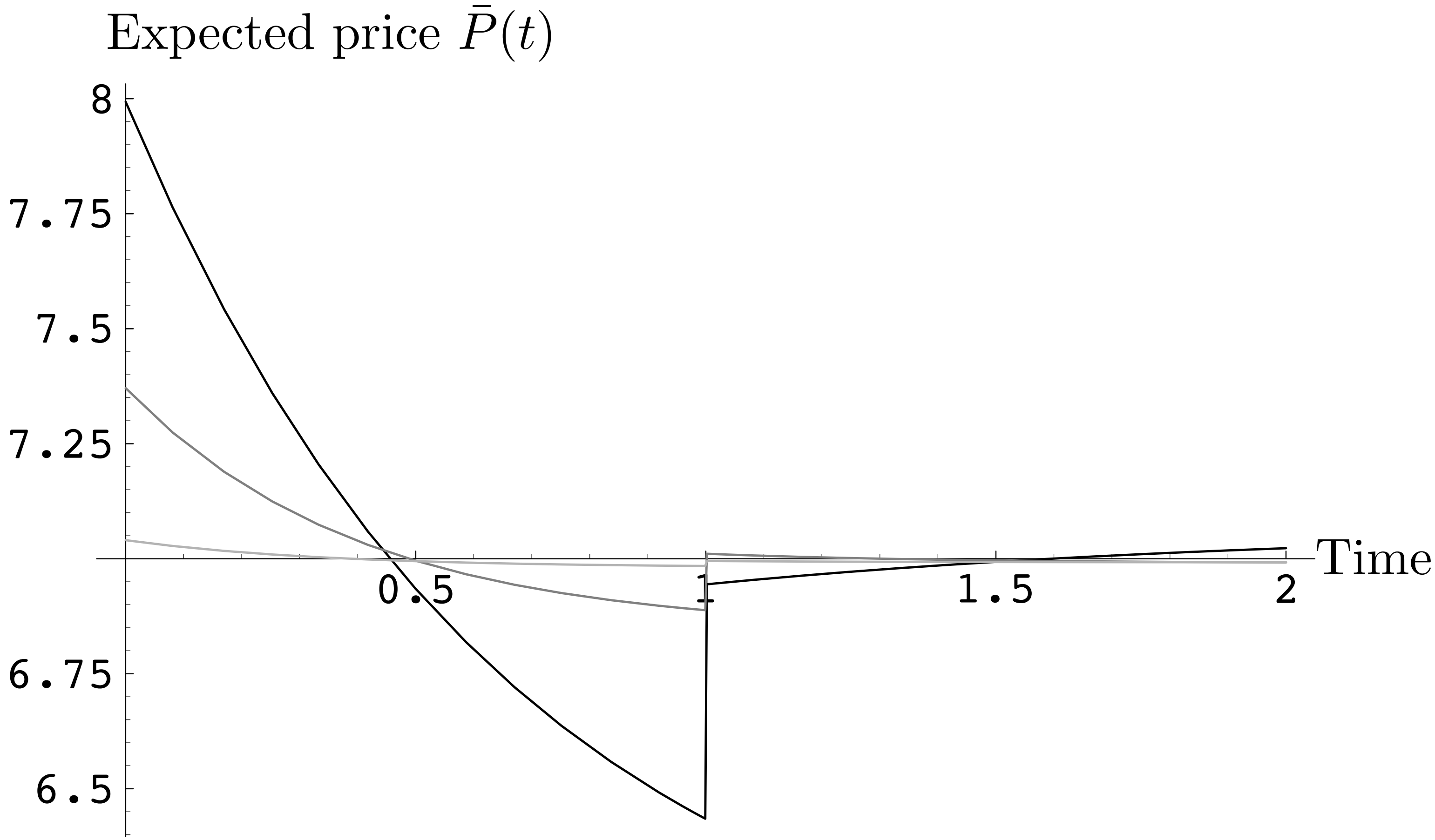
Upper grey line $= \lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1)$

Plastic market (large perm. impact)



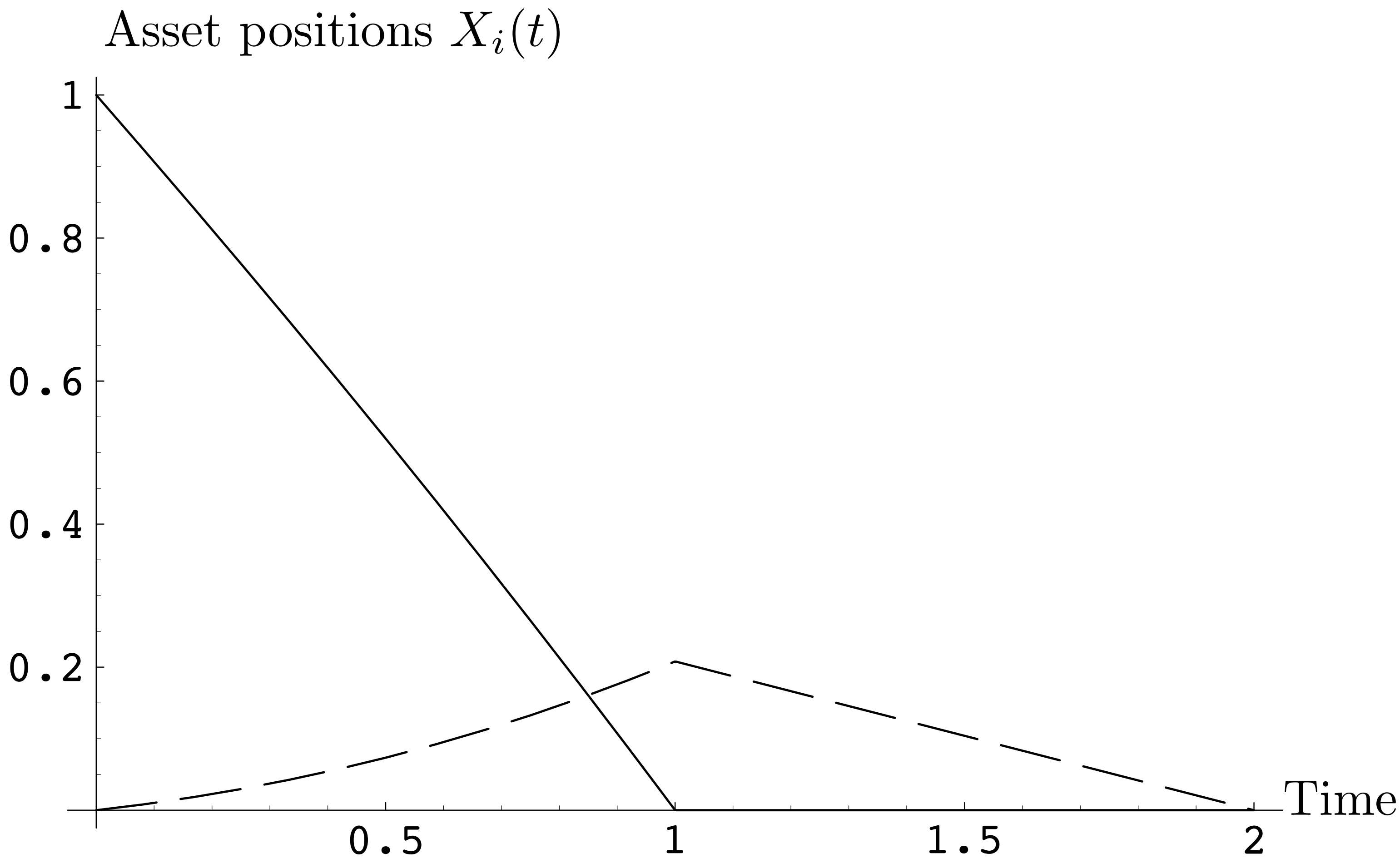
The grey line represents the limit $n \to \infty$. The return for the seller without predators is at the intersection of x- and y-axis.

Plastic market (large perm. impact)



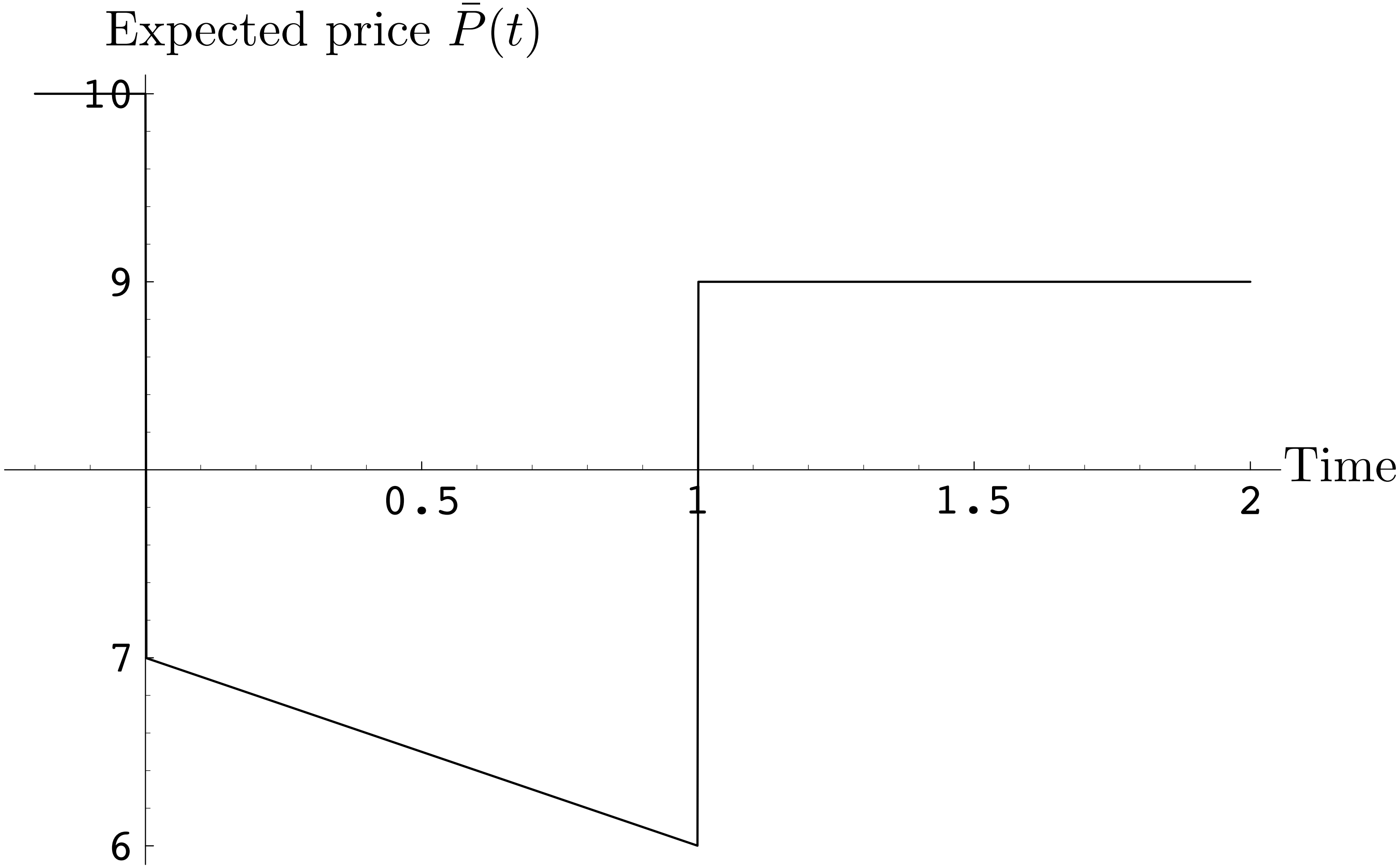
Black $\sim n = 2$, dark grey $\sim n = 10$, light grey $\sim n = 100$

Elastic market (large temp. impact) with one predator

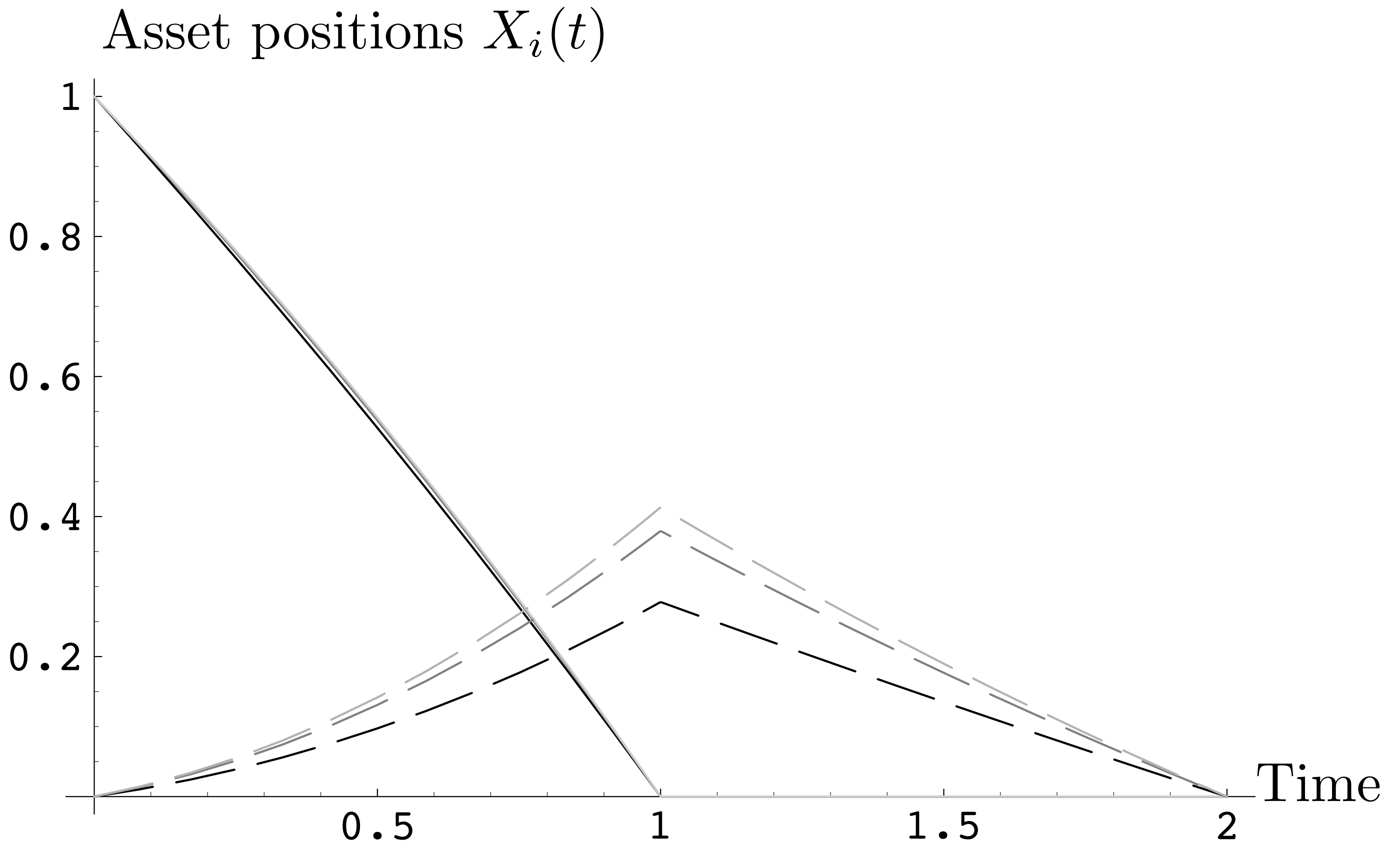


Solid line ∼ seller, dashed line ∼ predator

Elastic market (large temp. impact) without predators



Elastic market market (large temp. impact)

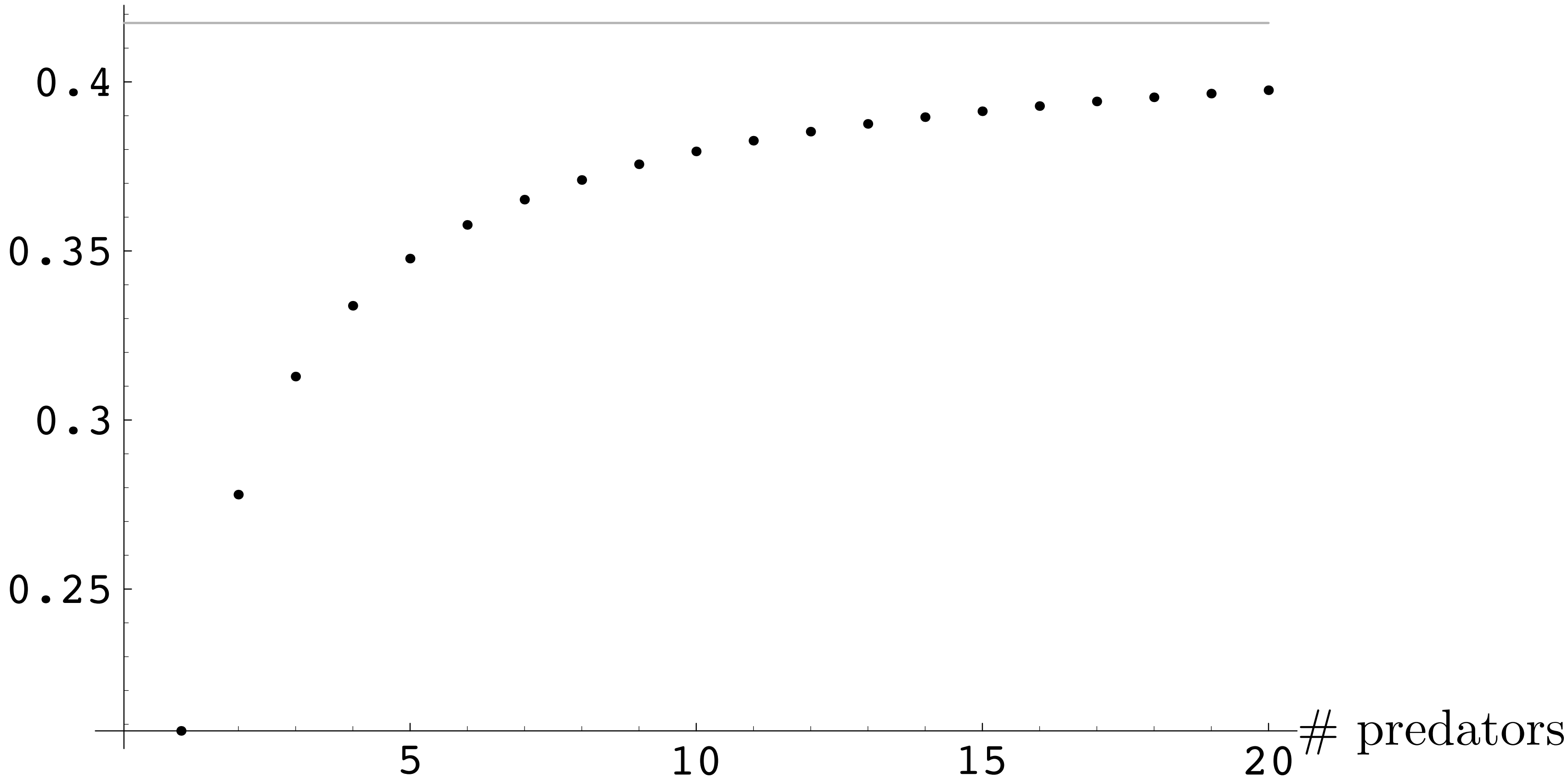


Solid lines ∼ seller, dashed lines ∼ n predators

Black ∼ $n = 2$, dark grey ∼ $n = 10$, light grey ∼ $n = 100$

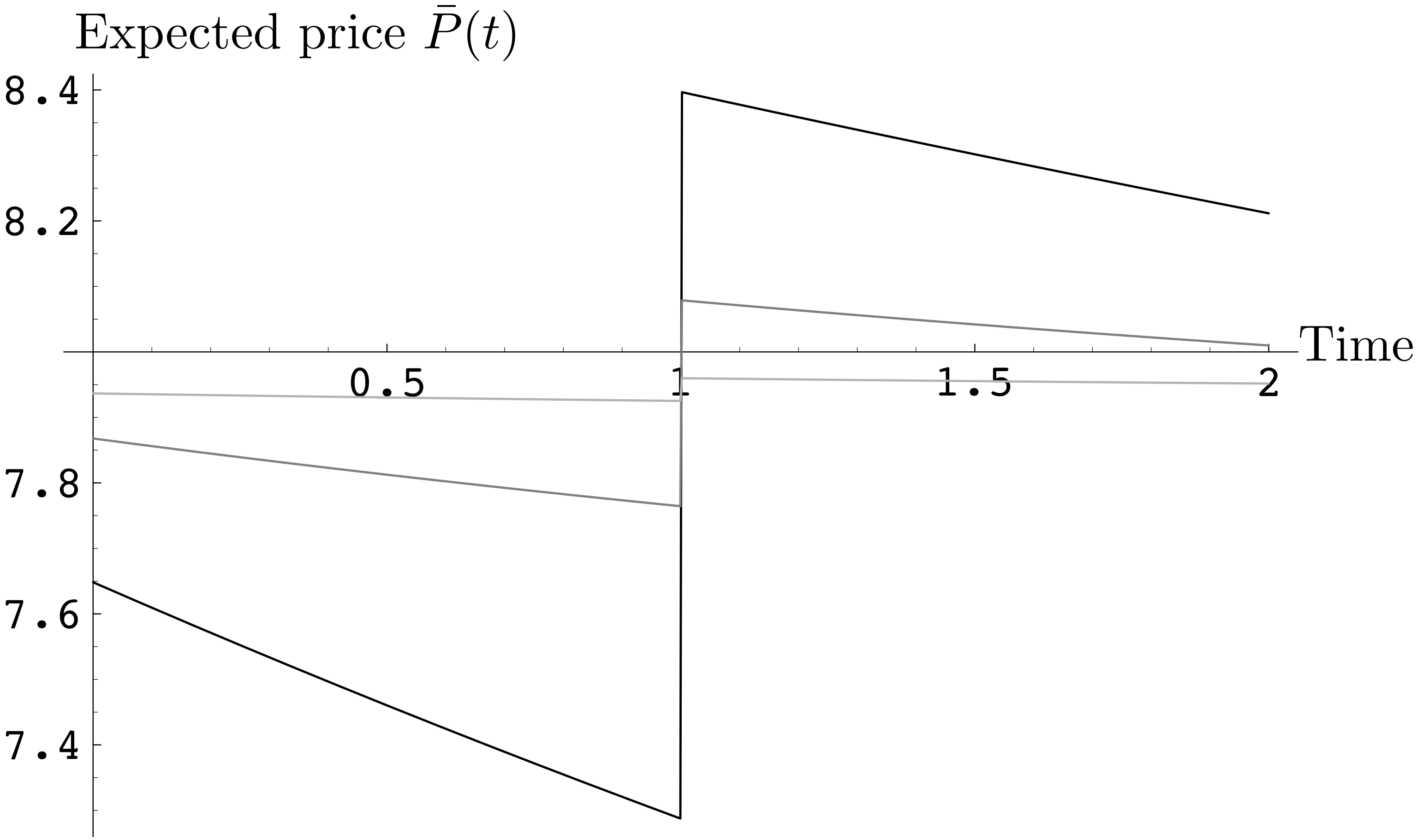
Elastic market (large temp. impact)

Joint asset position $\sum_{i=1}^{n} X_i(T_1)$ of all predators



The grey line represents the limit $\lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1)$

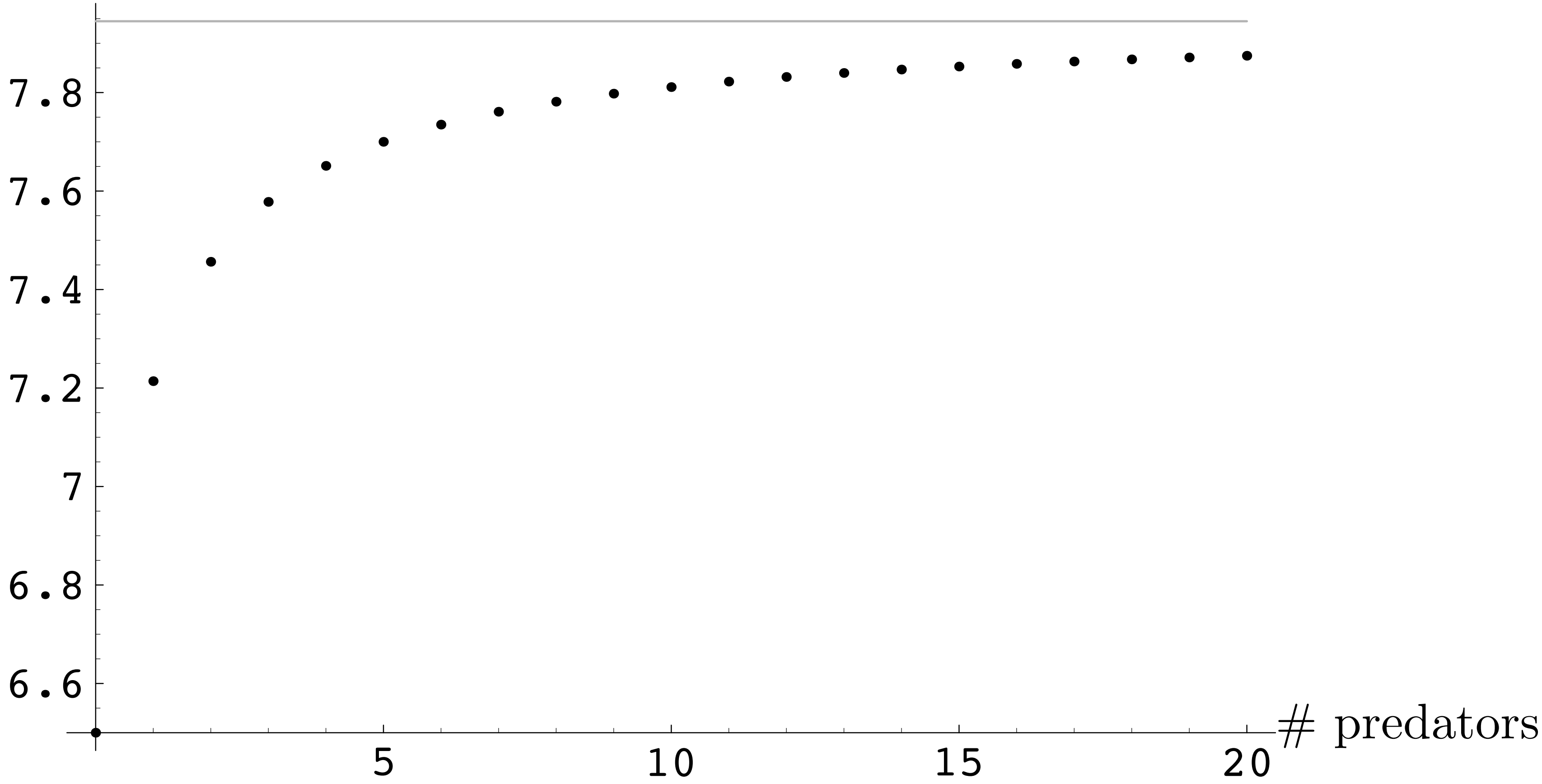
Elastic market (large temp. impact)



Black $\approx n = 2$, dark grey $\approx n = 10$, light grey $\approx n = 100$

Elastic market (large temp. impact)

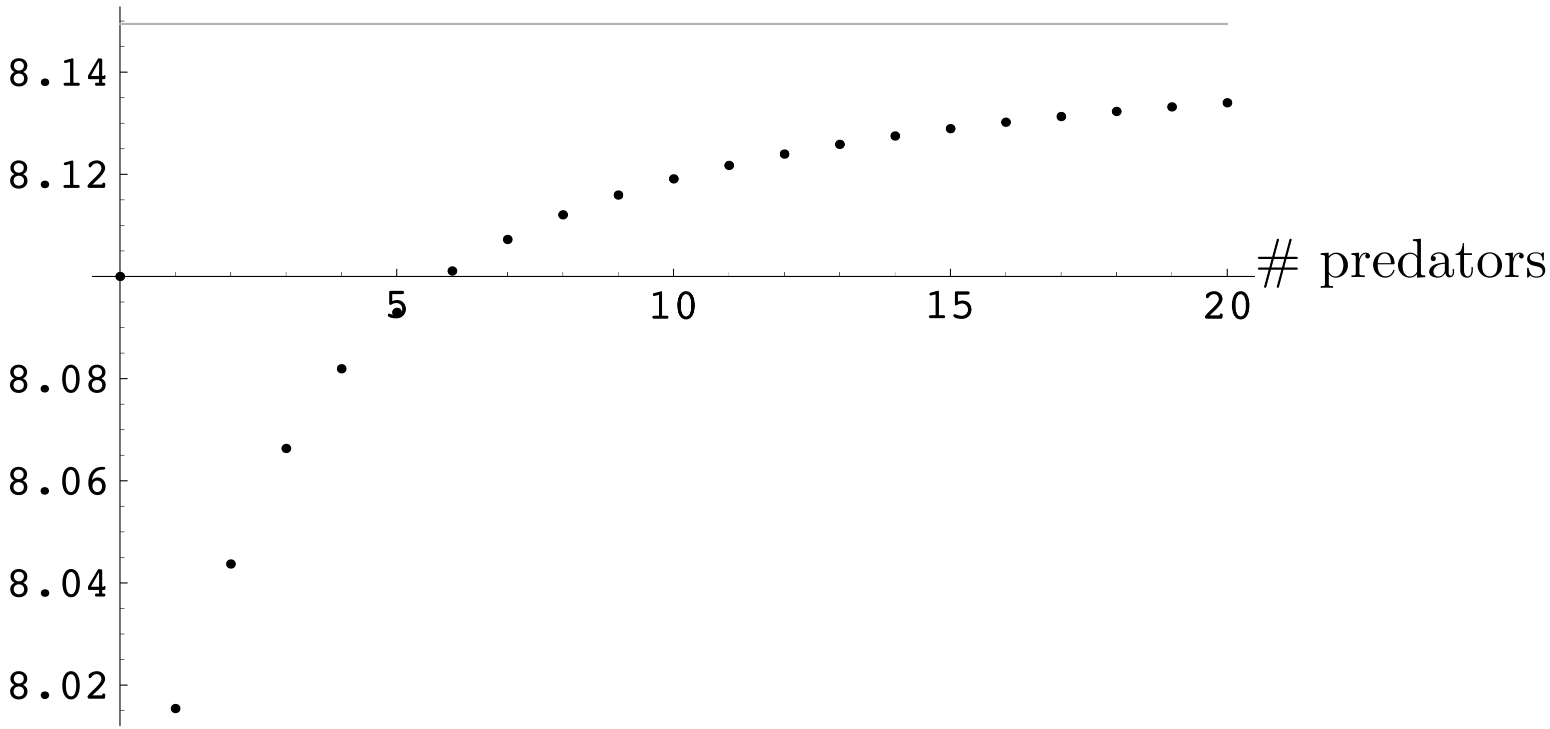
Expected return R_0 for the seller



The grey line represents the limit $n \to \infty$.

Moderate market ($\lambda \approx \gamma$)

Expected return R_0 for the seller



The grey line represents the limit $n \to \infty$. The return for the seller without predators is at the intersection of x- and y-axis.

Theorem 3.

- *For all n, the asset position of the combined asset positions of the competitors is increasing in $\gamma T_1/\lambda$*

- *As $n \uparrow \infty$, it converges to*

$$\lim_{n\to\infty}\sum_{i=1}^{n} X_i(T_1) = \lim_{n\to\infty} nX_1(T_1) = \frac{e^{\frac{\gamma(T_2-T_1)}{\lambda}}-1}{e^{\frac{\gamma T_2}{\lambda}}-1}X_0 > 0$$

- *For all n,*

$$\lim_{\gamma T_1/\lambda\downarrow 0} X_i(T_1) = \frac{T_2-T_1}{(n+1)T_2}X_0 > 0 \qquad \lim_{\gamma T_1/\lambda\uparrow\infty} X_i(T_1) = \frac{-2X_0}{n^3+4n^2+n-2} < 0$$

- *For all n, $\dot{X}_i(t)$ is increasing in t and decreasing in $\gamma T_1/\lambda$ with*

$$\dot{X}_i(0) = \frac{T_2-T_1}{(n+1)T_1T_2}X_0 > 0 \qquad \textit{for } \gamma T_1/\lambda = 0$$

Corollary 4.

There are $L \leq P \in]0, \infty]$ *such that*

- *For* $0 \leq \gamma T_1/\lambda \leq L$, *the competitors are pure liquidity providers, i.e.,* $X_i(t) \geq 0$ *for* $0 \leq t \leq T$
- *For* $L \leq \gamma T_1/\lambda \leq P$, *there is first predatory trading, then liquidity provision, i.e.,* $\dot{X}_i(0) \leq 0$ *and* $X_i(T_1) \geq 0$
- *For* $P < \gamma T_1/\lambda$, *there is pure predation, i.e.,* $X_i(T_1) < 0$

Theorem 5.
In competitive markets (i.e. in the limit $n \uparrow \infty$), the competitors are pure liquidity providers, i.e.,

$$\lim_{n\uparrow\infty}\sum_{i=1}^{n} X_i(t) > 0 \qquad \textit{for } 0 < t \le T_1$$

if and only if

$$\frac{T_2}{T_1} > -\frac{\log(2-e^{\gamma T_1/\lambda})^+}{\frac{\gamma}{\lambda}T_1}$$

Otherwise, they engage in intra-stage predatory trading (i.e., $\sum_i \dot{X}_i(0) < 0$)

Stealth trading: no predators, expected return

$$X_0(P_0 - \gamma X_0/2 - \lambda X_0/T_1).$$

Sunshine trading: large number of predators, expected return

$$X_0\Big(P_0 - \frac{\gamma X_0}{1-e^{-\gamma T_2/\lambda}}\Big)$$

Proposition 6. *For $n \uparrow \infty$, sunshine trading is superior to steath trading if*

$$\frac{1}{2} + \frac{\lambda}{\gamma T_1} > \frac{1}{1-e^{-\frac{\gamma}{\lambda}T_2}}.$$

For $T_2 \uparrow \infty$, a stealth algorithm is beneficial if

$$\frac{\gamma}{\lambda}T_1 < 2$$

Predatory trading vs. liquidity provision: anecdotal evidence

Conclusion

Have studied optimal execution problems on three different levels

- **Microscopic:** Order book models
- **Mesoscopic:** Expected utility maximization in stylized model
- **Macroscopic:** Multi-agent situation; stealth vs. sunshine trading, predation vs. liquidity provision

Thank you