Financial Modeling under Illiquidity: Finance, Economics, Mathematics

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and

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Outline

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Liquidity: What? and Why?

Illiquidity: Now you see it, now you don't

Asset prices in Economics: classical equilibrium theory

Market indifference prices and their dynamics

Optimal investment

Conclusions

What is (il)liquidity?

Economist.com FINANCE & ECONOMICS
ECONOMICS FOCUS

A fluid concept

Feb 8th 2007
From The Economist print edition

Just about everyone agrees that there's a lot of liquidity about—whatever it is

LIQUIDITY is everywhere. Depending on what you read, you may learn that the world's financial markets are awash with it, that there is a glut of it or even that there is a wall of it. But what exactly is it? Again depending on what you read, you may be told that "it is one of the most mentioned, but least understood, concepts in the financial market debate today" or that "there is rarely much clarity about what 'buoyant liquidity' actually means." An economics textbook may bring you clarity—or confusion. It is likely to define liquidity as the ease with which assets can be converted into money. Fine: but that is scarcely the stuff of dramatic metaphors. Liquidity thus defined is surely to be welcomed; floods, gluts and walls of water surely not.

Helpfully, Martin Barnes, of BCA Research, an economic research firm, has laid out three ways of looking at liquidity. The first has to do with overall monetary conditions: money supply, official interest rates and the price of credit. The second is the state of balance sheets—the share of money, or things that can be exchanged for it in a hurry, in the assets of firms, households and financial institutions. The third, financial-market liquidity, is close to the textbook definition: the ability to buy and sell securities without triggering big changes in prices.

...

Aspects of (il)liquidity

Kyle's (1985) characteristics of financial market liquidity:

Tightness: *the cost of turning around a position over a short period of time*
$\rightsquigarrow$ spread, transaction costs

Depth: *the size of an order flow innovation required to change prices a given amount*
$\rightsquigarrow$ these lectures

Resiliency: *the speed with which prices recover from a random, uninformative shock*
$\rightsquigarrow$ market manipulation, Alex Schied's lectures

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Topic of these lectures:

How to account for finite market depth when pricing and hedging financial derivatives?

Market prices depend on transactions

Economist.com **FINANCE & ECONOMICS**

A Japanese trading debacle

Please may I take it back?

Dec 14th 2005 | TOKYO
From The Economist print edition

Red faces all round

At the start of trading on December 8th Mizuho Securities placed an order on the TSE to sell 610,000 shares of J-Com, a small recruiting agency it was bringing to market that day, for ¥1 each: it had meant to sell one share for ¥610,000, the initial offer price. The false order, indeed, was for 40 times more shares than J-Com had outstanding.

Having tried frantically to cancel the order and failed, Mizuho scrambled to buy the shares it had sold but did not own. Some of the nicest sharks in finance, including Morgan Stanley, UBS, Nomura Securities and Nikko Citigroup, detected blood in the water. Meanwhile, as rumours swirled, investors sold the shares of brokers who might have made the mistake (Mizuho did not own up until trading ended). They also sold more broadly, calculating that a troubled broker would unload its own holdings to cover its losses. The Nikkei 225-share average registered its third-biggest daily fall of the year (though it recovered to end December 13th at its highest since 2000).

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1987 stock market crash: program trading?
LTCM crisis: Russian default
dotcom bubble: small volume stocks
subprime crisis: counterparty risk

Illiquidity: an issue when pricing and hedging derivatives

Knock-out call option on the stock with price process $P = (P_t)$:

$$H = (P_T - k)^+ 1_{\{\max_{0 \le t \le T} P_t < b\}}$$

Rationale for this product:

Secure small price k for stock at lower costs than a call!

Black-Scholes approach:

- P geometric Brownian motion:
$$P_0 = p,\ dP_t = P_t(\mu\, dt + \sigma\, dW_t)$$
- option price: $v(T, p) = \mathbb{E}^* e^{-rT} H$
- $\mathbb{P}^*$ martingale measure for $(e^{-rt} P_t)$:
$$d\mathbb{P}^* = \exp(-\lambda W_T - \tfrac{1}{2}\lambda^2 T)\, d\mathbb{P}$$
- $\lambda = (\mu - r)/\sigma$: market price of risk

Knock-out call indeed considerably less expensive than call...

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...but in practice significantly more difficult to hedge:

- replication of H à la Black-Scholes:

$$H = \underbrace{\mathbb{E}^* e^{-rT} H}_{=v(T,p)} + \int_0^T \Delta_t \, dP_t + \int_0^T (v(T-t, P_t) - \Delta_t P_t) r \, dt$$

  where

$$\Delta_t = \partial_p v(T-t, P_t) \quad : \text{'Delta' of the option}$$

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- ...and these costs should be taken into account when pricing the option: not done in the Black-Scholes approach!

Asset prices

Mathematical Finance:

- price dynamics **exogenous** as semimartingale models
- stochastic analysis

+ mathematically tractable
+ dynamic model: hedging
+ 'easy' to calibrate: volatility

– correlation between assets ad hoc

– only suitable for (very) *liquid markets* or *small investors*

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Economics:

- prices **endogeneous**: demand matches supply
- equilibrium theory

+ undeniably reasonable explanation for price formation
+ excellent qualitative properties
+ accounts for intricate interdependencies

– difficult to calibrate: preferences, endowments
– hedging strategies (essentially) not considered

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Problem:

How to bridge the gap between these price formation principles and still say something about hedging in illiquid markets?

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Some models proposed in the literature

Henceforth: interest rate $r = 0$ — for simplicity!

Cvitanic-Ma/-Cuoco: strategy dependent diffusion coefficients

$$dP_t = P_t(\mu_t(\theta_t)\,dt + \sigma_t(\theta_t)\,dW_t)$$

with $\theta_t =$ position at time t.

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- no immediate price impact from large transactions

Some models proposed in the literature

Cetin-Jarrow-Protter: series of independent auctions

$$P_t = p \exp(\sigma W_t + (\mu - \tfrac{1}{2}\sigma^2)t) e^{\eta \Delta\theta_t}$$

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+ 'local' model: no longterm effects $\rightsquigarrow$ tractable, most appropriate for markets with infrequent trades

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+ 'local' model: no longterm effects $\rightsquigarrow$ tractable, most appropriate for markets with infrequent trades
– liquidity effects disappear for absolutely continuous strategies
– option prices not sensitive to liquidity
– hedging strategies merely time averaged versions of Black-Scholes hedges $\rightsquigarrow$ impose constraints on hedging strategies

Some models proposed in the literature

Gökay-Soner: Binomial approximation to Cetin et al.

- consider discrete-time binomial model and solve super-replication problem by dynamic programming
- pass to diffusion-limit by properly rescaling time and space to find nonlinear pde for asymptotic superreplication price $\phi = \phi(t, p)$:

$$-\phi_t - \inf_{\beta \geq 0} \left\{ \frac{1}{2} p^2 \sigma^2 (\phi_{pp} + \beta) + \Lambda(t,p) p^2 \sigma^2 (\phi_{pp} + \beta)^2 \right\} = 0$$

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+ illiquidity causes strictly positive premium over Black-Scholes

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+ Presentation by Selim Gökay on Friday

Some models proposed in the literature

Rogers-Singh: Penalized quadratic optimization

- trading at rates $\dot{\theta}$ only
- seek to minimize sum of
  - expected illiquidity costs: $\mathbb{E}^* \int_0^T P_t \frac{\varepsilon}{2} \dot{\theta}_t^2 \, dt$
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$-$ portfolios not selffinancing

$-$ hedging criterion ad hoc

$-$ even a miniscule transaction causes ruin if carried out fast enough

Some models proposed in the literature

Frey, Papanicolaou & Sircar . . . : demand function

$$P_t = P_0 \exp(\sigma W_t + (\mu - \tfrac{1}{2}\sigma^2)t) e^{\eta\theta_t}$$

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+ immediate price impact from large transactions
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– if we allow for semimartingale strategies: liquidity effect disappears!?!

Wealth dynamics in Frey's model

Evolution of bank account

- family of continuous semimartingales $P(\vartheta) = (P(\vartheta; t))$ $(\vartheta \in \mathbb{R})$:

$$P_t^\vartheta = \text{asset price if our position at time } t \text{ is } \theta_t = \vartheta$$

- $\vartheta \mapsto P(\vartheta)$ smooth so that for $\theta = (\theta_t)$ a semimartingale strategy, the observed price process

$$P_t^\theta := P_t^\vartheta \Big|_{\vartheta=\theta_t} \quad (t \geq 0)$$

is a semimartingale.

- position in bank account resulting from strategy (θ_t):

$$\beta_t = -\int_0^t P_{t-}^\theta \, d\theta_t - \left[P^\theta, \theta\right]_t \quad (t \geq 0)$$

Wealth dynamics in Frey's model

Possible definitions of wealth:

- *book value* or *mark-to-market value*: $V_t^{\text{book}} = \beta_t^\theta + \theta_t \, P_t^\vartheta\big|_{\vartheta=\theta_t}$

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- *block liquidation value*: $V_t^{\text{block}} = \beta_t^\theta + \theta_t P_t^0$
- *realizable portfolio value* or *real wealth*: $V_t = \beta_t^\theta + L(\theta_t; t)$ where

$$L(\vartheta; t) := \int_0^\vartheta P(x; t)\, dx = \text{asymptotic liquidation proceeds}$$

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$$L(\vartheta; t) := \int_0^\vartheta P(x; t)\, dx = \text{asymptotic liquidation proceeds}$$

If $\vartheta \mapsto P_t^\vartheta$ is increasing (as it should be):

$$V_t^{\text{book}} \geq V_t \geq V_t^{\text{block}}$$

We will focus on the realizable portfolio value V!

Dynamics of the real wealth process

For any semimartingale strategy θ:

$$\begin{aligned} V_t - V_{0-} &= \int_0^t L(\theta_{s-}; ds) - \frac{1}{2}\int_0^t P'(\theta_{s-}; s)\, d\,[\theta]^c_s \\ &\quad - \sum_{0 \le s \le t} \int_{\theta_{s-}}^{\theta_s} \{P(\theta_s; s) - P(x; s)\}\, dx\,. \end{aligned}$$

Three components:

- $\int_0^t L(\theta_{s-}; ds)$: nonlinear stochastic integral describing profit or loss due to exogenous shocks
- $\sum_{0 \le s \le t} \int_{\theta_{s-}}^{\theta_s} \{P(\theta_s; s) - P(x; s)\}\, dx \ge 0$: transaction costs due to block orders
- $\frac{1}{2}\int_0^t P'(\theta_{s-}; s)\, d\,[\theta]^c_s \ge 0$: transaction costs due to 'intense trading'

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Definition

θ is called admissible if $\int_0^{\cdot} L(\theta_{s-}; ds) \ge \mathrm{const.}$.

Nonlinear stochastic integration

Given: 'smooth' family of continuous semimartingales

$$L(\vartheta) = (L(\vartheta; t))_{t \geq 0} \; (\vartheta \in \mathbb{R})$$

Define stochastic integral (see Kunita (1991)):

- for simple strategies $\theta = \sum_i \vartheta_{i+1} 1_{(s_i, s_{i+1}]}$ with $\vartheta_{i+1} \in L^0(\mathscr{F}_{s_i})$:

$$\int_0^t L(\theta_s; ds) := \sum_i \{L(\vartheta_{i+1}; s_{i+1} \wedge t) - L(\vartheta_{i+1}; s_i \wedge t)\}$$

- extend to general strategies by approximation

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Itô-Wentzell formula

If $L(\vartheta)$ $(\vartheta \in \mathbb{R})$ is smooth and $\theta = (\theta_t)$ a semimartingale, then also $L^\theta = (L(\theta_t; t))$ is a semimartingale and its dynamics are given by

$$\begin{aligned} L(\theta_t; t) - L(\theta_{0-}; 0) &= \int_0^t L(\theta_{s-}; ds) + \int_0^t L'(\theta_{s-}; s)\, d\theta_s \\ &+ \left[\int_0^{\cdot} L'(\theta_{s-}; ds), \theta\right]_t + \frac{1}{2}\int_0^t L''(\theta_{s-}; s)\, d\,[\theta]^c_s \\ &+ \sum_{0 \leq s \leq t} \{\Delta L(\theta_s; s) - L'(\theta_{s-}; s)\Delta\theta_s\}\,. \end{aligned}$$

Absence of arbitrage for the large investor

Recall:

$$V_t - V_{0-} = \int_0^t L(\theta_{s-}; ds) - \frac{1}{2}\int_0^t P'(\theta_{s-}; s)\, d\,[\theta]_s^c - \sum_{0\le s\le t}\int_{\theta_{s-}}^{\theta_s} \{P(\theta_s; s) - P(x; s)\}\, dx\,.$$

Proposition

If there exists a universal equivalent martingale measure, i.e., $\mathbb{P}^* \approx \mathbb{P}$ such that all P^ϑ $(\vartheta \in \mathbb{R})$ are $\mathbb{P}^*$-martingales, then there is no admissible semimartingale strategy θ such that

$V_T \geq V_0$ a.s. and ' $>$ ' holds with positive probability.

Approximate attainability

Recall:

$$V_t - V_{0-} = \int_0^t L(\theta_{s-}; ds) - \frac{1}{2}\int_0^t P'(\theta_{s-}; s)\, d\,[\theta]_s^c - \sum_{0 \le s \le t} \int_{\theta_{s-}}^{\theta_s} \{P(\theta_s; s) - P(x; s)\}\, dx\,.$$

Important observation:

Continuous strategies of bounded variation do *not* incur transaction costs since for these

$$V_t - V_{0-} = \int_0^t L(\theta_{s-}; ds)\,.$$

Question:

Which payoffs are attainable by such ‘tame’ strategies?

Approximate attainability

Definition

- $H \in L^0(\mathscr{F}_T)$ is called *approximately attainable for initial capital* v if for any $\varepsilon > 0$ there is θ^ε admissible for the large investor such that

$$V^{\theta^\varepsilon} \text{ with } V^{\theta^\varepsilon}_{0-} = v \text{ satisfies } \left| V^{\theta^\varepsilon}_T - H \right| \leq \varepsilon \ \mathbb{P}\text{–a.s.}$$

- $H \in L^0(\mathscr{F}_T)$ is called *attainable modulo transaction costs for initial capital* v if

$$H = v + \int_0^T L(\theta_s; ds)$$

for θ L–integrable with $\int_0^\cdot L(\theta_s; ds) \geq \mathrm{const.}$.

Approximate attainability

Theorem

Any contingent claim $H \in L^0(\mathscr{F}_T)$ which is attainable modulo transaction costs is approximately attainable with the same initial capital.

The scope of tame integrands

Approximation theorem for stochastic integrals

Assume L^ϑ $(\vartheta \in \mathbb{R})$ is a smooth family of semimartingales. Let θ be an L–integrable, predictable process and fix $\vartheta_0 \in L^0(\mathscr{F}_0)$, $\vartheta_T \in L^0(\mathscr{F}_{T-})$. Then, for any $\varepsilon > 0$, there exists a predictable process θ^ε with continuous paths of bounded variation such that $\theta_0^\varepsilon = \vartheta_0$, $\theta_T^\varepsilon = \vartheta_T$ and

$$\sup_{0\le t\le T}\left|\int_0^t L(\theta_s; ds) - \int_0^t L(\theta_s^\varepsilon; ds)\right| \le \varepsilon \quad \mathbb{P}\text{–a.s.}$$

Proof of approximation theorem

- **Lemma:** *For any given $\tau \le T$, $\vartheta_\tau \in L^0(\mathscr{F}_\tau)$, and $\varepsilon > 0$, there exists a predictable process $\theta^{\varepsilon,\tau,\vartheta_\tau}$ with continuous paths of bounded variation such that $\theta^{\varepsilon,\tau,\vartheta_\tau}_\tau = \vartheta_\tau$, $\theta^{\varepsilon,\tau,\vartheta_\tau}_T = \vartheta_T$ and*

$$\mathbb{P}\left[\sup_{\tau\le t\le T}\left|\int_\tau^t L(\theta_s, ds) - \int_\tau^t L(\theta^{\varepsilon,\tau,\vartheta_\tau}_s, ds)\right| \ge \varepsilon\right] \le \varepsilon\,.$$

- $\varepsilon_n := \varepsilon/2^n$ $(n = 0, 1, \ldots)$, $\tau_0 := 0$, $\theta^\varepsilon_0 := \vartheta_0$
- inductive extension: $\theta^\varepsilon := \theta^{\varepsilon_{n+1},\tau_n,\theta^\varepsilon_{\tau_n}}$ on $(\tau_n, \tau_{n+1}]$ where

$$\tau_{n+1} := \inf\left\{t \ge \tau_n \ : \ \left|\int_{\tau_n}^t L(\theta_s, ds) - \int_{\tau_n}^t L(\theta^\varepsilon_s, ds)\right| > \varepsilon_{n+1}\right\} \wedge T\,.$$

  and

$$\mathbb{P}\left[\tau_{n+1} < T\right] \le \varepsilon_{n+1} = \varepsilon/2^{n+1}$$

  $\rightsquigarrow$ continuous adapted process θ^ε of bounded variation with $\theta^\varepsilon_T = \vartheta_T$ — this θ^ε does the job!

Attainability for small and large investors

Assumption

- $P(\vartheta; t) = P(\vartheta; 0) + \int_0^t p_s^\vartheta \, dP_s^0$ for some $p^\vartheta \in L(P^0)$
- For $\mathbb{P} \otimes d[P^0]$–a.e. $(\omega, t) \in \Omega \times [0, T]$, the mapping $\vartheta \mapsto \int_0^\vartheta p_s^x(\omega) \, dx$ is surjective.

Wealth dynamics in terms of P^0:

$$L(\theta_t; dt) = \text{`} \int_0^{\theta_t} dP_t^x \, dx \text{'} = \left\{ \int_0^{\theta_t} p_t^x \, dx \right\} dP_t^0$$

Hence:

P^0-integrable small investor strategies ξ $\quad \leftrightsquigarrow \quad$ L-integrable 'strategies' θ

via

$$\xi_t = \int_0^{\theta_t} p_t^x \, dx$$

Description of attainable claims

Any claim $H \in L^0(\mathscr{F}_T)$ which is attainable in the small investor model P^0 is approximately attainable for the same initial capital in our large investor model.

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In particular:

Small investor will quote the **same** option price as the large investor!

No liquidity effect!

Illiquidity: Now you see it, now you don't

- illiquidity surprisingly hard to model
- ad hoc extensions of Black-Scholes often exhibit not necessarily desirable features:
    - constrained strategy spaces
    - small trades with high costs
- illiquidity effect may disappear through modeling loophole
- comparability to Black-Scholes?

Outline

Prices from demand and supply

Arrow, Debreu, Radner, . . . : Microeconomics 101

Equilibrium approach

- specify economy: economic agents' endowments and preferences
- allow these agents to trade: exchange economy
- consider pricing rules: market clearing

The economic agents

- $\mathscr{A}$ finite set of agents
- Endowment of agent $a \in \mathscr{A}$: e_a bounded $\mathscr{F}_T$-measurable random variable
- Preferences of agent $a \in \mathscr{A}$: utility function $u_a : \mathbb{R} \to \mathbb{R}$ for wealth at time T
  - $\tilde{e}_a$ considered better than e_a iff $\mathbb{E}u_a(\tilde{e}_a) > \mathbb{E}u_a(e_a)$
  - u_a is increasing: more is better
  - u_a is concave: risk aversion
  - u_a has bounded absolute risk aversion: $c \leq -\frac{u_a'(x)}{u_a''(x)} \leq C$
- Example: $u_a(x) = -\exp(-\alpha_a x)$ with $\alpha_a > 0$

The exchange economy

- Pricing rule: positive linear functional Π on L^∞
- Agent a has $\Pi(e_a)$ Euro to spend and will choose to buy $\tilde{e}_a^\Pi$, the solution to his utility maximization problem:

$$\mathbb{E}u_a(\tilde{e}_a^\Pi) = \max_{\tilde{e}_a \text{ such that } \Pi(\tilde{e}_a) \leq \Pi(e_a)} \mathbb{E}u_a(\tilde{e}_a)$$

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Does the market clear?

Can we find a pricing rule Π^* such that the induced allocation of wealth $(\tilde{e}_a^{\Pi^*})_{a \in \mathscr{A}}$ is feasible in the sense that

$$\sum_{a \in \mathscr{A}} \tilde{e}_a^{\Pi^*} \leq \sum_{a \in \mathscr{A}} e_a \quad ?$$

If so, Π^* will be called an equilibrium pricing rule and $(\tilde{e}_a^{\Pi^*})_{a \in \mathscr{A}}$ its equilibrium allocation of wealth.

Candidates for equilibria: Pareto optima

An allocation $(\tilde{e}_a)_{a\in\mathscr{A}}$ is called Pareto optimal if there is no other allocation $(\tilde{e}'_a)_{a\in\mathscr{A}}$ such that

$$\sum_{a\in\mathscr{A}} \tilde{e}'_a \leq \sum_{a\in\mathscr{A}} \tilde{e}_a$$

and

$$\mathbb{E}u_a(\tilde{e}'_a) \geq \mathbb{E}u_a(\tilde{e}_a) \text{ for all } a \in \mathscr{A} \text{ and ` } > \text{ ' for some } a \in \mathscr{A}.$$

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Lemma

Every equilibrium allocation is Pareto optimal.

Characterizations of Pareto optima

Lemma

Equivalent for an allocation $(\tilde{e}_a)_{a\in\mathscr{A}}$ with $\Sigma = \sum_{a\in\mathscr{A}} \tilde{e}_a$:

(i) $(\tilde{e}_a)_{a\in\mathscr{A}}$ is Pareto optimal.

(ii) Given the respective endowments $\tilde{e}_a$ $(a \in \mathscr{A})$ all agents will quote the same marginal indifference prices:

$$\Pi(X) = \frac{\mathbb{E}u_a'(\tilde{e}_a)X}{\mathbb{E}u_a'(\tilde{e}_a)} = \frac{\mathbb{E}u_b'(\tilde{e}_b)X}{\mathbb{E}u_b'(\tilde{e}_b)} \; (X \in L^\infty) \text{ for any } a, b \in \mathscr{A}\,.$$

(iii) $(\tilde{e}_a)_{a\in\mathscr{A}}$ is the solution to a social welfare problem:

$$\sum_{a\in\mathscr{A}} w_a \mathbb{E}u_a(\tilde{e}_a) \to \max \text{ subject to } \Sigma = \sum_{a\in\mathscr{A}} \tilde{e}_a$$

for suitable weights $w_a > 0$ $(a \in \mathscr{A})$ with $\sum_a w_a = 1$.

There are 1-1 correspondences: $w \leftrightarrow \Pi^w \leftrightarrow \tilde{e}^w$

Existence of an Arrow-Debreu equilibrium

Theorem

There exists an equilibrium pricing rule Π^* and an equilibrium allocation $(\tilde{e}_a^*)_{a\in\mathscr{A}}$.

Sketch of Proof: Consider the excess demand map

$$w = (w_a)_{a\in\mathscr{A}} \mapsto (\frac{1}{w_a}\Pi^w(\tilde{e}_a^w - e_a))_{a\in\mathscr{A}}$$

and use a fixed point argument to conclude that this map has a zero w^*: $\tilde{e}^* = \tilde{e}^{w^*}$, $\Pi^* = \Pi^{w^*}$; see, e.g., Dana-Le Van (1996). $\square$

Implementation as an Arrow-Radner equilibrium

Assume: $\mathbb{F}$ generated by B^*, Brownian motion under $\mathbb{P}^*$ with

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \frac{u_a'(e_a^*)}{\mathbb{E} u_a'(e_a^*)}$$

Then:

$$\tilde{e}_a^* = \Pi^*(e_a) + \int_0^T \eta_t^a \, dB_t^*$$

Hence: all agents can trade in a **single** security, e.g., with price process $P_t = B_t^*$ to attain their Arrow-Debreu allocation $\tilde{e}_a^*$.

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+ trading strategies can be considered ...

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Hence: all agents can trade in a **single** security, e.g., with price process $P_t = B_t^*$ to attain their Arrow-Debreu allocation $\tilde{e}_a^*$.

+ trading strategies can be considered ...
– ... but not in pre-specified securities
– and no real role to play for derivatives

$\rightsquigarrow$ We have to resort to equilibria in incomplete markets: fairly complicated, no gold standard available; see Magill-Quinzii (1996)

Equilibrium asset pricing?

+ Equilibrium prices match demand and supply.
+ Good qualitative properties: risk sharing/exchange, efficiency
+ Marginal prices provide asset prices in line with the economy's preferences.
+ Equilibria can be implemented via trading strategies: Radner

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- \+ Equilibrium prices match demand and supply.
- \+ Good qualitative properties: risk sharing/exchange, efficiency
- \+ Marginal prices provide asset prices in line with the economy's preferences.
- \+ Equilibria can be implemented via trading strategies: Radner
- – Unclear how to discuss hedging of contingent claims in this (essentially) static setting: too far away from Black-Scholes!

Outline

Liquidity: What? and Why?

Illiquidity: Now you see it, now you don't

Asset prices in Economics: classical equilibrium theory

Market indifference prices and their dynamics

Optimal investment

Conclusions

A large investor interacting with market makers

Our setting in a nutshell:

- market makers: economic agents required to quote prices for certain financial products
- large trader submits orders to market makers
- market makers fill orders of large trader and charge/pay him accordingly
- market makers continually hedge their positions
- market makers quote best prices which allow them to fill the large investors order without decreasing their expected utility

The simplest case

A single market maker ...

- a market maker accepts orders for a contingent claim paying

$$F = B_T \quad \text{at maturity} \quad T > 0$$

  where B is a Brownian motion: forward contract, cf. Bachelier model
- only additional investment opportunity: money market $r = 0$
- prices are quoted by indifference principle:

$$\text{position before transaction} \ \sim \ \text{position after transaction}$$

- market maker's preferences are modeled by exponential utility:

$$u(x) = -\exp(-\alpha x) \quad \text{where} \quad \alpha > 0 \text{ absolute risk aversion}$$

The simplest case

A single investor's wealth dynamics ...

- at time $t = 0$:
  - investor asks market maker for ϑ claims
  - market maker quotes indifference price $P_0(\vartheta)$:

$$u(0) = \mathbb{E}u(P_0(\vartheta) - \vartheta F) \text{ i.e. } P_0(\vartheta) = \vartheta B_0 + \frac{1}{2}\alpha\vartheta^2 T$$

- at time $t = 1$:
  - investor liquidates position ϑ
  - market maker pays indifference price $P_1(\vartheta)$:

$$P_1(\vartheta) = \vartheta B_1 + \frac{1}{2}\alpha\vartheta^2(T - 1)$$

- investor's P & L from $t = 0$ to $t = 1$, $\Delta t = 1$:

$$V_1 - V_0 = P_1(\vartheta) - P_0(\vartheta) = \underbrace{\vartheta(B_1 - B_0)}_{\text{liquid P \& L}} - \underbrace{\frac{1}{2}\alpha\vartheta^2\Delta t}_{\text{liquidity premium}}$$

The simplest case

As a result ...

- Continuous time wealth dynamics:

$$V_T(\theta) = \int_0^T \theta_s \, dB_s - \frac{1}{2}\alpha \int_0^T \theta_s^2 \, ds$$

- Small investments governed by Bachelier-model:

$$V_T(\varepsilon\theta) = \varepsilon \int_0^T \theta_s \, dB_s + o(\varepsilon)$$

- Reasonable wealth dynamics:

No arbitrage for *any* predictable strategy.

Pricing and hedging options

Hedging problem:

How can our investor hedge a contingent claim G depending on B?

Lemma

For every claim $G \in \sigma(B_s, s \leq T)$ such that $\mathbb{E} \exp(\alpha G) < \infty$, there exists a replicating portfolio. Its value process is given by

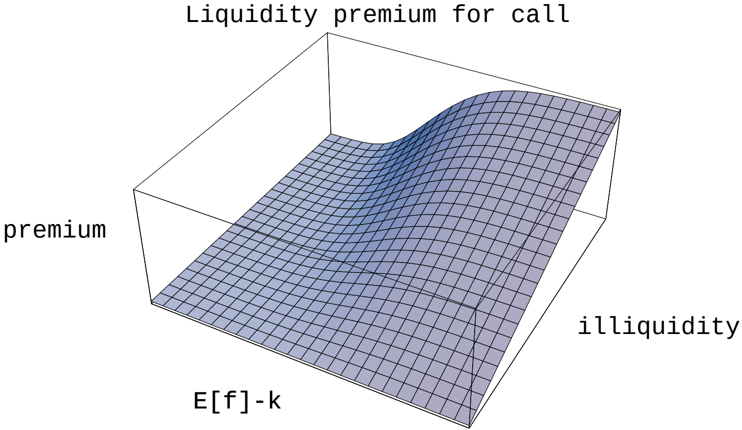
$$V_t = \frac{1}{\alpha} \log \mathbb{E} \left[\exp(\alpha G) \,|\, \mathscr{F}_t \right] .$$

- replication price for G is market maker's indifference price for $-G$, even though he is marketing F not G
- compare w/ Black-Scholes: risk-premium for call inherited from stock

Quantitative analysis of liquidity effects

Assume $F = B_T$ is the traded underlying and consider a call option on F maturing at T: $G = (F - k)^+$

What is the liquidity premium to be charged for a call option?



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Does illiquidity cause hedge ratios to increase or decrease?

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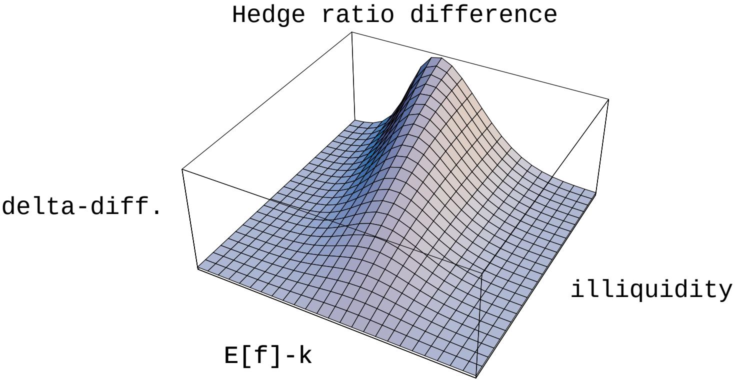

Optional decomposition

Superhedging?

What is the most cost effective way to super-replicate claims which do not merely depend on B?

Theorem (Said)

We have the superhedging duality

$$\inf\{V_0 \in \mathbb{R} \mid V_T \geq g \text{ for some strategy } \theta\} = \sup_{\mathbb{P}^*} \frac{1}{\alpha} \log \mathbb{E}^*[\exp(\alpha g)]$$

where the sup is taken over all martingale measures $\mathbb{P}^* \approx \mathbb{P}$ *for* B*. A superhedging strategy* θ *can be calculated from the multiplicative optional decomposition*

$$\sup_{\mathbb{P}^*} \mathbb{E}^*_t[\exp(\alpha g)] = \mathscr{E}\Big(\int \alpha\theta \, dW\Big)_t D_t \quad (0 \leq t \leq T)\,.$$

Utility maximization

Assume:

- under the investor's beliefs $\tilde{\mathbb{P}}$, B is a Brownian motion with drift $\lambda = \mu/\sigma \in \mathbb{R}$, $H = \sigma B_T$:

  $$dB_t = d\tilde{B}_t + \lambda\, dt \ \rightsquigarrow\ dV_t = \theta_t(\sigma d\tilde{B}_t + \mu\, dt) - \frac{1}{2}\alpha\theta_t^2\, dt$$

- investor has utility function $\tilde{U}$ and investment horizon T

Investment problem:

How to invest some given initial capital $V_0 = x$?

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- Hamilton-Jacobi-Bellman equation for $u = u(t, x)$:

  $$u(T, x) = \tilde{U}(x), \quad u_t + \frac{1}{2}\frac{\mu^2 u_x^2}{\sigma^2 u_{xx} - \alpha u_x} = 0$$

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- in case $\tilde{U}(x) = -\exp(-\beta x)$: value function

  $$u(t, x) = -\exp\left(-\frac{1}{2}\frac{\mu^2}{\sigma^2 + \alpha/\beta}(T - t) - \beta x\right)$$

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- in case $\tilde{U}(x) = -\exp(-\beta x)$: optimal investment

  $\theta(t,x) \equiv \dfrac{\mu}{\alpha + \sigma^2\beta}$ i.e. more 'conservative' than Merton

Non-equivalence of underlyings and derivatives

Recall: For $F = B_T$, every bounded contingent claim $\vartheta G \in \mathbb{D}^{1,2}$ is replicable and a hedging strategy is given by

$$\Delta_t^{\vartheta} = \mathbb{E}_t^{\alpha\vartheta}[D_t(\vartheta G)] = \frac{\mathbb{E}_t[\exp(\alpha\vartheta G) D_t(\vartheta G)]}{\mathbb{E}_t[\exp(\alpha\vartheta G)]}$$

Question:

If market maker deals in G, can we replicate ϑF?

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Question:

If market maker deals in G, can we replicate ϑF?
NOT IN GENERAL!

Counterexample: $G = (-k) \vee W_T \wedge k$ cannot replicate all $\vartheta f = \vartheta B_T$ $(\vartheta \in \mathbb{R})$ as

$$\sup_{\vartheta \in \mathbb{R}} \Delta_t^\vartheta < +\infty \quad \text{and} \quad \inf_{\vartheta \in \mathbb{R}} \Delta_t^\vartheta > -\infty \quad \text{for} \quad t < T$$

Intuition: Extreme long positions in G can be acquired only by paying essentially G's maximal possible payoff k. This payoff does not fluctuate, and so it becomes impossible to scale exposure to external shocks dB_t at will.

Does this generalize?

- more than one underlying?
- underlyings other than $F = B_T$?
- utility functions other than $u(x) = -\exp(-\alpha x)$?
- more than one market maker?
- risk management by market makers?
- more than one investor?
- ...

General setting

Financial model

- beliefs and information flow described by stochastic basis $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P})$
- marketed claims: European with payoff profiles $\psi_i \in L^0(\mathscr{F}_T)$ $(i = 1, \dots, I)$ possessing all exponential moments
- utility functions $u_m : \mathbb{R} \to \mathbb{R}$ $(m \in \mathscr{M})$ with bounded absolute risk aversion:

$$0 < c \le -\frac{u_m''(x)}{u_m'(x)} \le C < \infty$$

  $\rightsquigarrow$ similar to exponential utilities
- initial endowments $e_0^m \in L^0(\mathscr{F}_T)$ $(m \in \mathscr{M})$ have finite exponential moments and form a Pareto-optimal allocation

Pareto-optimal allocations

Recall:
$e = (e^m) \in L^0(\mathscr{F}_T, \mathbb{R}^{\mathscr{M}})$ is Pareto-optimal iff the large investor gets the same marginal indifference price quotes from all market makers, i.e., we have a universal marginal pricing measure $\mathbb{Q}(e)$ for the market:

$$\frac{d\mathbb{Q}(e)}{d\mathbb{P}} = \frac{u'_m(e^m)}{\mathbb{E}u'_m(e^m)} \quad \text{independent of} \quad m \in \mathscr{M}$$

Note:
Pareto-optimal allocations realized through trades among market makers $\rightsquigarrow$ complete Over The Counter (OTC)-market

A single transaction

- pre-transaction endowment of market makers: $e = (e^m)$ with total endowment $\Sigma = \sum_m e^m$
- investor submits an order for $q = (q^1, \ldots, q^I)$ claims and receives x in cash
- total endowment of market makers after transaction

$$\tilde{\Sigma} = \Sigma - (x + \langle q, \psi \rangle)$$

is redistributed among the market makers to form a new Pareto optimal allocation of endowments $\tilde{e} = (\tilde{e}^m)$

Obvious question:

How exactly to determine the cash transfer x and the new allocation $\tilde{e}$?

A single transaction

Theorem

There exists a unique cash transfer x *and a unique Pareto-optimal allocation* $\tilde{e} = (\tilde{e}^m)$ *of the total endowment* $\tilde{\Sigma} = \Sigma - (x + \langle q, \psi \rangle)$ *such that each market maker is as well-off after the transaction as he was before:*

$$\mathbb{E} u_m(\tilde{e}^m) = \mathbb{E} u_m(e^m) \quad (m \in \mathscr{M})\,.$$

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Note:

The cash transfer x can be viewed as the **market's indifference price** for the transaction: it is the minimal amount for which the market makers can accommodate the investor's order without anyone of them being worse-off.

$\leadsto$ most friendly market environment for our investor!

Information and price formation

Why don't market makers improve their utility?

At any moment, the market makers do not make guesses about or anticipate future trades of the investor.

$\Longleftrightarrow$ Any two strategies coinciding up to time t induce the same price dynamics up to t.

$\Longleftrightarrow$ The investor can split any order into a sequence of very small orders each of which is filled at the market's current marginal utility indifference price.

$\Longleftrightarrow$ The expected utilities of our market makers do not change.

Comparison to classical Arrow-Debreu setting

- their investor completely reveals his strategy at time 0
- market makers take this into account when forming Pareto allocation
- and thus gain in terms of utility

The wealth dynamics for simple strategies

When our investor follows a simple strategy

$$\theta_t = \sum_n \vartheta_n 1_{(t_{n-1},t_n]}(t) \quad \text{with} \quad \vartheta_n \in L^0(\mathscr{F}_{t_{n-1}})$$

we can proceed inductively to determine the corresponding cash balance process

$$X_t = \sum_n x_n 1_{(t_{n-1},t_n]}(t)$$

and (conditionally) Pareto-optimal allocations

$$E_t = \sum_n e_n 1_{(t_{n-1},t_n]}(t)\,.$$

In particular, we obtain the investor's terminal wealth mapping:

$$\theta \mapsto V_T(\theta) = \langle \theta_T, \psi \rangle + X_T = \sum_m e_0^m - \sum_m E_T^m$$

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Mathematical challenge:

How to consistently pass to general predictable strategies?

More on Pareto-optimal allocations

We need to keep track of those allocations!

Lemma

The following conditions are equivalent:

1. $e = (e^m)$ *is Pareto-optimal given* $\mathscr{F}_t$ *with total endowment* $\Sigma = \sum_m e^m$.
2. *There exist weights* $W_t = (W_t^m) \in L^0(\mathscr{F}_t, \mathscr{S})$ *such that* e *solves the social planner's allocation problem*

$$\max_{e\,:\,\sum_m e^m = \Sigma} \sum_m W_t^m \mathbb{E}\left[u_m(e^m) \,|\, \mathscr{F}_t \right],$$

where $\mathscr{S} = \{w \in \mathbb{R}_+^M \mid \sum_m w^m = 1\}$.

Moreover, there is actually a 1-1-correspondence between all Pareto allocations of Σ *and weights in* $\mathscr{S}$.

The technical key observation

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But: (W_t, X_t) changes whenever θ_t does: 'wild' dynamics!

Fortunately: Given $\vartheta = \theta_t$, (W_t, X_t) can be recovered from the vector of the market makers' expected utilities $u = U_t$:

$$W_t = W_t(u, \vartheta), \quad X_t = X_t(u, \vartheta)$$

— and these utilities evolve as martingales:

- no changes because of transactions: indifference pricing principle
- changes induced by arrival of new information: martingales

Convex duality

Theorem

The social planner's utility

$$r_t(w,x,\vartheta) = \max_{\alpha\,:\,\sum_m \alpha^m = \Sigma_0 - (x + \langle \vartheta, \psi \rangle)} \sum_m w^m \mathbb{E}\left[u_m(\alpha^m) \,|\, \mathscr{F}_t \right]$$

has the dual

$$\tilde{r}_t(u,y,\vartheta) = \sup_w \inf_x \left\{ \langle w, u \rangle + xy - r_t(w,x,\vartheta) \right\}$$

in the sense that

$$r_t(w,x,\vartheta) = \inf_u \sup_y \left\{ \langle w, u \rangle + xy - \tilde{r}_t(u,y,\vartheta) \right\}$$

and (w,x) *is a saddle point for* $\tilde{r}_t(u,y,\vartheta)$ *if and only if* (u,y) *is a saddle point for* $r_t(w,x,\vartheta)$*. In this case:*

$$w = \partial_u \tilde{r}_t(u,y,\vartheta),\; x = \partial_y \tilde{r}_t(u,y,\vartheta),\; u = \partial_w r_t(w,x,\vartheta),\; y = \partial_x r_t(w,x,\vartheta)$$

An SDE for the utility process

We need to understand the martingale dynamics of expected utilities.

Assumption

- *filtration generated by Brownian motion* B
- *contingent claims* ψ *and total initial endowment* Σ_0 *Malliavin differentiable with bounded Malliavin derivatives*
- *bounded prudence:* $\left| -\frac{u_m'''(x)}{u_m''(x)} \right| \leq K < +\infty$

Notation:

- $E(w, x, \vartheta) =$ Pareto allocation of $\Sigma_0 - (x + \langle \vartheta, \psi \rangle)$ with weights w
- $U_t(w, x, \vartheta) = (\mathbb{E}\left[u_m(E^m(w, x, \vartheta)) \,|\, \mathscr{F}_t \right])_{m \in \mathscr{M}}$
- $dU_t(w, x, \vartheta) = U(w, x, \vartheta; dt) = F_t(w, x, \vartheta)\, dB_t$

An SDE for the utility process

Theorem

For every simple strategy θ the induced process $u = (u_t)$ of expected utilities for our market makers solves the SDE

$$\begin{aligned} u_0 &= (\mathbb{E} u_m(e_0^m))_{m \in \mathscr{M}} \\ du_t &= U(W_t(u_t, \theta_t), X_t(u_t, \theta_t); dt) \\ &= G_t(u_t, \theta_t)\, dB_t, \end{aligned}$$

where

$$G_t(u, \vartheta) = F_t(W_t(u, \vartheta), X_t(u, \vartheta), \vartheta)\,.$$

Note:

This SDE makes sense for any predictable (sufficiently integrable) strategy θ!

The rest: Stability theory for SDEs

Corollary

For θ^n such that $\int_0^T (\theta_t^n - \theta_t)^2 \, dt \to 0$ in probability, the corresponding solutions u^n converge uniformly in probability to the solution u corresponding to θ.
In particular, we have a consistent and continuous extension of our terminal wealth mapping $\theta \mapsto V_T(\theta)$ from simple strategies to predictable, a.s. square-integrable strategies.

Sketch of Proof:

- Use Clark-Ocone-Formula to compute F_t.
- Use assumptions on u_m and bounds on Malliavin derivatives to control dependence of G on (u, ϑ).
- Get existence, uniqueness, stability of strong solutions to SDE.

$\square$

No arbitrage

Theorem

There is no arbitrage opportunity for the large investor among **all** *predictable strategies.*

No arbitrage

Theorem

There is no arbitrage opportunity for the large investor among **all** *predictable strategies.*

Sketch of Proof: For the large investor to have an arbitrage opportunity, some market makers have to lose in terms of expected utility.

However, utility processes are local martingales and bounded from above — thus submartingales! $\square$

Hedging of contingent claims

Problem

Large investor wishes to hedge against a claim H by dynamically trading the assets ψ available on the market.

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Large investor wishes to hedge against a claim H by dynamically trading the assets ψ available on the market.

Solution

If H has finite exponential moments and if $\psi = B_T$, then

$$\text{replication price of } H = \begin{cases} \text{market indifference price of } H \\ \text{if it was traded on the market} \end{cases}$$

and the integrand I in the Ito representation of the utility process U induced by corresponding Pareto allocation yields the hedging strategy θ via

$$G_t(u_t, \theta_t) = I_t \,.$$

Conclusion

- market illiquidity is a major problem in Finance on both the theoretical and the practical level.
- some models for hedging in illiquid markets
- difficult to avoid pitfalls: vanishing liquidity effects
- economic equilibrium theory yields sound pricing theory, but limited insights into hedging problems
- market indifference pricing allows for combination of equilibrium theory and Black-Scholes approach

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- neglects: a lot more — future research!

THANK YOU VERY MUCH!

Based on joint work

- with Dietmar Baum: Hedging and Portfolio Optimization in Financial Markets with a Large Trader, Mathematical Finance (2004), 14, 1–18.
- with Dmitry Kramkov: A large investor trading at market indifference prices, in preparation.

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