Backward Stochastic Differential Equations with Financial Applications (Part I)

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Outline

1. Introduction

An Example:

A standard "LQ" *stochastic control problem*:

$$\begin{cases} dX_t^u = (aX_t^u + bu_t)dt + dW_t, \qquad X_0^u = x; \\ J(u) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \{|X_t^u|^2 + |u_t|^2\}dt + |X_T^u|^2\right\}, \end{cases}$$

where W is a standard Brownian motion and $\mathbb{F} = \{\mathscr{F}_t^W\}_{t\geq 0}$ is the natural filtration generated by W; $u = \{u_t\}$ is the "*control" process*; and $J(u)$ is the "*cost functional*".

The problem:

Find $u^* \in \mathscr{U}_{ad} \subseteq L^2_{\mathbb{F}}(\Omega \times [0, T])$ such that $J(u^*) = \inf_{u\in\mathscr{U}_{ad}} J(u)$.

A necessary condition (Pontryagin's Maximum Principle):

Assume u^* is optimal.
Then $\forall \varepsilon > 0$ and $\forall v \in \mathscr{U}_{ad}$, one has $J(u^* + \varepsilon v) \geq J(u^*) \quad \Longrightarrow$

$$0 \leq \frac{d}{d\varepsilon} J(u^* + \varepsilon v)\bigg|_{\varepsilon=0} = \mathbb{E}\left\{\int_0^T \{X_t^{u^*}\xi_t + u_t^* v_t\}dt + X_T^{u^*}\xi_T\right\},$$

where $\xi = \frac{d}{d\varepsilon} X^{u^*+\varepsilon v}\big|_{\varepsilon=0}$ is the solution to the *variational equation:*

$$d\xi_t = \{a\xi_t + bv_t\}dt, \qquad \xi_0 = 0. \tag{1}$$

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A Lucky Guess (?):

Assume that η is the solution to the following "*adjoint equation*" of (1):

$$d\eta_t = -(a\eta_t + X_t^{u^*})dt, \qquad \eta_T = X_T^{u^*}. \tag{2}$$

Then, "integration by parts" yields

- $\xi_T \eta_T = \int_0^T \{-\xi_t X_t^{u^*} + b\eta_t v_t\} dt$
- $\int_0^T \{u_t^* v_t + b\eta_t v_t\} dt = \int_0^T \{X_t^{u^*} \xi_t + u_t^* v_t\} dt + X_T^{u^*} \xi_T$
- $\mathbb{E} \int_0^T \{u_t^* + b\eta_t\} v_t dt =$
  $\mathbb{E}\Big\{ \int_0^T \{X_t^{u^*} \xi_t + u_t^* v_t\} dt + X_T^{u^*} \xi_T \Big\} \geq 0$
- Since $v \in L^2_{\mathbb{F}}(\Omega \times [0, T])$ is arbitrary, $u_t^* = -b\eta_t$, $\forall t$, a.s.
- $u^* = -b\eta$ should have all the reasons to be an optimal control except...

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A Problem:

$u^* \notin \mathscr{U}_{ad}$! (since it is **not** adapted!!)

BSDE to the rescue:

Example

$$\begin{cases} dY_t = 0; \\ Y_T = \xi \in L^2(\mathscr{F}_T). \end{cases} \tag{3}$$

Same Problem: The unique "solution" $Y_t \equiv \xi$ is **not** adapted!

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The Solution:

Define $Y_t = \mathbb{E}\{\xi | \mathscr{F}_t\}$, $t \in [0, T]$. Then Y becomes an L^2-martingale, and by Martingale Representation Theorem (Itô, 1951), there exists $Z \in L^2_{\mathbb{F}}(\Omega \times [0, T])$ such that

$$Y_t = \mathbb{E}\{\xi\} + \int_0^t Z_t dW_t, \qquad t \in [0, T].$$

$$\Longrightarrow \qquad Y_t = \xi - \int_t^T Z_t dW_t, \quad t \in [0, T] \quad \text{— A BSDE!} \tag{4}$$

Back to the LQ problem:

Consider the modified adjoint equation (as a BSDE):

$$\begin{cases} d\eta_t = -(a\eta_t + X_t^{u^*})dt + Z_t dW_t, \\ \eta_T = X_T^{u^*}. \end{cases} \tag{5}$$

The Conclusion

Suppose that one can find a pair of process (η, Z) that is the solution to (5). Then define $u_t^* = -b\eta_t$, $\forall t$, we obtain an optimal control!

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Observation:

The "close-loop" system is then

$$\begin{cases} dX_t^{u^*} = (aX_t^* - b^2\eta_t)dt + dW_t, \\ d\eta_t = -(a\eta_t + X_t^{u^*})dt + Z_t dW_t, \\ X_0^{u^*} = x \qquad \eta_T = X_T^{u^*}, \end{cases} \qquad \text{— An FBSDE!}$$

A Brief History

- Bismut ('73) — Linear BSDEs (Maximum Principle)
- Pardoux-Peng ('90, '92) — Nonlinear BSDEs
- Antonelli ('93) — FBSDEs (Stochastic Recursive Utility — Duffie-Epstain ('92))
- Ma-Yong/Ma-Protter-Yong ('93,'94) — "Four Step Scheme"
- El Karoui-Kapoudjian-Pardoux-Peng-Quenez, Cvitanic-Karatzas, ('97) — BSDEs with reflections
- Ma-Yong ('96-'98) — BSPDEs
- Ma-Yong ('99) — Book (LNM 1702)

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Other Developments

- Lepeltier-San Martin ('97) — BSDEs with cont. coefficients
- Kobylanski ('01) — BSDEs with quadratic growth (in Z)
- Delarue ('02) — FBSDE with Lipschitz coefficients
- Ma-Zhang-Zheng ('08) — Weak solution and "FBMP"
- Soner-Touzi-Zhang (09?) — 2BSDEs
- ... ...

2. BSDEs/FBSDEs in Finance

The (Black-Scholes) market model:

$$\begin{cases} dS_t^0 = S_t^0 r_t dt, \quad S_0^0 = s^0, & \text{(Bond/Money Market)} \\ dS_t^i = S_t^i \Big\{ b_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j \Big\}, \ S_0^i = s^i, \ 1 \le i \le d, & \text{(Stocks)} \end{cases}$$

- S_t^0, S_t^i—prices of bond/(i-th) stocks (per share) at time t
- r_t—interest rate at time t
- $\{b_t^i\}_{i=1}^d$—appreciation rates at time t
- $[\sigma_t^{ij}]$—volatility matrix at time t

More general form of the underlying asset price:

$$dS_t = S_t\{b(t, S_t)dt + \sigma(t, S_t)dW_t\}, \quad S_0 = s.$$

The Wealth Equation:

Denote:

- Y_t—dollar amount of the total wealth of an investor at time t
- π_t^i—dollars invested in i-th stock at time t, $i = 1, \cdots, N$
- C_t—cumulated consumption up to time t

Then, the wealth process Y satisfies an SDE: for $t \in [0, T]$,

$$Y_t = y + \int_0^t \{r_s Y_s + \langle \pi_s, [b_s - r_s \mathbf{1}] \rangle\} ds + \int_0^t \langle \pi_s, \sigma_s dW_s \rangle - C_t,$$

where $\mathbf{1} \stackrel{\triangle}{=} (1, \cdots, 1)$.

The Contingent Claims:

Any $\xi \in \mathscr{F}_T$. In particular, $\xi = g(S_T)$ — *Options*. E.g.,

- $\xi = (S_T^1 - q)^+$—European call
- $\xi = (S_\tau^1 - q)^+$—American call (τ-stopping time)

European Options (Fixed exercise time T)

Define the "fair price" of an option to be

$$p = \inf\{v : \exists(\pi, C), \text{such that } \; Y_T^{y,\pi,C} \geq \xi\}.$$

Then (El Karoui-Peng-Quenez, '96), the price p and the "hedging strategy" (π, C) can be determined by:

- $C \equiv 0$, $p = Y_0 = y$, and $\pi_t = (\sigma_t^T)^{-1} Z_t$;
- (Y, Z) solves the BSDE:

$$Y_t = \xi - \int_t^T \{r_s Y_s + \langle\, Z_s, \sigma_s^{-1}[b_s - r_s \mathbf{1}]\,\rangle\} ds - \int_t^T \langle\, Z_s, dW_s \,\rangle\,.$$

Fair price for American Option:

$$p = \inf\{v : \exists(\pi, C), \text{such that } \; Y_\tau^{y,\pi,C} \geq g(S_\tau), \forall \tau\}.$$

American Options (El Karoui-Kapoudjian-Pardoux-Peng-Quenez, '97)

For $\xi = g(S_\tau)$, where τ is exercise time (any $\{\mathscr{F}_t\}$-stopping time). Then the price, hedging strategy, and the optimal exercise time are solved as:

- $p = Y_0 = y,\ C = 0,$
- (Y, Z, K) solves a *BSDE with reflection*:

$$\begin{cases} Y_t = g(S_T) - \displaystyle\int_t^T \{r_s Y_s + \langle\, Z_s, \sigma_s^{-1}[b_s - r_s \mathbf{1}]\,\rangle\} ds \\ \qquad\qquad - \int_t^T \langle\, Z_s, dW_s \,\rangle + K_T - K_t; \\ Y_t \geq g(S_t), \quad \forall t \in [0, T], a.s.; \displaystyle\int_0^T (Y_t - g(S_t)) dK_t = 0. \end{cases}$$

- The optimal exercise time is given by $\tau = \inf\{t > 0 : Y_t = g(S_t)\}$.

"Large Investors" Problem

Contrary to the Black-Scholes theory, one may assume that some investors are "large".

The price-wealth pair satisfies an FBSDE:

$$\begin{cases} dX_t = X_t\{b(t,X_t,Y_t,Z_t)dt + \sigma(t,X_t,Y_t,Z_t)dW_t\}, \\ dY_t = -\{r_tY_t + Z_t[b(t,X_t,Y_t,Z_t) - r_t\mathbf{1}]\}dt \\ \qquad\qquad -Z_t\sigma(t,\cdots)dW_t + C_T - C_t. \\ X_0 = x, \qquad Y_T = g(X_T). \end{cases}$$

- **Hedging without constraint** (Cvitanic-Ma, 1996)
- **Hedging with constraint** (Buckdahn-Hu, 1998)
- **American "game" option** (FBSDER, Cvitanic-Ma, 2000)

Stochastic Recursive Utility

Duffie-Epstain ('92) defined the "*SRU*" *by a BSDE:*

$$U_t = \Phi(Y_T) + \int_t^T f(s, c_s, U_s, V_s)ds - \int_t^T V_s dW_s,$$

- Y — wealth;
- Φ — utility function
- f — "standard driver" or "aggregator"
- $|V_t|^2 = \frac{d}{dt}\langle U \rangle_t$ — the "variability" process
- c — consumption (rate) process

- *Standard Utility:* $f(c, u, v) = \varphi(c) - \beta u$.
- *Uzawa Utility:* $f(c, u, v) = \varphi(c) - \beta(c)u$.
- *Generalized Uzawa Utility:* $f(c, u, v) = \varphi(c) - \beta u - \gamma|v|$, (Chen-Epstein (1999)).

Portfolio/Consumption Optimization Problems

General wealth equation with portfolio-consumption strategy (π, c):

$$dY_t = b(t, c_t, Y_t, \sigma_t^T \pi_t)dt - \langle \pi_t, \sigma_t dW_t \rangle . \qquad (6)$$

Portfolio/consumption optimization problem

Find (π, c) so as to maximize certain "utility":

$$U(y, \pi, c) \stackrel{\triangle}{=} E\Big\{\Phi(Y_T^{y,\pi,c}) + \int_0^T h(t, c_t, Y_t^{y,\pi,c})dt\Big\}.$$

— A stochastic control problem!

With Stochastic Recursive Utility:

$$U_0^{y,\pi,c} = E\Big\{\Phi(Y_T^{y,\pi,c}) + \int_0^T f(t, c_t, Y_t^{y,\pi,c}, U_t^{y,\pi,c}, V_t^{y,\pi,c})dt\Big\}.$$

$\Longrightarrow$ A stochastic control problem for FBSDEs!

Brennan-Schwartz's Term structure model: (1979)

$$\begin{cases} dr_t = \mu(r_t, R_t)dt + \alpha(r_t, R_t)dW_t \\ dR_t = \nu(r_t, R_t)dt + \beta(r_t, R_t)dW_t, \end{cases}$$

where r—short rate, R—consol rate (*consol* = *perpetual annuity*). This model was later disputed by M. Hogan, by counterexample, which leads to

Consol Rate Conjecture by Fisher Black:

Assume that the consol price $Y_t = R_t^{-1}$, where R is the consol rate. Then, under at most technical conditions, $\forall \mu$ and α, $\exists A(\cdot,\cdot)$ such that

$$\begin{cases} dr_t = \mu(r_t, Y_t)dt + \alpha(r_t, Y_t)dW_t \\ dY_t = (r_t Y_t - 1)dt + A(r_t, Y_t)dW_t, \end{cases} \tag{7}$$

- Assume $r_t = h(X_t)$ for some "factor" process X and $h(\cdot) > 0$,
- X satisfies an SDE depending on R (or equivalent Y).
- Then the term structure SDEs (7) becomes an FBSDE with infinite horizon:

$$\left\{ \begin{array}{l} dX_t = b(X_t, Y_t)dt + \sigma(X_t, Y_t)dW_t, \quad X_0 = x, \\ Y_t = E\left\{ \int_t^\infty e^{-\int_t^s h(X_u)du} ds \,\middle|\, \mathscr{F}_t \right\}, \quad t \in [0, \infty), \end{array} \right. \tag{8}$$

where Y is uniformly bounded for $t \in [0, \infty)$.

- Or equivalently,

$$\left\{ \begin{array}{l} dX_t = b(X_t, Y_t)dt + \sigma(X_t, Y_t)dW_t, \\ Y_t = (h(X_t)Y_t - 1)dt + A(X_t, Y_t)dW_t, \\ X_0 = x, \quad \mathrm{esssup}_\omega \sup_{t \in [0,\infty)} |Y_t(\omega)| < \infty. \end{array} \right. \tag{9}$$

This result (Duffie-Ma-Yong, 1993) was one of the early successful applications of FBSDE in finance, and the first application using the **Four Step Scheme**.

Theorem

Under some technical conditions, there exists a unique function $A(x,y) = -\sigma(x,y)^T\theta_x(x)$ *such that* (X,Y) *in (8) satisfies (9), and* θ *is the unique classical solution to the PDE:*

$$\frac{1}{2}\sigma\sigma^T(x,\theta)\theta_{xx} + b(x,\theta)\theta_x - h(x)\theta + 1 = 0.$$

Moreover, $Y_t = \theta(X_t)$ *for any* $t \in [0,\infty)$.

Consider a Backward SDE of the following general form:

$$Y_t = \xi + \int_t^T g(t, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \qquad (10)$$

where $\xi \in L^2(\mathscr{F}_T)$ is the *terminal condition* and $g(t, y, z)$ is the *generator*.

g-expectation via BSDE (Peng, '93)

- If the BSDE (10) is well-posed, then the solution mapping $\mathscr{E}^g : \xi \mapsto Y_0$ is called a *g-expectation*.
- For any $t \in [0, T]$, the *conditional g-expectation* of ξ given $\mathscr{F}_t$ is defined by $\mathscr{E}^g[\xi|\mathscr{F}_t] \stackrel{\triangle}{=} Y_t$.

Properties of g-expectations: Assume that $g|_{z=0} = 0$.

- **Constant-preserving:** $\mathscr{E}^g[\xi|\mathscr{F}_t] = \xi$, $\mathbb{P}$-a.s., $\forall\, \xi \in L^2(\mathscr{F}_t)$; In particular, $\mathscr{E}^g[c] = c$, $\forall\, c \in \mathbb{R}$;
- **Time-consistency:** $\mathscr{E}^g\left[\mathscr{E}^g[\xi|\mathscr{F}_t]\big|\mathscr{F}_s\right] = \mathscr{E}^g[\xi|\mathscr{F}_s]$, $\mathbb{P}$-a.s., $\forall\, 0 \le s \le t \le T$;
- **(Strict) Monotonicity:** If $\xi \ge \eta$, then
$$\mathscr{E}^g[\xi|\mathscr{F}_t] \ge \mathscr{E}^g[\eta|\mathscr{F}_t], \quad \mathbb{P}\text{-a.s.}, \quad t \in [0, T];$$
Moreover if "$=$" holds for some t, then $\xi = \eta$, $\mathbb{P}$-a.s.;
- **"Zero-one" Law:** $\mathscr{E}^g[\mathbf{1}_A\xi|\mathscr{F}_t] = \mathbf{1}_A\mathscr{E}^g[\xi|\mathscr{F}_t]$, $\mathbb{P}$-a.s., $\forall\, A \in \mathscr{F}_t$;
- **Translation Invariance:** If g is independent of y, then
$$\mathscr{E}^g[\xi + \eta|\mathscr{F}_t] = \mathscr{E}^g[\xi|\mathscr{F}_t] + \eta, \quad \mathbb{P}\text{-a.s.}, \quad \forall\, \eta \in L^2(\mathscr{F}_t).$$
- **Convexity:** If g is convex (in z), then so is $\mathscr{E}^g[\cdot|\mathscr{F}_t]$.

Axioms for Risk Measures (Artzner et al., Barrieu-El Karoui,...)

- A (static) RM is a mapping $\rho : \mathscr{X} \mapsto \mathbb{R}$ (for some space of random variables $\mathscr{X}$), s.t.,
  - **Monotonicity:** $\xi \le \eta \quad \Longrightarrow \quad \rho(\xi) \ge \rho(\eta)$;
  - **Translation Invariance:** $\rho(\xi + m) = \rho(\xi) - m, \quad m \in \mathbb{R}$;
  - **Coherent:** if
    - **Subadditivity:** $\rho(\xi + \eta) \le \rho(\xi) + \rho(\eta)$
    - **Positive homogeneity:** $\rho(\alpha\xi) = \alpha\rho(\xi)$, $\forall \alpha \ge 0$;
  - **Convex:** (Föllmer and Schied, '02)
    $\rho\big(\alpha\xi + (1-\alpha)\eta\big) \le \alpha\rho(\xi) + (1-\alpha)\rho(\eta)$, $\alpha \in [0,1]$.

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  - **Convex:** (Föllmer and Schied, '02) $\rho\big(\alpha\xi + (1-\alpha)\eta\big) \le \alpha\rho(\xi) + (1-\alpha)\rho(\eta)$, $\alpha \in [0,1]$.
- A (dynamic) RM is a family of mappings $\rho_t : \mathscr{X} \mapsto L^0(\mathscr{F}_t)$, $t \in [0, T]$, s.t. $\forall \xi, \eta \in \mathscr{X}$,
  - **Monotoncity:** If $\xi \le \eta$, then $\rho_t(\xi) \ge \rho_t(\eta)$, $\mathbb{P}$-a.s., $\forall\, t$;
  - **Translation Invariance:** $\rho_t(\xi + \eta) = \rho_t(\xi) - \eta$, $\forall \eta \in \mathscr{F}_t$.
  - ρ_0 is a static risk measure
  - $\rho_T(\xi) = -\xi$ for any $\xi \in \mathscr{X}$.

Example

- Worst-case Dynamic Risk Measure:

$$\rho_t(\xi) \stackrel{\triangle}{=} \operatorname*{esssup}_{Q\in\mathscr{P}_P} E_Q[-\xi|\mathscr{F}_t], \quad t\in[0,T],$$

- Entropic Dynamic Risk Measure: ◂ Convex RM

$$\rho^\gamma_t(\xi) = \gamma \ln\left\{E\left[e^{-\frac{1}{\gamma}\xi}\middle|\mathscr{F}_t\right]\right\}, \quad t\in[0,T].$$

- Convex Dynamic Risk Measure:

$$\rho_t(\xi) \stackrel{\triangle}{=} \operatorname*{esssup}_{Q\in\mathscr{P}_P}\left\{E_Q[-\xi|\mathscr{F}_t] - F_t(Q)\right\}, \quad t\in[0,T],$$

where F_t is the "penalty function" of ρ_t for any t.

- $\{\rho_t\}_{t\in[0,T]}$ is called convex (or coherent) if each ρ_t is. (e.g., Worst case — coherent; Entropic — convex.)
- $\{\rho_t\}_{t\in[0,T]}$ is said to be time-consistent if

$$\rho_0[\xi\mathbf{1}_A] = \rho_0[-\rho_t(\xi)\mathbf{1}_A], \qquad t\in[0,T], \xi\in\mathscr{X}, A\in\mathscr{F}_t.$$

g-Expectations and Risk Measures

Let g be generator with $g|_{z=0} = 0$, and is Lipschitz in (y, z).

- $\rho(\xi) \stackrel{\triangle}{=} \mathscr{E}^g[-\xi]$ defines a static risk measure on $\mathscr{X} = L^2(\mathscr{F}_T)$.
- $\rho_t(\xi) \stackrel{\triangle}{=} \mathscr{E}^g[-\xi|\mathscr{F}_t]$, $t \in [0, T]$, defines a dynamic risk measure on $\mathscr{X} = L^2(\mathscr{F}_T)$.
- The risk measure (resp. dynamic risk measure) is convex if g is convex in z.
- The risk measure (resp. dynamic risk measure) is coherent if g is further independent of y.

Question:

Does every risk measure have to be a g-expectation??

Nonlinear Expectations

Definition

Let $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\})$ be a given probability space. A functional $\mathscr{E} : L^2(\mathscr{F}_T) \mapsto \mathbb{R}$ is called a nonlinear expectation if it satisfies the following axioms:

- **Monotonicity:** $\xi \geq \eta$, P-a.s. $\Longrightarrow$
  - $\mathscr{E}[\xi] \geq \mathscr{E}[\eta]$
  - $\mathscr{E}[\xi] = \mathscr{E}[\eta] \iff \xi = \eta$, P-a.s.
- **Constant-preserving:** $\mathscr{E}[c] = c$, $c \in \mathbb{R}$.

A nonlinear expectation $\mathscr{E}$ is called $\{\mathscr{F}_t\}$-consistent if it satisfies

- for all $t \in [0, T]$ and $\xi \in L^2(\mathscr{F}_T)$, there exists $\eta \in \mathscr{F}_t$ such that
$$\mathscr{E}[\mathbf{1}_A \xi] = \mathscr{E}[\mathbf{1}_A \eta], \ \forall A \in \mathscr{F}_s$$
Will denote $\eta = \mathscr{E}\{\xi | \mathscr{F}_t\}$, for obvious reasons.

Nonlinear Expectations

Definition

An $\{\mathscr{F}_t\}$-consistant nonlinear expectation $\mathscr{E}$ is said to be dominated by $\mathscr{E}^{\mu} = \mathscr{E}^{g_\mu}$ $(\mu > 0)$ if

Convex RM

$$\mathscr{E}[\xi + \eta] - \mathscr{E}[\xi] \le \mathscr{E}^{\mu}[\eta], \qquad \forall \xi, \eta \in L^2(\mathscr{F}_T). \tag{11}$$

where $\mathscr{E}^{\mu} = \mathscr{E}^{g_\mu}$ is the g-expectation with $g \equiv \mu|z|$. Further, $\mathscr{E}$ is said to satisfy the translability condition if

$$\mathscr{E}[\xi + \alpha|\mathscr{F}_t] = \mathscr{E}[\xi|\mathscr{F}_t] + \alpha, \, \forall \, \xi \in L^2(\mathscr{F}_T), \quad \alpha \in L^2(\mathscr{F}_t).$$

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$$\mathscr{E}[\xi + \eta] - \mathscr{E}[\xi] \leq \mathscr{E}^{\mu}[\eta], \qquad \forall \xi, \eta \in L^2(\mathscr{F}_T). \tag{11}$$

where $\mathscr{E}^{\mu} = \mathscr{E}^{g_\mu}$ is the g-expectation with $g \equiv \mu|z|$. Further, $\mathscr{E}$ is said to satisfy the translability condition if

$$\mathscr{E}[\xi + \alpha|\mathscr{F}_t] = \mathscr{E}[\xi|\mathscr{F}_t] + \alpha, \, \forall \, \xi \in L^2(\mathscr{F}_T), \quad \alpha \in L^2(\mathscr{F}_t).$$

Representation Theorem (Coquet et al. '02)

If $\mathscr{E}$ is a translable $\{\mathscr{F}_t\}$-expectation dominated by $\mathscr{E}^{\mu}$, for some $\mu > 0$, then $\exists!$ (deterministic) function g, independent of y, such that $|g(t,z)| \leq \mu|z|$, and that

$$\mathscr{E}[\xi] = \mathscr{E}^{g}[\xi], \qquad \text{for all } \xi \in L^2(\mathscr{F}_T).$$

Representing Risk Measures as g-Expectations

A direct consequence of the Representation Theorem for nonlinear expectation is the following representation theorem for dynamic coherent risk measures.

Some Facts:

Let $\{\rho_t\}$ be a dynamic, coherent, time-consistent risk measure on $\mathscr{X} \triangleq L^2(\mathscr{F}_T)$. Define $\mathscr{E}_t(\xi) \triangleq \rho_t(-\xi)$, $t \in [0, T]$; and $\mathscr{E} \triangleq \mathscr{E}_0$. Then

- $\mathscr{E}$ is a *nonlinear expectation*
- $\mathscr{E}_t\{\cdot\} = \mathscr{E}\{\cdot|\mathscr{F}_t\}$ is the *nonlinear conditional expectation* ("time-consistency" $\Longrightarrow$ "$\{\mathscr{F}_t\}$-consistency"!)
- Consequently, if $\mathscr{E}$ is further $\mathscr{E}^\mu$-dominated for some $\mu > 0$, then there exists a unique Lipschitz generator g such that

$$\rho_0(\xi) = \mathscr{E}^g(-\xi), \qquad \rho_t(\xi) = \mathscr{E}^g\{-\xi|\mathscr{F}_t\}, \quad \forall \xi \in L^2(\mathscr{F}_T).$$

3. Wellposedness of BSDEs

Consider the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \tag{12}$$

where $\xi \in L^2_{\mathscr{F}_T}(\Omega; \mathbb{R}^n)$, W is a d-dimensional Brownian motion.

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Some spaces:

For any $\beta \geq 0$ and Euclidean space $\mathscr{H}$, define

- $\mathscr{S}^2_\beta(0, T; \mathscr{H})$ to be the space of all $\mathscr{H}$-valued, continuous, $\mathbb{F}$-adapted processes X, such that $\mathbb{E}\Big\{\sup_{0\leq t\leq T} e^{\beta t}|X_t|^2\Big\} < \infty$
- $\mathbb{H}^2_\beta(0, T; \mathbb{E})$ to be the space of all $\mathscr{H}$-valued, $\mathbb{F}$-adapted processes X such that $\mathbb{E}\Big\{\int_0^T e^{\beta t}|X_t|^2 dt\Big\} < \infty$
- $\mathscr{N}_\beta[0, T] \stackrel{\triangle}{=} \mathscr{S}^2_\beta(0, T; \mathbb{R}^n) \times \mathbb{H}^2_\beta(0, T; \mathbb{R}^{n\times d})$

Main result:

Assumption

f is Lipschitz in (y, z) with a uniform Lipschitz constant $L > 0$.

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Under the above assumptions on f, for any $\xi \in L^2_{\mathscr{F}_T}(\Omega; \mathbb{R}^n)$, (12) admits a unique solution $(Y, Z) \in \mathscr{N}_0[0, T]$.

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Theorem

Under the above assumptions on f, for any $\xi \in L^2_{\mathscr{F}_T}(\Omega; \mathbb{R}^n)$, (12) admits a unique solution $(Y, Z) \in \mathscr{N}_0[0, T]$.

Observations:

- Since $\mathscr{N}_\beta[0, T]$ is equivalent to $\mathscr{N}_0[0, T]$, we need only find the solution $(Y, Z) \in \mathscr{N}_\beta(0, T)$ for some $\beta > 0$.
- $\forall (y, z) \in \mathscr{N}[0, T]$, let $h(\cdot) \stackrel{\triangle}{=} f(\cdot, y_\cdot, z_\cdot) \in L^2_{\mathbb{F}}(\Omega \times [0, T]; \mathbb{R}^n)$. Then, $M_t \stackrel{\triangle}{=} E\{\xi + \int_0^T h(s)ds \big| \mathscr{F}_t\}$, $t \in [0, T]$ is an $L^2(\mathbb{F})$-martingale.

The First Step:

- By the Mart. Rep. Thm, $\exists Z \in L^2_{\mathscr{F}}(0, T; \mathbb{R}^{n\times d})$, such that

$$M_t = M_0 + \int_0^t Z_s dW_s, \qquad \forall t \in [0, T]. \tag{13}$$

- Define $Y_t \stackrel{\triangle}{=} M_t - \int_0^t h(s)ds$. Then $M_0 = Y_0$, and

$$\xi + \int_0^T h(s)ds = M_T = Y_0 + \int_0^T Z_s dW_s.$$

- Consequenly,

$$\begin{aligned} Y_t &= M_t - \int_0^t h(s)ds = Y_0 + \int_0^t Z_s dW_s - \int_0^t h(s)ds \\ &= \xi + \int_0^T h(s)ds - \int_0^T Z_s dW_s - \int_0^t h(s)ds + \int_0^t Z_s dW_s \\ &= \xi + \int_t^T h(s)ds - \int_t^T Z_s dW_s. \end{aligned} \tag{14}$$

- For any $(y, z) \in \mathscr{N}_\beta[0, T]$, let (Y, Z) be the solution to (14).
- Applying Itô's formula to $F(t, Y_t) = e^{\beta t}|Y_t|^2$, then taking expectation and applying Fatou:

$$\mathbb{E}\Big\{e^{\beta t}|Y_t|^2\Big\} + \beta\mathbb{E}\int_t^T e^{\beta s}|Y_s|^2 ds + \mathbb{E}\int_t^T e^{\beta s}|Z_s|^2 ds$$
$$\leq e^{\beta T}\mathbb{E}|\xi|^2 + 2\mathbb{E}\int_t^T e^{\beta s}\langle\, Y_s, h(s)\,\rangle\, ds.$$

- Using the trick: $2ab \leq \varepsilon a^2 + b^2/\varepsilon,\ \forall \varepsilon > 0$, we have

$$\mathbb{E}\Big\{e^{\beta t}|Y_t|^2\Big\} + (\beta - \frac{1}{\varepsilon})\mathbb{E}\int_t^T e^{\beta s}|Y_s|^2 ds + \mathbb{E}\int_t^T e^{\beta s}|Z_s|^2 ds$$
$$\leq e^{\beta T}\mathbb{E}|\xi|^2 + \varepsilon\mathbb{E}\int_t^T e^{\beta s}|h(s)|^2 ds.$$

- Continuing from before, one has

$$\begin{cases} \|Y\|^2_{\mathbb{H}^2_\beta} \le \dfrac{\varepsilon}{(\varepsilon\beta - 1)}\Big\{e^{\beta T}\mathbb{E}|\xi|^2 + \varepsilon\|h\|^2_{\mathbb{H}^2_\beta}\Big\}; \\ \\ \|Z\|^2_{\mathbb{H}^2_\beta} \le e^{\beta T}\mathbb{E}|\xi|^2 + \varepsilon\|h\|^2_{\mathbb{H}^2_\beta}. \end{cases} \tag{15}$$

- Using Burkholder-Davis-Gundy's inequality, one then derive that

$$\|Y\|^2_{\mathscr{S}^2_\beta} \le 2(1 + C_1(\beta,\varepsilon))e^{\beta T}\mathbb{E}|\xi|^2 + 2\varepsilon C_1(\beta,\varepsilon)\|h\|^2_{\mathbb{H}^2_\beta}, \tag{16}$$

where $C_1(\beta,\varepsilon) \stackrel{\triangle}{=} 1 + \dfrac{\varepsilon}{\varepsilon\beta - 1} + 2(1+C)^2$, and C is the universal constant in the Burkholder-Davis-Gundy inequality.
$\Longrightarrow \quad (Y, Z) \in \mathscr{N}_\beta[0, T]$.

Furthermore,

- For $(y, z), (\bar{y}, \bar{z}) \in \mathscr{N}_\beta[0, T]$, let $(Y, Z), (\bar{Y}, \bar{Z}) \in \mathscr{N}_\beta[0, T]$ be the corresponding solutions of (14), respectively.
- Define $\widehat{\zeta} = \zeta - \bar{\zeta}$, $\zeta = y, z, Y, Z$, and $H(s) = f(s, y_s, z_s) - f(s, \bar{y}_s, \bar{z}_s)$. Then,

$$|H(s)| \le L(|\widehat{y}_s| + |\widehat{z}_s|), \quad \widehat{Y}_T = \widehat{\xi} = 0;$$

- Choosing $\beta = \beta(\varepsilon)$ and $\varepsilon > 0$ small enough, show that

$$\|(\widehat{Y}, \widehat{Z})\|^2_{\mathscr{N}_\beta[0,T]} \le \widetilde{C}(\varepsilon)\|(\widehat{y}, \widehat{z})\|^2_{\mathscr{N}_\beta[0,T]},$$

where $\widetilde{C}(\varepsilon) < 1$. Thus the mapping $(y, z) \mapsto (Y, Z)$ is a contraction on $\mathscr{N}_\beta[0, T]$, proving the theorem.

Comparison Theorems

Suppose that (Y^i, Z^i), $i = 1, 2$ are solutions to the following two BSDEs: for $t \in [0, T]$,

$$Y_t^i = \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i)ds - \int_t^T Z_s^i dW_s, \quad i = 1, 2. \tag{17}$$

Question:

Assume that

- $\xi^1 \geq \xi^2$, $\mathbb{P}$-a.s.;
- $f^1(t, y, z) \geq f^2(t, y, z)$, $\forall (t, y, z)$.

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Can we conclude that $Y_t^1 \geq Y_t^2$, $\forall t \in [0, T]$?

Answer:

Yes, provided f^2 is uniformly Lipschitz in (y, z)!!

Define $\Delta Y \stackrel{\triangle}{=} Y^1 - Y^2$ and $\Delta Z \stackrel{\triangle}{=} Z^1 - Z^2$. Then

$$f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^2, Z_t^2) = \delta f(t) + \alpha(t)\Delta Y_t + \langle \beta(t), \Delta Z_t \rangle,$$

where $\delta f(t) \stackrel{\triangle}{=} f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^1, Z_t^1)$, and

$$\begin{cases} \alpha(t) = \dfrac{f^2(t, Y_t^1, Z_t^1) - f^2(t, Y_t^2, Z_t^1)}{\Delta Y_t} \mathbf{1}_{\{\Delta Y_t \neq 0\}}; \\ \beta^i(t) = \dfrac{f^2(t, Y_t^2, Z_t^{1,i}) - f^2(t, Y_t^2, Z_t^{2,i})}{\Delta Z_t^i} \mathbf{1}_{\{\Delta Z_t^i \neq 0\}}, \end{cases}$$

In other words, we have

$$\begin{aligned} \Delta Y_t &= \Delta\xi + \int_t^T \{\delta f(s) + \alpha(s)\Delta Y_s + \langle \beta(s), \Delta Z_s \rangle\} ds \\ &\quad - \int_t^T \langle \Delta Z_s, dW_s \rangle . \end{aligned}$$

Note:

Since f^2 is uniformly Lipschitz, both α and β are bounded, adapted processes!!

Two Tricks:

1. (Change of Measure:) Define $\Theta(t) = \exp\left\{ -\int_0^t \beta(s)dW_s - \frac{1}{2}\int_0^t |\beta_s|^2 ds \right\}$; and
$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \Theta(T).$$
Since β is bounded, by Girsanov Theorem we know that Θ is a $\mathbb{P}$-martingale, and $W_t^1 \stackrel{\triangle}{=} W_t + \int_0^t \beta_s ds$ is a $\mathbb{Q}$-Brownian motion.

2. (Exponentiating:) Define $\Gamma_t = \exp\left\{ -\int_0^t \alpha(s)ds \right\}$. Then applying Itô we have, for $t \in [0, T]$,

$$\Gamma_T \Delta Y_T - \Gamma_t \Delta Y_t = -\int_t^T \Gamma_s \delta f(s) ds + \int_t^T \Gamma_s \left\langle \Delta Z_s, dW_s^1 \right\rangle .$$

Now taking conditional expectation on both sides above, we have

$$\Gamma_t \Delta Y_t = \mathbb{E}^{\mathbb{Q}}\left\{ \Gamma_T \Delta\xi + \int_t^T \Gamma_s \delta f(s) ds \middle| \mathscr{F}_t \right\}, \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.}$$

Since $\Delta\xi = \xi^1 - \xi^2 \geq 0$, $\mathbb{P}$-a.s. (hence $\mathbb{Q}$-a.s.!); and

$$\delta f(t) = f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^1, Z_t^1) \geq 0, \quad \forall t, \ \mathbb{Q}\text{-a.s.},$$

we conclude that $\Gamma_t \Delta Y_t \geq 0$, $\forall t$, $\mathbb{Q}$-a.s. This implies that $\Delta Y_t \geq 0$, $\forall t$, $\mathbb{P}$-a.s., proving the comparison theorem. ■

BSDEs with Continuous Coefficients

Theorem (Lepeltier-San Martin, 1997)

Assume $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^{d+1})$ m'able function, s.t. for fixed t, ω, the mapping $(y,z) \mapsto f(t,\omega,y,z)$ is continuous, and $\exists K > 0$, s.t. $\forall (t,\omega,y,z)$,

$$|f(t,w,y,z)| \le K(1+|y|+|z|).$$

Then for any $\xi \in L^2(\Omega, \mathscr{F}_T, P)$ the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \qquad (18)$$

has an adapted solution $(Y,Z) \in H^2(\mathbb{R}^{d+1})$, where Y is a continuous process and Z is predictable.

Also, there is a minimal solution $(\underline{Y}, \underline{Z})$ of equation (1), in the sense that for any other solution (Y,Z) of (1) we have $\underline{Y} \le Y$.

Lemma 1

Let $f : \mathbb{R}^p \to \mathbb{R}$ be a continuous function with linear growth, that is: $\exists K > 0$ such that $\forall x \in \mathbb{R}^p \quad |f(x)| \le K(1 + |x|)$. Then the sequence of functions

$$f_n(x) = \inf_{y \in \mathbb{Q}^p} \{f(y) + n|x - y|\} \tag{19}$$

is well defined for $n \ge K$ and it satisfies:

(i) Linear growth: $\forall x \in \mathbb{R}^p \quad |f_n(x)| \le K(1 + |x|)$;

(ii) Monotonicity in n: $\forall x \in \mathbb{R}^p \quad f_n(x) \nearrow$;

(iii) Lipschitz condition: $\forall x, y \in \mathbb{R}^p \quad |f_n(x) - f_n(y)| \le n|x - y|$;

(iv) strong convergence: if $x_n \to x$ as $n \to \infty$, then $f_n(x_n) \to f(x)$, as $n \to \infty$.

Proof of Lemma 1

- Since $f_n \le f$ ($\Longrightarrow$ $f_n(x) \le K(1+|x|)$), and $f_n(x) \ge \inf_{y\in\mathbb{Q}^p}\{-K - K|y| + K|x-y|\} = -K(1+|x|)$, $\Longrightarrow$ (i) holds.
- (ii) is evident from the definition of the sequence (f_n).
- $\forall\varepsilon > 0$, choose $y_\varepsilon \in \mathbb{Q}^p$ so that

$$\begin{aligned} f_n(x) &\ge f(y_\varepsilon) + n|x-y_\varepsilon| - \varepsilon \\ &\ge f(y_\varepsilon) + n|y-y_\varepsilon| + n|x-y_\varepsilon| - n|y-y_\varepsilon| - \varepsilon \\ &\ge f(y_\varepsilon) + n|y-y_\varepsilon| - n|x-y| - \varepsilon \\ &\ge f_n(y) - n|x-y| - \varepsilon. \end{aligned}$$

Exchanging the roles of x and y, and since ε is arbitrary we deduce that $|f_n(x) - f_n(y)| \le n|x-y|$, proving (iii).

Proof of Lemma 1

- To see (iv), assume $x_n \to x$ as $n \to \infty$. For every n, let $y_n \in \mathbb{Q}^p$ be such that

$$f(x_n) \geq f_n(x_n) \geq f(y_n) + n|x_n - y_n| - 1/n.$$

  Since $\{x_n\}$ is bounded and f has linear growth, we deduce that $\{y_n\}$ is bounded, and so is $\{f(y_n)\}$.
  Consequently $\overline{\lim}_n\, n|y_n - x_n| < \infty$, and in particular $y_n \to x$, as $n \to \infty$. Moreover,

$$f(x_n) \geq f_n(x_n) \geq f(y_n) - 1/n,$$

  from which the result follows. ∎

Proof of the Theorem

Define, for fixed (t, ω), a sequence $f_n(t, \omega, y, z)$, associated to f by Lemma 1; and $h(t, \omega, y, z) = K(1 + |y| + |z|)$. Then consider the following two BSDEs:

$$
\begin{aligned}
Y_t^n &= \xi + \int_t^T f_n(s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dWs, \quad n \geq K \\
U_t &= \xi + \int_t^T h(U_s, V_s)ds - \int_t^T V_s dW_s.
\end{aligned}
$$

By Comparison Theorem we obtain that

$$
\forall n \geq m \geq K \qquad Y^m \leq Y^n \leq U \qquad dt \otimes dP - a.s.
$$

$\Longrightarrow \exists A > 0$, depending only on K, T and $\mathbb{E}(\xi^2)$, s.t.

$$
\|U\| \leq A, \ \|V\| \leq A, \quad \text{and hence} \quad \forall n \geq K, \ \|Y^n\| \leq A.
$$

Proof of the Theorem

Claim: $\|Z^n\| \leq A$ as well.

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Let $\lambda^2 > K$, and applying Itô to $(Y^n_t)^2$:

$$\begin{aligned}\xi^2 &= (Y^n_t)^2 - 2\int_t^T Y^n_s f_n(s, Y^n_s, Z^n_s)ds + 2\int_t^T Y^n_s Z^n_s dW_s \\ &\quad + \int_t^T (Z^n_s)^2 ds.\end{aligned}$$

Taking expectation on both sides, we deduce

$$\mathbb{E}((Y^n_t)^2) + \mathbb{E}\int_t^T (Z^n_s)^2 ds = \mathbb{E}(\xi^2) + 2\mathbb{E}\int_t^T Y^n_s f_n(s, Y^n_s, Z^n_s)ds.$$

Therefore we obtain from the uniform linear growth condition on f_n (see (i) of Lemma 1), for $t = 0$

$$\|Z^n\|^2 \le \mathbb{E}(\xi^2) + 2K\|Y^n\|^2 + 2K\mathbb{E}\int_0^T |Y^n_s|(1 + |Z^n_s|)ds.$$

Proof of the Theorem

Using $2a \leq a^2\lambda^2 + \frac{1}{\lambda^2}$ and $2ab \leq a^2\lambda^2 + \frac{b^2}{\lambda^2}$, we have

$$2K|Y_s^n|(1+|Z_s^n|) \leq K\{\frac{1}{\lambda^2} + 2\lambda^2|Y_s^n|^2 + \frac{1}{\lambda^2}|Z_s^n|^2\},$$

and

$$\|Z^n\|^2 \leq \mathbb{E}(\xi^2) + \frac{KT}{\lambda^2} + 2K(\lambda^2+1)\|Y^n\|^2 + \frac{K}{\lambda^2}\|Z^n\|^2.$$

Since $\lambda^2 > K$ we deduce for $n \geq K$

$$\|Z^n\|^2 \leq \frac{\mathbb{E}(\xi^2) + KT/\lambda^2 + 2K(\lambda^2+1)B^2}{1 - K/\lambda^2} \triangleq A,$$

proving the claim. ■

Proof of the Theorem

Now fix $n_0 \geq K$. Since $\{Y^n\}$ is increasing and bounded in $\mathbb{H}^2(\mathbb{R})$, it converges in $\mathbb{H}^2(\mathbb{R})$ to a limit Y. Then, for $n, m \geq n_0$:

$$
\begin{aligned}
&\mathbb{E}(|Y_0^n - Y_0^m|^2) + \mathbb{E}\int_0^T |Z_u^n - Z_u^m|^2 du \\
= \;\; & 2\mathbb{E}\int_0^T (Y_u^n - Y_u^m)(f_n(u, Y_u^n, Z_u^n) - f_m(u, Y_u^m, Z_u^m))du.
\end{aligned}
$$

Applying Cauchy-Schwartz, and noting the uniform linear growth of $\{f_n\}$ and boundedness of $\{\|(Y^n, Z^n)\|\}$ we obtain

$$
\text{for all } n, m \geq n_0, \qquad \|Z^n - Z^m\|^2 \leq 2C\|Y^n - Y^m\|.
$$

$\Longrightarrow$ $\{Z^n\}$ is Cauchy in $\mathbb{H}^2(\mathbb{R}^d)$, and thus converge to $Z \in \mathbb{H}(\mathbb{R}^d)$.

It then can be checked that, possibly along a subsequence: as $n \to \infty$, $\mathbb{P}$-almost surely,

$$\begin{aligned}\sup_{t\le T}|Y_t^n - Y_t| \;&\le\; \int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|ds \\ &\quad + \sup_{t\le T}\Big|\int_t^T Z_s^n dW_s - \int_t^T Z_s dW_s\Big| \to 0,\end{aligned}$$

$\Longrightarrow$ Y is continuous, and since $\{Y^n\}$ is monotone, by Dini the convergence is uniform.
$\Longrightarrow$ One can then pass all the necessary limits to show that (Y, Z) is an adapted solution of the original equation (18).

Finally, let $(\hat{Y}, \hat{Z})$ any H^2 solution of (18). By Comparison Thm we get that $\forall n$ $Y^n \le \hat{Y}$ and therefore $Y \le \hat{Y}$ proving that Y is the minimal solution. ∎

We now consider the following BSDE with Reflection (cf. e.g., El Karoui-Kapoudjian-Pardoux-Peng-Quenez, 1997):

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t, \\ Y_t &\geq S_t, \quad t \in [0, T], \end{aligned} \tag{20}$$

where

- S_t, $t \in [0, T]$ is the *obstacle* process, which is assumed to be continuous, and $\mathbb{E}\{\sup_{0\leq t\leq T} |S_t|^2\} < \infty$; and is given as a parameter of the equation.
- K_t, $t \in [0, T]$ is the *reflecting* process, which is assumed to be continuous and increasing, and satisfies:
$$K_0 = 0, \qquad \int_0^T (Y_t - S_t)dK_t = 0;$$
and it is defined as a part of the solution to the BSDE (20)(!)

Recall the well-known Skorohod Problem:

Let x be a continuous function on $[0,\infty)$ such that $x_0 \geq 0$. Then there exists a unique pair (y,k) of functions on $[0,\infty)$ such that

- $y = x + k$;
- $y_t \geq 0$, $\forall t$;
- $t \mapsto k_t$ is continuous, increasing, $k_0 = 0$, and $\int_0^\infty y_t dk_t = 0$.

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It is known that the solution to the Skorohod Problem for x has an explicit form: $k_t = \sup_{s \leq t} x_s^-$, $t \geq 0$. In the BSDE case we have

Proposition

Let (Y, Z, K) be a solution to the BSDE (20). Then for all $t \in [0, T]$, it holds that

$$K_T - K_t = \sup_{t \leq u \leq T} \left\{ \xi + \int_u^T f(s, Y_s, Z_s)ds - \int_u^T Z_s dW_s - S_u \right\}^-.$$

Proposition

Let (Y, Z, K) be a solution to (20). Then

- for each $t \in [0, T]$,

$$Y_t = \operatorname*{esssup}_{v \in \mathscr{T}_t} \mathbb{E}\Big\{ \int_t^v f(s, Y_s, Z_s)ds + S_v \mathbf{1}_{\{v<T\}} + \xi \mathbf{1}_{\{v=T\}}, \Big\},$$

  where $\mathscr{T}_t$ is the set of all stopping times v, s.t. $t \le v \le T$.

- Suppose further that the obstacle process S is an Itô process:

$$S_t = S_0 + \int_0^t U_s ds + \int_0^t \langle V_s, dW_s \rangle, \quad t \ge 0,$$

  where $U,\ V \in L^2_{\mathbb{F}}([0, T] \times \Omega)$. Then
  - $Z_t = V_t$, $d\mathbb{P} \otimes dt$-a.e. on the set $\{Y_t = S_t\}$;
  - $0 \le dK_t \le \mathbf{1}_{\{Y_t = S_t\}}[f(t, S_t, V_t) + U_t]^- dt$.

A Priori Estimates

Lemma 1

Let (Y, Z, K) be a solution to (20). Then $\exists C > 0$ such that

$$\mathbb{E}\Big\{\sup_{0\le t\le T} Y_t^2 + \int_0^T |Z_t|^2 dt + K_T^2\Big\} \le C\mathbb{E}\Big\{\xi^2 + \int_0^T f^2(t,0,0)dt + \sup_{0\le t\le T}(S_t^+)^2\Big\}. \quad (21)$$

Proof. First apply Itô's formula to get

$$Y_t^2 + \int_t^T |Z_s|^2 ds = \xi^2 + 2\int_s^T Y_s f(s, Y_s, Z_s)ds + 2\int_s^T Y_s dK_s - 2\int_s^T Y_s \langle Z_s, dW_s \rangle .$$

Then use $\int_0^T (Y_t - S_t)dK_t = 0$, Hölder, and Gronwall. ∎

A Priori Estimates

Lemma 2

Let (Y^i, Z^i, K^i), $i = 1, 2$ be solutions to BSDEs (20) with parameters (ξ^i, f^i, S^i), $i = 1, 2$, respectively. Then $\exists C > 0$ s.t.

$$
\begin{aligned}
&\mathbb{E}\Big\{\sup_{0\le t\le T}(\Delta Y_t)^2 + \int_0^T |\Delta Z_t|^2 dt + (\Delta K_T)^2\Big\} \\
&\le C\mathbb{E}\Big\{(\Delta\xi)^2 + \int_0^T [\Delta f(t,0,0)]^2 dt\Big\} \\
&+ C\Big[\mathbb{E}\Big(\sup_{0\le t\le T}(\Delta S_t^+)^2\Big)\Big]^{1/2}\Psi_T^{1/2},
\end{aligned}
\tag{22}
$$

where $\Delta X = X^1 - X^2$, for $X = \xi$, f, S, Y, Z, and K; and $\Psi_T = \mathbb{E}\Big\{\sum_{i=1}^2(|\xi^i|^2 + \int_0^T |f^i(t,0,0)|^2 dt + \sup_{0\le t\le T}|S_t^i|^2)\Big\}$.

A Priori Estimates

Lemma 2

Let (Y^i, Z^i, K^i), $i = 1, 2$ be solutions to BSDEs (20) with parameters (ξ^i, f^i, S^i), $i = 1, 2$, respectively. Then $\exists C > 0$ s.t.

$$\begin{aligned}
&\mathbb{E}\Big\{\sup_{0\le t\le T}(\Delta Y_t)^2 + \int_0^T |\Delta Z_t|^2 dt + (\Delta K_T)^2\Big\} \\
&\le C\mathbb{E}\Big\{(\Delta\xi)^2 + \int_0^T [\Delta f(t,0,0)]^2 dt\Big\} \qquad (22)\\
&+C\Big[\mathbb{E}\Big(\sup_{0\le t\le T}(\Delta S_t^+)^2\Big)\Big]^{1/2}\Psi_T^{1/2},
\end{aligned}$$

where $\Delta X = X^1 - X^2$, for $X = \xi$, f, S, Y, Z, and K; and $\Psi_T = \mathbb{E}\Big\{\sum_{i=1}^2(|\xi^i|^2 + \int_0^T |f^i(t,0,0)|^2 dt + \sup_{0\le t\le T}|S_t^i|^2)\Big\}$.

Note: The uniqueness of BSDE follows directly from Lemma 2!

Comparison Theorem

Theorem

Let (Y^i, Z^i, K^i), $i = 1, 2$ be solutions to BSDEs (20) with parameters (ξ^i, f^i, S^i), $i = 1, 2$, respectively. Suppose that

- $\xi^1 \le \xi^2$,
- $f^1 \le f^2$,
- $S^1_t \le S^2_t$, $0 \le t \le T$, a.s.

Then $Y^1_t \le Y^2_t$, $0 \le t \le T$, a.s.

Proof. Apply Itô's formula to $|(\Delta Y_t)^+|^2$, and taking expectation. Then use the fact that $Y^1 > S^2_t \ge S^1_t$ on $\{\Delta Y_t > 0\}$ to get

$$\int_t^T (\Delta Y_t)^+ (dK^1_t - dK^2_t) = -\int_t^T (\Delta Y_t)^+ dK^2_t \le 0.$$

Then apply Gronwall to get $(\Delta Y_t)^+ \equiv 0 \Longrightarrow Y^1 \le Y^2$. ∎

The existence and uniqueness of the adapted solution to the reflected BSDE (20) can be proved using a standard Picard iteration (see EK-K-P-P-Q). However, the following "*penalization*" method has been used more often for its clarity on the structure of the solution.

Penalization Scheme

For each $n \in \mathbb{N}$, let (Y^n, Z^n) be the solution to the unconstrained BSDE:

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds + K_T^n - K_t^n - \int_t^T \langle Z_s^n, dW_s \rangle, \quad (23)$$

where $K_t^n \stackrel{\triangle}{=} n \int_0^t (Y_s^n - S_s)^- ds$, $t \in [0, T]$.

Well-Posdedness of the Reflected BSDEs

One can show that (as unconstrained BSDE):

- $\mathbb{E}\{\sup_{0\le t\le T}|Y^n|^2\} < \infty$
- $\exists C > 0$, such that

$$\mathbb{E}\Big\{\sup_{0\le t\le T}|Y^n_t|^2 + \int_0^T |Z^n_t|^2dt + (K^n_T)^2\Big\} \le C.$$

- Since $f_n = f + n(y - S_t)^- \le f_{n+1}$, by Comparison Theorem, $Y^n_t \le Y^{n+1}_t$, $0 \le t \le T$, a.s. $\Longrightarrow Y^n \uparrow Y$.
- By Fatou, one has $\mathbb{E}\Big\{\sup_{0\le t\le T}|Y_t|^2\Big\} \le C$.
- Apply DCT to get $\mathbb{E}\int_0^T (Y_t - Y^n_t)^2 dt \to 0$, as $n \to \infty$.
- Since $\mathbb{E}\Big\{\sup_t|(Y^n_t - S_t)^-|^2\Big\} \to 0$, as $n \to \infty$ (not trivial!!), it follows that $\{(Y^n, Z^n)\}$ is Cauchy in $L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}\times\mathbb{R}^d)$.

Well-Posdedness of the Reflected BSDEs

- Thus $\{(Y^n, Z^n)\} \subset L^2_{\mathbb{F}}(\Omega; \mathbb{C}([0,T]; \mathbb{R}) \times L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^d)$ is Cauchy, and $\{K^n\}$ is Cauchy in $L^2_{\mathbb{F}}(\Omega; \mathbb{C}([0,T]; \mathbb{R})$ as well

  $\Longrightarrow$ The limit (Y, Z, K) (of $\{(Y^n, Z^n, K^n)\}$) must satisfy (20)

- To check the "flat-off" condition, note that

  $$\mathbb{E}\Big\{\sup_t |(Y_t - S_t)^-|^2\Big\} = \lim_n \mathbb{E}\Big\{\sup_t |(Y^n_t - S_t)^-|^2\Big\} = 0$$

  $\Longrightarrow Y_s \geq S_t,\ \forall t \Longrightarrow \int_0^T (Y_t - S_t)dK_t \geq 0$.

- On the other hand, since

  $$\int_0^T (Y^n_t - S_t)dK^n_t = -n\int_0^T [(Y^n_t - S_t)^-]^2 dt \leq 0,$$

  $\Longrightarrow \int_0^T (Y_t - S_t)dK_t = \lim_n \int_0^T (Y^n_t - S_t)dK^n_t \leq 0$

  $\Longrightarrow \quad \int_0^T (Y_t - S_t)dK_t = 0$. ■

Consider the BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T]. \tag{24}$$

We assume that the generator f takes the following form:

$$f(t, y, z) = a_0(t, y, z)y + F_0(t, y, z), \tag{25}$$

where for constants $\beta_0 < \alpha_0$, it holds for all $(y, z) \in \mathbb{R}^{1+d}$ that

(H1) $\begin{cases} \beta_0 \le a_0(t, y, z) \le \alpha_0; \\ |F_0(t, y, z)| \le k + c(|y|)|z|^2; \end{cases}$ $dt \times d\mathbb{P}$-a.s.

Note:

Under (H1) f grows linearly in y, but quadratically in z!

Theorem (Kobylanski, 2000)

Suppose that the coefficient f satisfies (H1), with α_0, β_0, $k \in \mathbb{R}$, and $c : \mathbb{R}^+ \mapsto \mathbb{R}^+$ being continuous and increasing.
Then, for any $\xi \in L^\infty(\mathscr{F}_T)$, the BSDE (24) admits at least one solution $(Y, Z) \in L^\infty_{\mathbf{F}}(\Omega; \mathbb{C}([0, T]; \mathbb{R})) \times \mathscr{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$.

BSDEs with Quadratic Growth

Theorem (Kobylanski, 2000)

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The Power of "Exponential (Hopf-Cole) Transformation"

Consider a simple quadratic BSDE:

$$Y_t = \xi + \int_t^T \frac{1}{2}|Z_s|^2 ds - \int_t^T Z_s dW_s, \quad t \in [0, T]. \tag{26}$$

Define $y = \exp[Y]$, $z = yZ$. Then, the BSDE (26) becomes

$$y_t = \exp[\xi] - \int_t^T z_s dW_s, \quad t \in [0, T].$$

Suppose that the assumption (H1) is replaced by

(H0) $$\begin{cases} a_0(t,y,z) \le \alpha_0; \\ |F_0(t,y,z)| \le b(t) + C(|y|)|z|^2, \end{cases} \qquad dt \times d\mathbb{P}\text{-a.s.}$$

where α_0 is constant and $b \in L^1([0,T])$. Then the following *a priori estimates* hold:

- Assume that $\xi \in L^\infty_{\mathscr{F}_T}(\Omega)$, then

$$Y_t \le \Big[\sup_{\Omega}(\xi)\Big]^+ e^{\int_t^T a_s ds} + \int_t^T b_s e^{\int_t^s a_\lambda d\lambda} ds; \tag{27}$$

- for some constant $K > 0$, $\mathbb{E}\int_0^T |Z_s|^2 ds \le K$;
- $\|Y\|_\infty \le \|\xi\|_\infty + \dfrac{\|b\|_\infty}{|\alpha_0|}$.

A Priori Estimates

Suppose that the assumption (H1) is replaced by

(H0) $$\begin{cases} a_0(t,y,z) \le \alpha_0; \\ |F_0(t,y,z)| \le b(t) + C(|y|)|z|^2, \end{cases} \qquad dt \times d\mathbb{P}\text{-a.s.}$$

where α_0 is constant and $b \in L^1([0,T])$. Then the following *a priori estimates* hold:

- Assume that $\xi \in L^\infty_{\mathscr{F}_T}(\Omega)$, then

$$Y_t \ge \Big[\inf_\Omega(\xi)\Big]^- e^{\int_t^T a_s ds} - \int_t^T b_s e^{\int_t^s a_\lambda d\lambda} ds; \qquad (28)$$

- for some constant $K > 0$, $\mathbb{E}\displaystyle\int_0^T |Z_s|^2 ds \le K$;
- $\|Y\|_\infty \le \|\xi\|_\infty + \dfrac{\|b\|_\infty}{|\alpha_0|}$.

Idea of the Proof.

- Define the RHS of (27) by φ, then φ satisfies the ODE:

$$\varphi_t = \Big[\sup_{\Omega}(\xi)\Big]^+ \int_t^T (a_s\varphi_s + b_s)ds, \quad t \in [0, T].$$

- Let Φ be a C^2-function to be determined. Applying Itô to get

$$\begin{aligned}\Phi(Y_t - \varphi_t) &= \Phi(Y_T - \varphi_T) \\ &+ \int_t^T \Phi'(Y_s - \varphi_s)[f(s, Y_s, Z_s) - (a_s\varphi_s + \beta_s)]ds \qquad (29) \\ &- \int_t^T \frac{1}{2}\Phi''(Y_s - \varphi_s)|Z_s|^2 ds - \int_t^T \Phi'(Y_s - \varphi_s)Z_s dW_s.\end{aligned}$$

A Priori Estimates

- Denote $M = \|Y\|_\infty + \|\varphi\|_\infty$, and choose

$$\Phi(u) = \begin{cases} e^{2Cu} - 1 - 2Cu - 2C^2u^2, & u \in [0, M] \\ 0 & u \in [-M, 0]. \end{cases}$$

  One can check that

  - $\Phi(u) \geq 0$ and $\Phi(u) = 0 \iff u \leq 0$
  - $\Phi'(u) \geq 0$
  - $0 \leq u\Phi'(u) \leq 2(M+1)C\Phi(u)$
  - $C\Phi'(u) - \frac{1}{2}\Phi''(u) \leq 0.$

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  - $C\Phi'(u) - \frac{1}{2}\Phi''(u) \leq 0.$

- Applying these to (29) we get, with $k_t \stackrel{\triangle}{=} a_t^+ 2(M+1)C$,

$$0 \leq \Phi(Y_t - \varphi_t) \leq \int_t^T k_s \Phi(Y_s - \varphi_s) ds - \int_t^T \Phi'(Y_s - \varphi_s) Z_s dW_s,$$

- Taking expectation and applying Gronwall one shows that $\mathbb{E}\{\Phi(Y_t - \varphi_t)\} = 0 \Longrightarrow \Phi(Y_t - \varphi_t) = 0 \Longrightarrow Y_t \leq \varphi_t$.

The L^2-bound for Z can be proved by considering

$$\Phi(u) = \frac{1}{2C^2}\Big[\exp(2C(u+M)) - (1+2C(u+M))\Big],$$

where $M = \|Y\|_\infty$. Indeed, since

- $\Phi(u) \geq 0$, $\Phi'(u) \geq 0$
- $0 \leq u\Phi'(u) \leq \frac{M}{C}(e^{4CM} - 1) \stackrel{\triangle}{=} K_0$
- $\frac{1}{2}\Phi''(u) - C\Phi'(u) = 1$,

Setting $\varphi \equiv 0$ in (29) we can check

$$0 \leq \Phi(Y_0) \leq \Phi(Y_T) + K_0 \int_t^T a_s^+ ds - \int_t^T |Z_s|^2 ds - \int_t^T \Phi'(Y_s) Z_s dW_s,$$

$$\Longrightarrow \mathbb{E}\int_0^T |Z_s|^2 ds \leq \Phi(M) + K_0 \|a^+\|_{L^1} \stackrel{\triangle}{=} K. \qquad \blacksquare$$

Monotone Stability

Proposition

Suppose that $\{(f^n, \xi^n)\}$ are a sequence of parameters such that

- $f^n \to f$ locally uniformly on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$;
- $\xi^n \to \xi$ in L^∞.
- $\exists k : \mathbb{R}^+ \to \mathbb{R}^+$, $k \in L^1([0, T])$, such that for some $C > 0$,

$$|f^n(t, y, z)| \le k_t + C|z|^2, \qquad \forall n \in \mathbb{N}, (t, y, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d.$$

- $(Y^n, Z^n) \in L^\infty_{\mathbf{F}}(\Omega; \mathbb{C}([0, T]; \mathbb{R})) \times \mathscr{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$ such that $\{Y^n\} \nearrow$ and $\|Y^n\|_\infty \le M$.

Then $\exists (Y, Z) \in L^\infty_{\mathbf{F}}(\Omega; \mathbb{C}([0, T]; \mathbb{R})) \times \mathscr{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$ such that

$$\lim_{n\to\infty} Y^n = Y, \text{ uniformly on } [0, T]; \quad Z^n \to Z \text{ in } \mathscr{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d),$$

and (Y, Z) solves BSDE (24). ■

Main Points:

- $\{Y^n\}$ is monotone and bounded $\Longrightarrow \exists Y$, s.t., $Y^n \to Y$ (pointwisely).
- $\{Z^n\}$ is bounded in $L^2([0,T]\times\Omega) \Longrightarrow$ it has a weakly convergent subsequence, denoted by itself.

Main Points:

- $\{Y^n\}$ is monotone and bounded $\Longrightarrow \exists Y$, s.t., $Y^n \to Y$ (pointwisely).
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Want to Show:

- $\{Z^n\}$ converges Strongly in $L^2([0,T]\times\Omega)$ (Mazur's Theorem)
- $\{Y^n\}$ converges Uniformly in t (Dini's Theorem)

Consequently, one can then show that

- $\displaystyle\int_t^T f^n(s, Y_s^n, Z_s^n)ds \to \int_t^T f(s, Y_s, Z_s)ds$; and
- $\displaystyle\int_t^T Z_s^n dW_s \to \int_t^T Z_s dW_s$, and thus (Y, Z) solves the BSDE.

Proof of the Main Result

Assumptions:

There exists α_0, $\beta_0 \in \mathbb{R}$, $B, C \in \mathbb{R}^+$, such that for all $(t, y, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$,

$$f(t, y, z) = a_0(t, y, z)y + F_0(t, y, z),$$

where

- $\beta_0 \leq a_0(t, y, z) \leq \alpha_0$,
- $F_0(t, y, z)| \leq B + C|z|^2$.

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There exists α_0, $\beta_0 \in \mathbb{R}$, $B, C \in \mathbb{R}^+$, such that for all $(t, y, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$,

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- $\beta_0 \leq a_0(t, y, z) \leq \alpha_0$,
- $F_0(t, y, z)| \leq B + C|z|^2$.

1. Exponential Change. Define $\mathbf{y}_t \triangleq e^{2CY_t}$ and $\mathbf{z}_t \triangleq 2C\mathbf{y}_t Z_t$. Then, by Itô one can check that $(\mathbf{y}, \mathbf{z})$ solves the BSDE (24) with parameters:

- $\mathbf{y}_T = e^{2C\xi}$;
- $F(t, y, z) \triangleq 2C \cdot yf\Big(s, \dfrac{\ln(y)}{2C}, \dfrac{z}{2C \cdot y}\Big) - \dfrac{1}{2}\dfrac{|z|^2}{y}$

Proof of the Main Result

2. Truncation. Define a C^∞ function $\psi : \mathbb{R} \mapsto [0,1]$ by

$$\psi(u) = \begin{cases} 1, & \text{if } u \in [e^{-2CM}, e^{2CM}]; \\ 0, & \text{if } u \notin [e^{-2C(M+1)}, e^{2C(M+1)}]. \end{cases}$$

Now, define $\widetilde{F}(t,y,z) \stackrel{\triangle}{=} \psi(y)F(t,y,z)$, and let

$$\begin{aligned} \ell^+(y) &\stackrel{\triangle}{=} \psi(y)(\alpha_0 y \ln(y) + 2CBy); \\ \ell^-(y,z) &\stackrel{\triangle}{=} \psi(y)\Big(\beta_0 y \ln(y) - 2CBy - \frac{|z|^2}{y}\Big). \end{aligned}$$

Then it is easily checked that

$$\ell^-(y,z) \le \widetilde{F}(t,y,z) \le \ell(y), \qquad \forall (t,y,z). \tag{30}$$

Note:

The function $y \mapsto \ell^+(y)$ is bounded and Lipschitz!

3. Approximation. For any $n \in \mathbb{N}$, find $\widetilde{F}^n \in C_b^\infty$ such that

$$\widetilde{F} + \frac{1}{2^{n+1}} \le \widetilde{F}^n \le \widetilde{F} + \frac{1}{2^n}.$$

Then, let $\phi_n \in C^\infty$ be s.t. $\phi(u) = \begin{cases} 1, & 0 \le u \le n; \\ 0, & u \ge n+1. \end{cases}$ Define

$$F^n(t,y,z) \stackrel{\triangle}{=} \widetilde{F}^n(t,y,z)\phi_n(|y|+|z|) + \Big(\ell^+(y) + \frac{1}{2^n}\Big)[1 - \phi_n(|y|+|z|)].$$

Note:

- F^n's are uniformly Lipschitz (in (y,z));
- For any $n \in \mathbb{N}$ and all (t,y,z), it holds that

$$\widetilde{F}(t,y,z) \le \widetilde{F}^n(t,y,z) \le F^n(t,y,z) \le \ell^+(y) + \frac{1}{2^n}. \qquad (31)$$

4. Synthesis. Denote $(\mathbf{y}^n, \mathbf{z}^n)$ to be solution to BSDE$(F^n, e^{2C\xi})$, via standard theory.

- For n large enough
  - $F^n(t, e^{2CM}, 0) \le 0$, and $e^{2CM} \ge e^{2C\xi}$;
  - $F^n(t, e^{-2CM}, 0) \ge 0$, and $e^{-2CM} \le e^{2C\xi}$,
- Since $y_t \equiv e^{2CM}$, $z_t \equiv 0$ (resp. $y_t \equiv e^{-2CM}$, $z_t \equiv 0$) are solutions to the BSDE$(e^{2CM}, 0)$ (resp. BSDE$(e^{-2CM}, 0)$), by the standard Comparison Theorem we conclude:

$$e^{-2CM} \le \mathbf{y}^{n+1} \le \mathbf{y}^n \le e^{2CM}.$$

- Define $Y_t^n \triangleq \frac{\ln(\mathbf{y}_t^n)}{2C}$, $Z_t^n \triangleq \frac{\mathbf{z}_t^n}{2C\mathbf{y}_t^n}$, and

$$\begin{aligned} f^n(t,y,z) &\triangleq \frac{F^n(t, e^{2Cy}, 2Ce^{2Cy}z)}{2Ce^{2Cy}} + C|z|^2 \\ \widetilde{f}^n(t,y,z) &\triangleq \frac{\widetilde{F}^n(t, e^{2Cy}, 2Ce^{2Cy}z)}{2Ce^{2Cy}} + C|z|^2; \end{aligned}$$

Then (Y^n, Z^n) is the solution to BSDE$(\widetilde{f}^n, \xi)$, $n \in \mathbb{N}$, and Y^n's are monotone, since y^n's are!

- Since $\widetilde{f}^n \to \widetilde{f}$ and $f^n \to f$, uniformly on compacts, we can first apply the Monotone Stability Theorem, we know that $\exists (Y, Z) \in L^\infty_{\mathbf{F}}(\Omega; \mathbb{C}([0,T]; \mathbb{R})) \times \mathscr{H}^2_{\mathbf{F}}([0,T]; \mathbb{R}^d)$ such that (Y, Z) solves BSDE$(\widetilde{f}, \xi)$.
- One can then show that $\|Y\|_\infty \leq M$ as was done in the a priori estimate, and note that

$$\widetilde{f}(t, y, z) = f(t, y, z), \qquad \text{whenever } |y| \leq M$$

(by the nature of the truncation), we conclude that (Y, Z) solves BSDE(f, ξ), proving the existence. ■

Uniqueness

We shall assume that the generator f satisfies the following assumptions throughout the uniqueness discussion.

(H2) For some constants M and $C > 0$, and positive functions $l(\cdot)$ and $k(\cdot)$, it holds for all $t \in \mathbb{R}^+$, $y \in [-M, M]$, and $z \in \mathbb{R}^d$ that

$$\begin{cases} |f(t,y,z)| \le l(t) + C|z|^2, & a.s., \\ \left|\dfrac{\partial f}{\partial z}(t,y,z)\right| \le k(t) + C|z|^2, & a.s., \end{cases} \tag{32}$$

(H3) For some constant $\varepsilon > 0$ and $C_\varepsilon > 0$, it holds for all $t \in \mathbb{R}^+$, $y \in \mathbb{R}$, and $z \in \mathbb{R}^d$ that

$$\frac{\partial f}{\partial y}(t,y,z) \le l_\varepsilon(t) + C|z|^2, \qquad a.s. \tag{33}$$

Uniqueness

Comparison Theorem

Let (Y^i, Z^i), $i = 1, 2$ be two solutions of BSDE(f^i, ξ^i), $i = 1, 2$. Assume that

- $\xi^1 \le \xi^2$, a.s., and $f^1 \le f^2$;
- For all $\varepsilon > 0$ and $M > 0$, there exist functions $l, l_\varepsilon \in L^1$, $k \in L^2$, and constant $C \in \mathbb{R}$, such that either f^1 or f^2 satisfies both (H2) with l, k, and C and (H3) with l_ε and ε.

Then if (Y^1, Z^1) [resp. $(Y^2, Z^2) \in L^\infty(\cdots) \times L^2(\cdots)$] is a sub-solution (resp. super-solution) of the BSDEs with parameters (f^1, ξ^1) (resp. (f^2, ξ^2)), one has

$$Y^1_t \le Y^2_t, \qquad \forall t \in \mathbb{R}^+, \quad a.s.$$

Proof. Lengthy. (cf. Kobylanski (2000)) ■

A Quick Summary

We have studied following types of BSDEs beyond the standard ones:

- BSDEs with continuous coefficients
- BSDEs with reflections
- BSDEs with quadratic growth

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Some Variations

- Reflected BSDEs with continuous coefficients — Matoussi (1997), Hamadene-Matoussi-Lepeltier (1997)
- BSDEs with superlinear-quadratic coefficients — Lepelier-San Martin (1998)
- Reflected BSDE with superlinear-quadratic coefficients — Kobylanski-Lepeltier-Quenez-Torres (2001)
- ......

Converse Comparison Theorem

Consider the BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \geq 0.$$

We know that "$\xi^1 \geq \xi^2$" $\oplus$ "$f^1 \geq f^2$" $\implies$ "$Y_t^1 \geq Y_t^2$, $t \geq 0$"

Question:

Under what condition $Y^1 \geq Y^2$ implies $f^1 \geq f^2$?

Main Assumptions

(A1) The random field $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ is uniformly Lipschitz in (y, z), uniformly in (t, ω).

(A2) $t \mapsto f(t, 0, 0)$, is a square-integrable adapted process.

(A3) $f(t, y, 0) = 0$

(A4) $t \mapsto f(t, y, z)$ is continuous.

Converse Comparison Theorem

Theorem (Briand-Coquet-Hu-Mémin-Peng, 2000)

Assume (A1)–(A4), and assume further that for any $\xi \in L^2(\mathscr{F}_T)$, it holds that $Y_t^1(\xi) \le Y_t^2(\xi)$, for all $t \in [0, T]$, $\mathbb{P}$-a.s.
Then $\mathbb{P}$-almost surely,

$$f_0^1(t, y, z) \le f_0^1(t, y, z), \quad \forall (y, z) \in \mathbb{R} \times \mathbb{R}^6.$$

Main Tricks:

- Choose $\xi_\varepsilon = y + z(W_{t+\varepsilon} - W_t)$, $\varepsilon > 0$; and denote $Y_T^\varepsilon \stackrel{\triangle}{=} Y_t(\xi_\varepsilon)$;
- Show that $\frac{1}{\varepsilon}(Y_t^\varepsilon - y) \to g(t, y, z)$, as $\varepsilon \to 0$;
- Then $Y^{1,\varepsilon} \le Y^{2,\varepsilon} \implies \quad g_1 \le g_2$.

This is based on the works of Briand and Hu (2005-08).
Consider the BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \qquad t \in [0, T] \quad (34)$$

Main Assumptions

$\exists \beta \geq 0, \gamma > 0$, $\alpha \in L^0_{\mathbb{F}}([0, T] \times \Omega)$, and $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\varphi(0) = 0$, such that $\mathbb{P}$-a.s.,

(i) For all $t \in [0, T]$, $(y, z) \mapsto f(t, y, z)$ is continuous;

(ii) (Monotoniciy in y) $\forall (t, z)$,

$$y[f(t, y, z) - f(t, 0, z)] \leq \beta |y|^2;$$

(iii) (Quadratic growth): $\forall (t, y, z)$,

$$|f(t, y, z) \leq \alpha(t) + \varphi(|y|) + \frac{\gamma}{2}|z|^2.$$

Main Purpose:

Find the adapted solution (hopefully unique!) to the BSDE (34), with terminal value ξ satisfying: $\mathbb{E}\{e^{\gamma|\xi|}\} < \infty$ (ξ is said to have "*exponential moment of order γ*"), for some or all $\gamma > 0$.

A Trick: Consider $U(t, |Y_t|) = e^{\gamma\psi(t,|Y_t|)}$, where ψ is a smooth function to be determined. Applying Itô $\oplus$ Tanaka:

$$\frac{dU(t,|Y_t|)}{\gamma U(t,|Y_t|)} = \{-\psi_x(t,|Y_t|)sgn(Y_t)f(t,Y_t,Z_t) + \psi_t(t,|Y_t|) + \frac{\gamma}{2}\psi_x(t,|Y_t|)^2|Z_t|^2\}dt + \frac{1}{2}\psi_{xx}(t,|Y_t|)|Z_t|^2 dt + \psi_x(t,|Y_t|)dL_t + \psi_x(t,|Y_t|)sgn(Y_t)Z_t \cdot dW_t,$$

where L is the local time of Y at zero.

Since

$$\begin{aligned} sgn(Y_t)f(t, Y_t, Z_t) &= sgn(Y_t)[f(t, Y_t, Z_t) - f(t, 0, Z_t)] \\ &\quad + sgn(Y_t)f(t, 0, Z_t) \\ &\leq \beta|Y_t| + \alpha(t) + \frac{\gamma}{2}|Z_t|^2, \end{aligned}$$

assuming $\psi_x(t, x) \geq 1$ for $x \geq 0$, one has

$$\begin{aligned} &\psi_x(t, |Y_t|)sgn(Y_t)f(t, Y_t, Z_t) - \psi_t(t, |Y_t|) - \frac{\gamma}{2}\psi_x(t, |Y_t|)^2|Z_t|^2 \\ &\leq \psi_x(t, |Y_t|)[\alpha(t) + \beta|Y_t|] - \psi_t(t, |Y_t|). \end{aligned}$$

Since

$$\begin{aligned} sgn(Y_t)f(t, Y_t, Z_t) &= sgn(Y_t)[f(t, Y_t, Z_t) - f(t, 0, Z_t)] \\ &\quad + sgn(Y_t)f(t, 0, Z_t) \\ &\le \beta|Y_t| + \alpha(t) + \frac{\gamma}{2}|Z_t|^2, \end{aligned}$$

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$$\psi_x(t, |Y_t|)sgn(Y_t)f(t, Y_t, Z_t) - \psi_t(t, |Y_t|) - \frac{\gamma}{2}\psi_x(t, |Y_t|)^2|Z_t|^2$$
$$\le \psi_x(t, |Y_t|)[\alpha(t) + \beta|Y_t|] - \psi_t(t, |Y_t|).$$

Idea:

Look for ψ that solves the first order PDE for $(t, x) \in [s, T] \times \mathbb{R}$:

$$\psi_t(t, x) - (\alpha(t) + \beta x)\psi_x(t, x) = 0, \quad \psi(s, x) = x. \tag{35}$$

The solution to the characteristic equation of (35):

$$v(u;t,x) = x + \int_u^t [\alpha(r) + \beta v(r;t,x)]dr, \quad 0 \le u \le t. \qquad (36)$$

is $v(s;t,x) = xe^{\beta(t-s)} + \int_s^t \alpha(r)e^{\beta(r-s)}dr$, $0 \le s \le t \le T$.

Since $\frac{d}{du}\psi(u, v(u;t,x)) = 0$, we have for $s \le t \le T$,

$$\psi(t,x) = \psi(t, v(t;t,x)) = \psi(s, v(s;t,x)) = v(s;t,x).$$

$\Longrightarrow$ $\psi_x(t,x) \ge 1$ and $\psi_{xx}(t,x) \ge 0$!!

A Key Estimate

$$\begin{aligned} e^{\gamma|Y_s|} &= U(s,|Y_s|) \\ &\le U(t,|Y_t|) - \int_s^t \gamma U(r,|Y_r|)\psi_x(r,|Y_r|)sgn(Y_r)Z_r dW_r. \end{aligned}$$

Quadratic BSDEs with Unbounded Terminal Value

Theorem (Existence)

Assume that the main assumption holds. Assume also that $\xi + |\alpha|_1$ has an exponential moment of order $\gamma e^{\beta T}$, then the BSDE (34) has a solution (Y, Z) such that

$$|Y_t| \le \frac{1}{\gamma} \log \mathbb{E}\Big\{ \exp\Big\{\gamma e^{\beta(T-t)}|\xi| + \gamma \int_t^T \alpha(r) e^{\beta(r-t)} dr\Big\} \Big| \mathscr{F}_t \Big\}.$$

Note:

The Comparison Theorems (whence uniqueness) for quadratic BSDE were only proved for the bounded terminal value case, based essentially on the fact that in that case the process $Z \bullet W$ is a "BMO Martingale". Since this fact fails in the unbounded terminal case, a new idea is needed!

Assumption (A2)

There exist two constants $\gamma > 0$ and $\beta \geq 0$, and a non-negative, progressively measurable process $\alpha(t)$, $t \geq 0$, such that,

- $\forall t \in [0, T]$, $\forall y \in \mathbb{R}$, the mapping $z \mapsto f(t, y, z)$ is convex;
- $\forall (t, z) \in [0, T] \times \mathbb{R}$,

$$|f(t, y, z) - f(t, y', z')| \leq \beta |y - y'|, \quad \forall (y, y') \in \mathbb{R}^2;$$

- f satisfies the growth condition:

$$|f(t, y, z)| \leq \alpha(t) + \beta |y| + \frac{\gamma}{2} |z|^2;$$

- $|\alpha|_1$ has exponential moment of all order.

Comparison Theorem

Let (Y, Z) and (Y', Z') be two solution to (34) w.r.t. terminal conditions ξ and ξ', generators f and f', respectively. Assume that

- for any $\lambda > 0$, $\mathbb{E}\Big\{e^{\lambda Y^*} + e^{\lambda Y'^*}\Big\} < \infty$, where $Y^* = \sup_{t\in[0,T]} |Y_t|$;
- $\xi \le \xi'$, $\mathbb{P}$-a.s.;
- $f(t, y, z) \le f'(t, y, z)$, $\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$;
- f satisfies (A2).

Then $Y_t \le Y'_t$, for all $t \in [0, T]$, $\mathbb{P}$-a.s. Furthermore, if $Y_0 = Y'_0$, then

$$\mathbb{P}\Big\{\xi' = \xi, \quad \int_0^T (f' - f)(t, Y'_t, Z'_t)dt = 0\Big\} > 0.$$

Main Trick:

- For any $\theta \in (0,1)$, consider $\eta^\theta = \eta - \theta\eta'$, $\eta = \xi, Y, Z$.
- Let $A_t = \int_0^t \alpha(s)ds$, then we have

$$e^{A_t} Y_t^\theta = e^{A_T} Y_T^\theta + \int_t^T e^{A_s} F_s ds - \int_t^T e^{A_s} Z_s^\theta dW_s,$$

where, denoting $\delta f(t) \stackrel{\triangle}{=} (f - f')(t, Y_t', Z_t')$,

$$\begin{aligned} F_t &= (f(t, Y_t, Z_t) - \theta f'(t, Y_t', Z_t')) - \alpha(t) Y_t^\theta \\ &= (f(t, Y_t, Z_t) - f(t, Y_t', Z_t)) \\ &\quad + (f(t, Y_t', Z_t) - \theta f(t, Y_t', Z_t')) + \theta \delta f(t). \end{aligned}$$

- Using the convexity of f in z, one has

$$f(t, Y_t', Z_t) \le \theta f(t, Y_t', Z_t') + (1-\theta) f\left(t, Y_t', \frac{Z_t^\theta}{1-\theta}\right)$$

- Using the growth condition to get

$$f(t, Y'_t, Z_t) \le \theta f(t, Y'_t, Z'_t) + (1-\theta)(\alpha(t) + \beta|Y'_t|) + \frac{\gamma}{1-\theta}|Z^\theta_t|^2.$$

$$\Longrightarrow F_t \le (1-\theta)(\alpha(t) + 2\beta|Y'_t|) + \frac{\gamma}{2(1-\theta)}|Z^\theta_t|^2 + \theta\delta f(t).$$

- Denote $P_t = e^{ce^{A_t}Y^\theta_t}$, $Q_t = cP_tZ^\theta_t e^{A_t}$, then

$$P_t = P_T + c\int_t^T P_s e^{A_s}\Big(F_s - \frac{ce^{A_s}}{2}|Z^\theta_t|^2\Big)ds - \int_t^T Q_s dW_s.$$

$$\Longrightarrow \quad Y^\theta_t \le \frac{1-\theta}{\gamma}\log\mathbb{E}\Big\{\exp\Big\{\gamma e^{2\beta T}\Big(|\xi| + \int_t^T G(s, |Y'_s|)ds\Big)\Big\}\Big|\mathscr{F}_t\Big\}.$$

- Letting $\theta \to 1$, one obtains $Y_t \le Y'_t$! ■

Recall the Entropic dynamic risk measure. Entropic RM

- It is shown by Barrieu-El Karoui ('04) that $\{\rho_t^\gamma(\xi)\}_{t\in[0,T]}$ is the unique solution of the following quadratic BSDE:

$$\rho_t^\gamma(\xi) = -\xi + \frac{1}{2\gamma}\int_t^T |Z_s|^2 ds - \int_t^T Z_s dB_s, \quad \forall\, t \in [0, T],$$

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- But the generator $g = \frac{1}{2\gamma}|z|^2$ is *quadratic*, and hence NOT a consequence of the representation theorem!
- In fact, in this case the "domination" (11) fails. E.g., $\gamma = 1$:

$$\rho_0(\xi+\eta)-\rho_0(\xi) = \eta+\frac{1}{2}\int_0^T (|Z_s^2+Z_s|^2-|Z_s^2|^2)ds-\int_0^T Z_s dB_s.$$

where $Z = Z^1 - Z^2$. But $\frac{1}{2}(|z^2 + z|^2 - |z^2|^2) \le |z|^2 + \frac{1}{2}|z^2|^2$ cannot be dominated by any (quadratic g).

- In fact one needs to consider a quadratic BSDE:
$$Y_t = \xi + zB_\tau + \int_t^T g(s, Z_s)ds - \int_t^T Z_s dB_s, \qquad (37)$$
where $\xi \in L^\infty(\mathscr{F}_T)$, $z \in \mathbb{R}^d$, and $\tau \in \mathscr{M}_{0,T}$.
- Although $\xi + zB_\tau$ is unbounded, it does have exponential moment of all orders (recall the moment generating function of a Brownian motion), and the BSDE is convex in z. Thus the previous existence and uniqueness applies!
- An easier way: Set $\widetilde{Y}_t = Y_t - zB_{t\wedge\tau}$, $\widetilde{Z}_t = Z_t - z\beta 1_{\{t\le\tau\}}$, then (37) becomes
$$\widetilde{Y}_t = \xi + \int_t^T g(s, \widetilde{Z}_s + z\beta 1_{\{s\le\tau\}})ds - \int_t^T \widetilde{Z}_s dB_s. \qquad (38)$$
Since $\xi \in L^\infty(\mathscr{F}_T)$, the BSDE (38) is uniquely solvable.
- The "domination" problem is more subtle, need to invoke the "BMO" theory (see, Hu-Ma-Peng-Song, 2008).

4. Wellposedness of FBSDEs

Solution Methods for FBSDEs:

General FBSDEs: for $t \in [0, T]$,

$$\begin{cases} X_t = x + \int_0^t b(s, \Theta_s)ds + \int_0^t \sigma(s, \Theta_s)dW_s; \\ Y_t = g(X_T) + \int_t^T f(s, \Theta_s)ds + \int_t^T Z_s dW_s, \end{cases} \tag{39}$$

where $\Theta_s = (X_s, Y_s, Z_s)$.

Objective:

For any given $T > 0$, and $x \in \mathbb{R}^n$, find an **F**-adapted, square-integrable process (X, Y, Z) that satisfies (39) on $[0, T]$, P-a.s.

◂ Contraction

An Example:

Consider the following simple FBSDE:

$$\begin{cases} dX_t = Y_t dt + dW_t, & X_t = x \\ dY_t = -X_t dt + Z_t dW_t, & Y_T = -X_T \end{cases} \tag{40}$$

- Suppose that (40) has an adapted solutions (X, Y, Z)
- letting $x_t = EX_t$, $y_t = EY_t$ one has

$$\begin{cases} dx_t = y_t dt, & x_0 = x \\ dy_t = -x_t dt, & y_T = -x_T \end{cases}$$

- Solving, $\dot{x}_T + x_T = x(\cos T - \sin T) + C(\cos T + \sin T)$.
- If $T = k\pi + \frac{3\pi}{4}$, then $0 = y_T + x_T = \sqrt{2}x \iff x = 0(!)$.

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- Solving, $\dot{x}_T + x_T = x(\cos T - \sin T) + C(\cos T + \sin T)$.
- If $T = k\pi + \frac{3\pi}{4}$, then $0 = y_T + x_T = \sqrt{2}x \Longleftrightarrow x = 0(!)$.

Warning

The example shows that an FBSDE is **not** always solvable over an arbitrary duration!

A Simple Case

Consider a simple FBSDE:

$$\begin{aligned} dX_t &= b(t, X_t, Z_t)dt + \sigma(Z_t)dW_t \\ dY_t &= h(t, X_t, Y_t)dt + Z_t dW_t, \quad t \in [0, T], \\ X_0 &= x, Y_T = g(X_T). \end{aligned}$$

Assume that

- b and h are Lipschitz in (X, Y, Z) with constant L,
- σ is Lipschitz in z with constant L_1,
- g is Lipschitz in x with constant L_0

Define
$\|(Y, Z)\|_{\overline{\mathscr{N}}[0,T]} \stackrel{\triangle}{=} \sup_{t\in[0,T]} \left\{E|Y(t)|^2 + E\int_t^T |Z(s)|^2 ds\right\}^{1/2}$, and
let $\overline{\mathscr{N}}[0, T]$ be the completion of $\mathscr{N}[0, T]$ in L^2.

For a given $(Y, Z) \in \overline{\mathscr{N}}[0, T]$, define $\Gamma(Y, Z) = (\overline{Y}, \overline{Z})$ as follows. First solve an FSDE for X:

$$\begin{cases} dX_t = b(t, X_t, Z_t)dt + \sigma(Z_t)dW_t, & t \in [0, T], \\ X_0 = x. \end{cases}$$

and then solve the BSDE

$$\begin{cases} d\overline{Y}_t = h(Y_t, Z_t)dt + \overline{Z}_t dW_t, & t \in [0, T], \\ \overline{Y}_T = g(X_T). \end{cases}$$

We shall see when Γ could be a contraction mapping.

So take $(Y^i, Z^i) \in \overline{\mathscr{N}}[0, T]$, $i = 1, 2$, and denote X^i and $(\overline{Y}^i, \overline{Z}^i)$ be the corresponding solutions above. Denote $\Delta\xi = X^1 - X^2$, $\xi = X, Y, Z, \overline{Y}, \overline{Z}$.

A Simple Case

Applying Itô:

$$
\begin{aligned}
E|\Delta X_t|^2 &\le \mathbb{E}\int_0^t \Big\{2L|\Delta X_s|\Big(|\Delta X_s|+|\Delta Z_s|\Big)+L_0^2|\Delta Z_s|^2 ds\\
&\le \mathbb{E}\int_0^t \Big\{C_\varepsilon\Big(|\Delta X_s|^2+|\Delta Y_s|^2\Big)+(L_0^2+\varepsilon)|\Delta Z_s|^2\Big\}ds.
\end{aligned}
$$

$$
\Longrightarrow \mathbb{E}|\Delta X_t|^2 \le e^{C_\varepsilon T}\mathbb{E}\int_0^T \Big\{C_\varepsilon|\Delta Y_s|^2+(L_0^2+\varepsilon)|\Delta Z_s|^2\Big\}ds.
$$

Similarly one has

$$
\begin{aligned}
&\mathbb{E}|\Delta \overline{Y}_t|^2+\mathbb{E}\int_t^T |\Delta \overline{Z}_s|^2 ds \le e^{C_\varepsilon T}\Big\{\widetilde{C}_\varepsilon \mathbb{E}\int_0^T |\Delta Y_s|^2 ds\\
&+[\varepsilon+(L_1^2+T)(L_0^2+\varepsilon)e^{C_\varepsilon T}]\mathbb{E}\int_0^T |\Delta Z_s|^2 ds\Big\}\\
\le\ & e^{C_\varepsilon T}[\widetilde{C}_\varepsilon T+\varepsilon+(L_1^2+T)(L_0^2+\varepsilon)e^{C_\varepsilon T}]\|(\Delta Y,\Delta Z)\|^2_{\mathscr{N}[0,T]},
\end{aligned}
$$

By choosing $\varepsilon > 0$ small enough then choosing $T > 0$ small enough, we obtain

$$\|(\Delta\overline{Y}, \Delta\overline{Z}\|_{\overline{\mathscr{N}}[0,T]} \leq \alpha\|(\Delta Y, \Delta Z)\|_{\overline{\mathscr{N}}[0,T]},$$

for some $0 < \alpha < 1$, whenever $L_0 L_1 < 1$.

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for some $0 < \alpha < 1$, whenever $L_0 L_1 < 1$.
Namely, the mapping Γ is contraction if

- T small;
- $L_0 L_1 < 1$.

This method was used by Antonelli ('92), Pardoux-Tang ('96), Cvitanic-Ma ('98)... A more general version can be found in Ma-Yong (LMN, 1999). Consider the FBSDE (39).

Basic Assumptions:

(A1) b, h, and σ are continuous, $\mathbf{F}$-adapted random fields with linear growth in (x, y, z), and $\exists k_1, k_2 \geq 0$ and $\gamma \in \mathbb{R}$ s.t. for all (t, ω) and $\theta \stackrel{\triangle}{=} (x, y, z), \theta_i \stackrel{\triangle}{=} (x_i, y_i, z_i)$, and $\theta_0 \stackrel{\triangle}{=} (x, y)$,

$$
\begin{aligned}
&|b(\omega, t, \theta_1) - b(\omega, t, \theta_2)| \leq K|\theta_1 - \theta_2|;\\
&\langle\, h(\omega, t, x, y_1, z) - h(\omega, t, x, y_2, z), y_1 - y_2 \,\rangle \leq \gamma|y_1 - y_2|^2;\\
&|h(\omega, t, x_1, y, z_1) - h(\omega, t, x_2, y, z_2)|\\
&\qquad\qquad\qquad\qquad \leq K(|x_1 - x_2| + \|z_1 - z_2\|);\\
&|\sigma(\omega, t, \theta_1) - \sigma(\omega, t, \theta_2)|^2 \leq K^2|\theta_0^1 - \theta_0^2|^2 + k_1^2|z_1 - z_2|^2;\\
&|g(\omega, x_1) - g(\omega, x_2)| \leq k_2|x_1 - x_2|.
\end{aligned}
$$

Method of Contraction Mapping

Denote, for any constants C_1, C_2, C_3, $C_4 > 0$, and $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\lambda_1 &= \lambda - 2K - K(2 + C_1^{-1} + C_2^{-1}) - K^2; \\
\lambda_2 &= -\lambda - 2\gamma - K(C_3^{-1} - C_4^{-1}), \\
\mu(\alpha, T) &= K(C_1 + K)B(\lambda_2, T) + \frac{A(\lambda_2, T)}{\alpha}(KC_2 + k_1^2),
\end{aligned}$$

where $A(\lambda, t) = e^{-(\lambda \wedge 0)t}$ and $B(\lambda, t) = \dfrac{1 - e^{-\lambda t}}{\lambda}$.

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Theorem

Assume (A1), and that $0 \le k_1 k_2 < 1$. Assume also that one of the following holds for some constants C_1—C_3, and $C_4 = \frac{1-\alpha_0}{K}$:

- $k_2 = 0$; $\exists \alpha_0 \in (0, 1)$ *such that* $\mu(\alpha_0, T)KC_3 < \lambda_1$;
- $k_2 > 0$; $\lambda_1 = \frac{KC_3}{k_2^2}$; $\exists \alpha_0 \in (0, 1)$ *such that* $\mu(\alpha_0^2, T)k_2^2 < 1$.

Then the FBSDE (39) has a unique adapted solution over $[0, T]$.

Method of Contraction Mapping

Note:

The "compatibility condition": $0 \leq k_1 k_2 < 1$ is essential!

- If $0 \leq k_1 k_2 < 1$, then there exists $T_0 > 0$ such that for all $0 < T \leq T_0$, the FBSDE (39) is uniquely solvable on $[0, T]$.

Method of Contraction Mapping

Note:

The "compatibility condition": $0 \le k_1 k_2 < 1$ is essential!

- If $0 \le k_1 k_2 < 1$, then there exists $T_0 > 0$ such that for all $0 < T \le T_0$, the FBSDE (39) is uniquely solvable on $[0, T]$.
- This condition is indispensable! For example, consider

$$\begin{cases} X_t = \displaystyle\int_0^t Z_s dW_s; \\ Y_t = (X_T + \xi) - \displaystyle\int_t^T Z_s dW_s, \end{cases} \tag{41}$$

where ξ is an $\mathscr{F}_T$-measurable, L^2 random variable. This FBSDE has **no** adapted solution on any interval $[0, T]$! Indeed, if (X, Y, Z) were an adapted solution, let $\eta = Y - X$, then $\eta_t \equiv \xi$, $\forall t \in [0, T]$. The $\mathbb{F}$-adaptedness of η then leads to that ξ is a constant(!). But this is obviously not necessarily ture.

Denote, for $t \in [0, T)$,

- $\mathbf{H}(t, T) = L^2_{\mathbb{F}}(t, T; \mathbb{R})$,
- $\mathbf{H}^{\mathbf{c}}(t, T)$ — elements in $\mathbf{H}(t, T)$, with continuous paths
- $\forall \lambda \in \mathbb{R}$, $\xi \in \mathbf{H}(t, T)$, define $\|\xi\|^2_{t,\lambda} \stackrel{\triangle}{=} \mathbb{E} \int_t^T e^{-\lambda s} |\xi(s)|^2 ds$.

  $\Longrightarrow \mathbf{H}_\lambda(t, T) \stackrel{\triangle}{=} \{\xi \in \mathbf{H}(t, T) : \|\xi\|_{t,\lambda} < \infty\} = \mathbf{H}(t, T)$
- For $\xi \in \mathbf{H}^{\mathbf{c}}(t, T)$, $\lambda \in \mathbb{R}$, and $\beta > 0$, define

$$|\!|\!| \xi |\!|\!|_{t,\lambda,\beta} \stackrel{\triangle}{=} e^{-\lambda T} \mathbb{E}|\xi_T|^2 + \beta \|\xi\|^2_{t,\lambda},$$

and let $\mathbf{H}_{\lambda,\beta}(t, T)$ be the completion of $\mathbf{H}^{\mathbf{c}}(t, T)$ under norm $|\!|\!| \cdot |\!|\!|_{t,\lambda,\beta}$. Then for any λ and β, $\mathbf{H}_{\lambda,\beta}(t, T)$ is a Banach space.

The Solution Mapping:

Define $\Gamma : \mathbf{H^c} \mapsto \mathbf{H^c}$ defined as follows: for fixed $x \in \mathbb{R}^n$, let $\overline{X} \stackrel{\triangle}{=} \Gamma(X)$ be the solution to the FSDE:

$$\overline{X}_t = x + \int_0^t b(s, \overline{X}_s, Y_s, Z_s)ds + \int_0^t \sigma(s, \overline{X}_s, Y_s, Z_s)dW_s, \quad (42)$$

where (Y, Z) ia the solution to the BSDE:

$$Y_t = g(X_T) + \int_t^T h(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \quad (43)$$

Need to show that Γ is a contraction on $\mathbf{H}_{\lambda, \bar{\lambda}_1}$ for some $\bar{\lambda}_1$.

Let $X^1, X^2 \in \mathbf{H^c}$; and let $\overline{X}^i$ and (Y^i, Z^i), $i = 1, 2$, be the corresponding solutions to (42) and (43), respectively. Denote $\Delta\xi = \xi^1 - \xi^2$, for $\xi = X, Y, Z$. Then one shows that (with $C_4 = \frac{1-\alpha}{K}$)

$$e^{-\lambda T} E|\Delta\overline{X}_T|^2 + \bar{\lambda}_1 \|\Delta\overline{X}\|_\lambda^2 \le \mu(\alpha, T)\{k_2^2 e^{-\lambda T} E|\Delta X_T|^2 + KC_3\|\Delta X\|_\lambda^2\}. \tag{44}$$

where

$$\mu(\alpha, T) \stackrel{\triangle}{=} K(C_1 + K)B(\bar{\lambda}_2, T) + \frac{A(\bar{\lambda}_2, T)}{\alpha}(KC_2 + k_1^2);$$

and

$$\begin{cases} \bar{\lambda}_1 = \lambda - K(2 + C_1^{-1} + C_2^{-1}) - K^2; \\ \bar{\lambda}_2 = -\lambda - 2\gamma - K(C_3^{-1} + C_4^{-1}). \end{cases} \tag{45}$$

Fix $C_4 = \frac{1-\alpha_0^2}{K}$.

(i) If $k_2 = 0$, then (44) leads to

$$\|\Delta\overline{X}\|_\lambda^2 \le \frac{\mu(\alpha, T)KC_3}{\bar{\lambda}_1}\|\Delta X\|_\lambda^2.$$

Find $\alpha \in (0,1)$ so that $\mu(\alpha, T)KC_3 < 1 \quad \Longrightarrow \Gamma$ is a contraction mapping on $(H, \|\cdot\|_\lambda)$.

(ii) If $k_2 > 0$, then we can solve λ from (45) and $\bar{\lambda}_1 = KC_3/k_2^2$, (44) gives

$$|\!|\Delta\overline{X}|\!|^2_{\lambda^0,\bar{\lambda}_1} \le \mu(\alpha_0^2, T)k_2^2 |\!|\Delta X|\!|^2_{\lambda^0,\bar{\lambda}_1},$$

Let C_i, $i = 1,2,3$ and $\alpha_0 \in (k_1k_2, 1)$ be such that $\mu(\alpha_0^2, T)k_2^2 < 1 \Longrightarrow\ \Gamma$ is a contraction on $\mathbf{H}_{\lambda,\bar{\lambda}_1}$. ■

Method of Stochastic Control

Purpose: *Solve FBSDEs over arbitrary interval* $[0, T]$*!*

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Consider the stochastic control problem with

- **State equations:**

$$\begin{cases} X_t = x + \int_s^t b(r, X_r, Y_r, Z_r)dr + \int_s^t \sigma(r, X_r, Y_r, Z_r)dW_r, \\ Y_t = y - \int_s^t h(r, X_r, Y_r, Z_r)dr - \int_s^t \widehat{\sigma}(r, X_r, Y_r, Z_r)dW_r, \end{cases}$$

  with Z being the **control process**, and

- **Cost functional**

$$J_T(s, x, y; Z) \stackrel{\triangle}{=} E_{s,x,y} \left|g(X_T) - Y_T\right|^2;$$

- **Value function**

$$V_T(s, x, y) \stackrel{\triangle}{=} \inf_Z J_T(s, x, y; Z).$$

Objective:

$\forall x \in \mathbb{R}^n$, $\forall T > 0$, find $y \in \mathbb{R}^m$ and $Z^* \in L^2_{\mathbb{F}}([0, T]; \mathbb{R}^{m\times d})$, such that

$$J_T(0, x, y; Z^*) \overset{(1)}{=} V_T(0, x, y) \overset{(2)}{=} 0.$$

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Remark

- $(1) =$ Existence of optimal control (relaxed control);
- (Hard!) Note that $V_T(s,x,y)$ is only a *viscosity solution* of a fully nonlinear PDE (Hamilton-Jacobi-Bellman equation). If we define the "Nodal set" of V_T as

$$\mathscr{N}(V_T) \overset{\triangle}{=} \{(t,x,y) : V_T(t,x,y) = 0\},$$

  Then (2) amounts to saying that

$$\forall x \in \mathbb{R}^n, T > 0, \ \mathscr{N}(V_T) \cap \{(0,x,y) : y \in \mathbb{R}^m\} \neq \emptyset.$$

A Worked-out Case (Ma-Yong, 1993)

Assume that b, h, and σ satisfies some standard conditions (e.g., Lipschitz, linear growth, ...), and that

- σ and h are independent of Z ($k_1 = 0!$)
- σ is non-degenerate. I.e., , $\exists \mu > 0$ such that $\sigma\sigma^T \geq \mu I$.

Then, it holds that

$$\mathscr{N}(V_T) = \{(t, x, \theta(t, x)) | (t, x) \in [0, T] \times \mathbf{R}^n\},$$

where θ is the *classical solution* of the following PDE:

$$\begin{cases} \theta_t + \frac{1}{2}\mathrm{tr}\left\{\sigma(x,\theta)\sigma^T(x,\theta)\theta_{xx}\right\} + \langle b(x,\theta), \theta_x\rangle + h(x,\theta) = 0; \\ \theta(T, x) = g(x). \end{cases} \tag{46}$$

In other words, $V_T(s, x, \theta(s, x)) \equiv 0$, $\forall (s, x)$; and if we let $y = \theta(0, x)$, then $V_T(0, x, y) = 0$.

A Deeper Thinking...

In light of the previous theorem, it is natural to conjecture that $Y_t = \theta(t, X_t)$ for all $t \in [0, T]$, for some function θ.

Question:

Is there a direct method to figure out the function θ?

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Question:

Is there a direct method to figure out the function θ?

A Heuristic Argument:

- Assume θ is "smooth" and apply Itô's formula:

$$\begin{aligned} dY_t &= d\theta(t, X_t) \\ &= \Big\{\theta_t(t, X_t) + \langle \theta_x(t, X_t), b(t, X_t, \theta(t, X_t), Z_t)\rangle \\ &\quad + \frac{1}{2}\mathrm{tr}\left[\theta_{xx}(t, X_t)\sigma\sigma^T(t, X_t, \theta(t, X_t))\right]\Big\}dt \\ &\quad + \langle \theta_x(t, X_t), \sigma(t, X_t, \theta(t, X_t), Z_t)dW_t\rangle, \end{aligned}$$

- Comparing this to the BSDE in (39)!

Step 1: Find a "smooth" function $z = z(t, x, y, p)$ so that

$$p\sigma(t, x, y, z(t, x, y, p)) + z(t, x, y, p) = 0, \qquad (47)$$

Step 1: Find a "smooth" function $z = z(t, x, y, p)$ so that

$$p\sigma(t, x, y, z(t, x, y, p)) + z(t, x, y, p) = 0, \tag{47}$$

Step 2: Using z above, solve the quasilinear parabolic system for $\theta(t, x)$:

$$\begin{cases} 0 = \theta_t + \frac{1}{2}\mathrm{tr}\left[\theta_{xx}\sigma\sigma^T(t, x, \theta)\right] + \langle\, b(\cdot, z(\cdot, \theta_x)), \theta_x\,\rangle \\ \qquad + h(t, x, \theta, z(t, x, \theta, \theta_x)), \\ \theta(T, x) = g(x), \qquad x \in \mathbb{R}^n \end{cases} \tag{48}$$

Step 3: Setting

$$\begin{cases} \tilde{b}(t,x) = b(t,x,\theta(t,x), z(t,x,\theta(t,x),\theta_x(t,x))) \\ \\ \tilde{\sigma}(t,x) = \sigma(t,x,\theta(t,x)), \end{cases} \tag{49}$$

Solve the FSDE:

$$X_t = x + \int_0^t \tilde{b}(s,X_s)ds + \int_0^t \tilde{\sigma}(s,X_s)dW_s. \tag{50}$$

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$\Longrightarrow$ **DONE!**

Theorem (Ma-Protter-Yong, '94)

Assume that $d = n$; and that

- σ *is independent of* z;
- b, σ, h, *and* g *are smooth, and their first order derivatives in* (x, y, z) *are bounded by a common constant* $L > 0$;
- $\exists$ *continuous function* $\nu > 0$ *and constant* $\mu > 0$ *such that*

$$\begin{cases} \nu(|y|) \le \sigma(t,x,y)\sigma(t,x,y)^T \le \mu I; \\ \\ |b(t,x,0,0)| + |h(t,x,0,z)| \le \mu \end{cases}$$

- g *is bounded in* $C^{2+\alpha}(\mathbf{R}^n)$ *for some* $\alpha \in (0,1)$.

Then, the quasilinear PDE (48) admits a unique classical solution θ *which has uniformly bounded derivatives* θ_x *and* θ_{xx}; *and the FBSDE (39) has a unique adapted solution, constructed via steps (49)—(51).*

More generally....

Theorem

Assume that (47) admits a unique solution z, and (48) admits a classical solution θ with bounded θ_x and θ_{xx}. Assume that z, b, σ are Lipschitz with linear growth in (x, y, p), uniformly in (t, x, y) and locally uniformly in p. Then the processes defined in (51) give an adapted solution to the FBSDE (39).

Moreover, if h is also uniform Lipschitz in (x, y, z), σ is bound, and there exists a constant $\beta > 0$ such that

$$|(\sigma(s,x,y,z) - \sigma(s,x,y,z'))^T \theta_x^k(s,x)| \leq \beta|z - z'|, \qquad (52)$$

for all (s, x, y, z), then the adapted solution to (39) is unique.

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Assume that (47) admits a unique solution z, and (48) admits a classical solution θ with bounded θ_x and θ_{xx}. Assume that z, b, σ are Lipschitz with linear growth in (x, y, p), uniformly in (t, x, y) and locally uniformly in p. Then the processes defined in (51) give an adapted solution to the FBSDE (39).
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Remark

The dependence of σ on z will complicate both the existence and the uniqueness of the solution to an FBSDE (recall FBSDE (41))!

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- explicit solution (especially the component Z!)
- numerically "feasible".

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Method of Continuation

Benefits of Previous Methods:

- explicit solution (especially the component Z!)
- numerically "feasible".

Shortcomings of Previous Methods

- non-degeneracy of σ
- high regularity of the coefficients
- all coefficients have to be deterministic (PDE)

The purpose of the Method of continuation is to replace the smoothness conditions on the coefficients by some structural condition. E.g., the "*Monotonicity Conditions*".

Still consider the FBSDE (39), and allow even the coefficients to be random(!).

The Monotonicity condition

The coefficients (h, b, σ, g) satisfy the following *monotonicity conditions*: $\exists \beta > 0$ such that

$$\begin{cases} \langle\, U(t,\theta_1) - U(t,\theta_2), \theta_1 - \theta_2 \,\rangle \le -\beta \|\theta_1 - \theta_2\|^2; \\ \\ \langle\, g(x_1) - g(x_2), x_1 - x_2 \,\rangle \ge \beta |x_1 - x_2|^2, \end{cases} \tag{53}$$

where $\theta = (x, y, z)$, and $U = (h, b, \sigma)$.

Let $(h^i, b^i, \sigma^i, g^i)$, $i = 1, 2$ be two sets of coefficients. For any $(h^0, b^0, \sigma^0) \in L^2_{\mathbf{F}}(\Omega \times [0, T])$, $g_0 \in L^2_{\mathscr{F}_T}(\Omega)$, and $\alpha \in (0, 1)$, consider the FBSDE $(\alpha; h^0, b^0, \sigma^0, g^0)$:

$$\begin{cases} dX^\alpha_t = \{(1-\alpha)b^1(t, \Theta^\alpha_t) + \alpha b^2(t, \Theta^\alpha_t) + b^0_t\}dt \\ \qquad\qquad + \{(1-\alpha)\sigma^1(t, \Theta^\alpha_t) + \alpha\sigma^2(t, \Theta^\alpha_t) + \sigma^0_t\}dW_t \\ dY^\alpha_t = \{(1-\alpha)h^1(t, \Theta^\alpha_t) + \alpha h^2(t, \Theta^\alpha_t) + h^0_t\}dt \\ \qquad\qquad + Z^\alpha_t dW_t \\ X^\alpha_0 = x, \quad Y^\alpha_T = (1-\alpha)g^1 + \alpha g^2 + g^0 \end{cases} \tag{54}$$

where $\Theta^\alpha = (X^\alpha, Y^\alpha, Z^\alpha)$.

The Continuation Step:

Show that, there exists an $\varepsilon_0 > 0$, such that for any $\alpha \in [0,1)$,

- *If FBSDE($\alpha; h^0, b^0, \sigma^0, g^0$) is solvable for all $(h^0, b^0, \sigma^0, g^0)$, then FBSDE($\alpha + \varepsilon_0; h^0, b^0, \sigma^0, g^0$) is solvable for all $(h^0, b^0, \sigma^0, g^0)$.*
- *Consequently, the solvability of FBSDE($h^1, b^1, \sigma^1; g^1$) ($\alpha = 0$) will imply the solvability of any FBSDE($h^2, b^2, \sigma^2; g^2$) ($\alpha = 1$) as long as the coefficients $(h^2, b^2, \sigma^2; g^2)$ verify the continuation step!*

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Theorem (Hu-Peng, '96)

Under the monotonicity condition, the FBSDE (39) admits a unique adapted solution.

Monotonicity condition vs. Four Step Scheme

Consider the following *decoupled* FBSDE:

$$\begin{cases} dX_t = X_t dt + dW_t, & X_0 = x; \\ dY_t = X_t dt + Z_t dW_t, & Y_T = X_T. \end{cases}$$

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The monotonicity condition **does not** hold in this case:

$$\begin{aligned} \langle U(\theta_1) - U(\theta_2), \theta_1 - \theta_2 \rangle &= |x_1 - x_2|^2 + \langle x_1 - x_2, y_1 - y_2 \rangle \\ &\le C\|\theta_1 - \theta_2\|^2. \end{aligned}$$

However, the (quasilinear) PDE

$$\begin{cases} 0 = \theta_t + \frac{1}{2}\theta_{xx} + x\theta_x - x, \\ \theta(T, x) = x, \qquad x \in \mathbb{R}^n \end{cases}$$

has a unique solution $\theta(t,x) \equiv x$! That is, $Y_t \equiv X_t$ and $Z_t \equiv 1$ solves the FBSDE (uniquely)!

An Extended form of Four Step Scheme

Restrictions of the Method presented:

- Contraction Mapping — Small duration
- Four Step Scheme — High regularity of the coefficients(thus exclusively Markovian)
- Continuation — Monotonicity of the coefficients (could not even cover the simple Lipschitz case!)

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Question:

Can we improve the methods above by combining them?

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Question:

Can we improve the methods above by combining them?

Answer:

Yes! — F. Delarue (2001) combined the method of Contraction mapping with the Four Step Scheme, and extended latter to the case when coefficients need only be Lipschitz!

Consider the FBSDE:

$$\begin{cases} X_t = \xi + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s)dW_s \\ Y_t = g(X_T) + \int_t^T h(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \end{cases} \tag{55}$$

Main Assumptions

- W is an $\mathbb{F}$-BM, but $\mathbb{F}^W \subset \mathbb{F}$ (denote $\mathscr{F}_t^0 = \mathscr{F}_0 \vee \mathscr{F}_t^W$, $\forall t$);
- All coefficients are deterministic, and are of linear growth;
- b is uniformly Lipschitz in (y, z), monotone in x;
- f is uniformly Lipschitz in (x, z), monotone in y;
- g is uniformly Lipschitz in x;
- σ is uniformly Lipschitz in (x, y);

An Extended form of Four Step Scheme

Theorm (Existence and uniqueness in small time duration)

Assume that the main assumptions are all in force. Then

- For every $\xi \in L^2(\mathscr{F}_0; \mathbb{R}^d)$, the solution (X, Y, Z) to FBSDE(55) satisfies
  - (X, Y) has continuous paths;
  - $\mathbb{E}\Big\{ \sup_{t \in [0,T]} |X_t|^2 + \sup_{t \in [0,T]} |Y_t|^2 \Big\} < \infty.$
- $\exists T_K^0 > 0$, depending only on the common Lipschitz constant of the coefficients K, such that for every $T < T_K^0$ and for every $\xi \in L^2(\mathscr{F}_0; \mathbb{R}^d)$, the FBSDE has a unique solution.

Note: The relaxation of the filtration is possible because of a martingale representation theorem by Jacod-Shiryaev.

A slightly modified form of the small duration case is to consider the following FBSDE for $0 \le t \le s \le T$:

$$\begin{cases} X_s = \xi + \int_t^s b(r, X_r, Y_r, Z_r)dr + \int_t^s \sigma(r, X_r, Y_r)dW_r \\ Y_s = g(X_T) + \int_s^T h(r, X_r, Y_r, Z_r)ds - \int_s^T Z_r dW_r. \end{cases} \tag{56}$$

Then for $T \le T_K^0$, there exists a unique solution to (56). Denote the solution by $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$, for $s \in [t, T]$, and extend it to $[0, T]$ by setting

$$X_s^{t,x} = x, \quad Y_s^{t,x} = Y_t^{t,x}, \quad Z_s^{t,x} = 0, \qquad s \in [0, t].$$

We define the (deterministic) mapping $(t, x) \mapsto Y_t^{t,x}$ by $\theta(t, x)$.

First note that for some constants C_1, C_2, $C_3 > 0$, depending only on K, it holds that

$$\mathbb{E}\Big\{\sup_{0\le s\le T}|X_s^{t,x}|^2 + \sup_{0\le s\le T}|Y_s^{t,x}| + \int_0^T |Z_s^{t,x}|^2 ds\Big\} \le C_1(1+|x|^2); \tag{57}$$

$$\mathbb{E}\Big\{\sup_{0\le s\le T}|X_s^{t,x} - X_s^{t',x'}|^2 + \sup_{0\le s\le T}|Y_s^{t,x} - Y_s^{t',x'}| + \int_0^T |Z_s^{t,x} - Z_s^{t,x}|^2 ds\Big\} \le C_2|x-x'|^2 + C_3(1+|x|^2)|t-t'|. \tag{58}$$

Consequently,

- $|\theta(t,x)|^2 \le C_1(1+|x|^2)$;
- $|\theta(t',x') - \theta(t,x)| \le C_2|x-x'|^2 + C_3(1+|x|^2)|t-t'|$
- $\forall t \in [0, T]$, and $\forall \xi \in L^2(\mathscr{F}_t; \mathbb{R}^n)$, $\exists$ a $\mathbb{P}$-null set $N \in \mathscr{F}_0$ s.t.

$$Y_s^{t,\xi}(\omega) = \theta(s, X_s^{t,\xi}(\omega)), \qquad \forall s \in [t, T], \ \forall \omega \notin N.$$

Continuous Dependence on Coefficients (Stability)

Theorem

Assume that the main assumptions are all in force, and assume that $T \le T_K^0$. Let $(b_n, h_n, g_n, \sigma_n)$ be a family of coefficients satisfying the same assumptions as (b, h, g, σ) with the same Lipschitz constants, such that $(b_n, h_n, g_n, \sigma_n) \to (b, h, g, \sigma)$ pointwisely. Then

$$\mathbb{E}\Big\{ \sup_{0\le s\le T} |X_s^{n,0,\xi} - X_s^{0,\xi}|^2 + \sup_{0\le s\le T} |Y_s^{n,0,\xi} - Y_s^{0,\xi'}| + \int_0^T |Z_s^{n,0,\xi} - Z_s^{0,\xi}|^2 ds \Big\} \to 0, \qquad \text{as } n \to \infty.$$

Consequently, $\theta_n(t,x) \to \theta(t,x)$ uniformly on compacta in $[0,T] \times \mathbb{R}^d$.

Some Important Facts

Recall the quasi-linear PDE in Four Step Scheme

$$
\begin{cases}
0 = \theta_t + \frac{1}{2} tr \left[\theta_{xx}\sigma\sigma^T(t,x,\theta) \right] + \langle\, b(\cdot,\theta,\theta_x\sigma(\cdot,\theta)), \theta_x \,\rangle \\
\qquad + h(t,x,\theta,\theta_x\sigma(t,x,\theta)), \\
\theta(T,x) = g(x), \qquad x \in \mathbb{R}^d
\end{cases}
\tag{59}
$$

We know if

- all coefficients are in $\mathbb{C}_b^\infty$, and
- $\xi^T(\sigma\sigma^T)\xi \geq c|\xi|^2$, $\forall \xi \in \mathbb{R}^d$, for some $c > 0$.

Then the PDE (59) admits a unique bounded solution $\theta \in \mathbb{C}^{1,2}$ with bounded first and second order derivatives.

The Solution Scheme

On the other hand, if θ is a (smooth) solution to the PDE (59), then we define

$$\begin{aligned}\widetilde{b}(t,x) &\stackrel{\triangle}{=} b(t,x,\theta(t,x),\theta_x(t,x)\sigma(t,x,\theta(t,x))),\\ \widetilde{\sigma}(t,x) &\stackrel{\triangle}{=} \sigma(t,x,\theta(t,x)).\end{aligned}$$

For any $t \in [0,T]$ and $\xi \in L^2(\mathscr{F}_t;\mathbb{R}^d)$, let $X^{t,\xi}$ denote the solution to the forward SDE:

$$X_s = \xi + \int_t^s \widetilde{b}(r,X_r)dr + \int_t^s \widetilde{\sigma}(r,X_r)dW_r, \quad s \in [t,T],$$

and define $Y_s^{t,\xi} = \theta(r, X_s^{t,\xi})$, $Z_s^{t,\xi} = \theta_x(s, X_s^{t,\xi})\sigma(s,X_s,\theta(s,X_s))$. Then, whenever $T - t < T_K^0$, $(X^{t,\xi}, Y^{t,\xi}, Z^{t,\xi})$ should be the the unique solution to the FBSDE(46) on $[t,T]$, starting from ξ.

A Problem:

Under only Lipschitz assumptions, the PDE(59) DOES NOT have smooth solutions in general!

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The Solution Scheme:

- Approximate (b, h, g, σ) by $(b_n, h_n, g_n, \sigma_n) \in \mathbb{C}^\infty$
- For each n, find $\theta^n \in \mathbb{C}^{1,2}$ to the PDE (59), with bounded first and second order derivatives, such that

$$|\theta^n(t,x)| \le C_1, \; |\theta^n(t,x) - \theta^n(t',x')| \le C_2|x-x'| + C_3|t-t'|^{1/2}.$$

- By "*Continuous Dependence*": $\theta^n \to \theta$, $|\theta(t,x)| \le C_1$, and

$$|\theta(t,x) - \theta(t',x')| \le C_2|x-x'| + C_3|t-t'|^{1/2}.$$

- Construct a "global" solution via θ.

The Solution Scheme

Note:

The function θ may not be obtained by a simple Arzela-Ascoli argument, because the lack of "equi-continuity" in the variable t and the uniform bound of the second derivatives.

The following "running-down" induction defines the function θ on $[0, T] \times \mathbb{R}^d$:

- Partition the interval $[0, T]$ into $0 = t_0 < t_1 < \cdots < t_N = T$, s.t. $t_{i+1} - t_i = T/N < T_K^0$.
- Consider the following FBSDEs on $[t, t_{i+1}]$, $i = N-1, \cdots, 1$:

$$\begin{aligned} X_s &= \xi + \int_t^s b(r, X_r, Y_r, Z_r)dr + \int_t^s \sigma(r, X_r, Y_r)dW_r \\ Y_s &= \theta(t_{i+1}, X_{t_{i+1}}) + \int_s^{t_{i+1}} h(r, X_r, Y_r, Z_r)ds - \int_s^{t_{i+1}} Z_r dW_r. \end{aligned}$$

- Then $\theta(t, x) = Y_t^{t,x,i}$, for $t \in [t_i, t_{i+1}]$ is the desired function.

The Solution Scheme

Once the "decoupling machine" θ is defined, then the following "running-up" induction gives the desired solution on $[0, T]$:

- For $0 \le s \le t_1$, let $(X^{(0)}, Y^{(0)}, Z^{(0)})$ solve the FBSDE:

$$
\begin{aligned}
X_s^{(0)} &= x + \int_0^s b(r, \Theta_r^{(0)})dr + \int_t^s \sigma(r, X_r^{(0)}, Y_r^{(0)})dW_r \\
Y_s^{(0)} &= \theta(t_1, X_{t_1}^{(0)}) + \int_s^{t_{i+1}} h(r, \Theta^{(0)})ds - \int_s^{t_1} Z_r^{(0)} dW_r.
\end{aligned}
$$

- For $t_{k-1} \le s \le t_k$, let $(X^{(k)}, Y^{(k)}, Z^{(k)})$ solve the FBSDE:

$$
\begin{aligned}
X_s^{(k)} &= X_{t_{k-1}}^{(k-1)} + \int_{t_{k-1}}^s b(r, \Theta_r^{(k)})dr + \int_{t_{k-1}}^s \sigma(r, X_r^{(k)}, Y_r^{(k)})dW_r \\
Y_s^{(k)} &= \theta(t_k, X_{t_k}^{(k)}) + \int_s^{t_k} h(r, \Theta^{(k)})ds - \int_s^{t_k} Z_r^{(k)} dW_r.
\end{aligned}
$$

- Then, to complete the "patch-up", one needs only check: $X_{t_k}^{(k-1)} = X_{t_k}^{(k)}$, $Y_{t_k}^{(k)} = \theta(t, X_{t_k}^{(k)}) = \theta(t, X_{t_k}^{(k-1)}) = Y_{t_k}^{(k-1)}$!

5. Some Important facts

Denote $X^{t,x}$ to be the solution to an SDE on $[t, T]$:

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r.$$

Then under appropriate regularity conditions the function

$$u(t,x) \triangleq E_{t,x}\left\{g(X_T)e^{\int_t^T c(X_s)ds} + \int_t^T e^{\int_t^r c(X_s)ds} f(r, X_r)dr\right\}$$

is a (probablistic) solution to the (linear) PDE:

$$\begin{cases} u_t + \frac{1}{2}tr\left[u_{xx}\sigma\sigma^T(x)\right] + \langle\, b(x), u_x\,\rangle + c(x)u + f(t,x) = 0, \\ u(T,x) = g(x), \qquad x \in \mathbb{R}^n. \end{cases} \tag{60}$$

Feynman-Kac formula (the linear case)

Denote $X^{t,x}$ to be the solution to an SDE on $[t, T]$:

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r.$$

Then under appropriate regularity conditions the function

$$u(t,x) \triangleq E_{t,x}\left\{ g(X_T)e^{\int_t^T c(X_s)ds} + \int_t^T e^{\int_t^r c(X_s)ds} f(r,X_r)dr \right\}$$

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Question:

Is it possible to extend the Feynman-Kac formula to the case where PDE obove is nonlinear in u (or even u_x)?

Non-linear Feynman-Kac Formula via BSDEs

Consider FBSDEs defined on the subinterval $[t, T] \subseteq [0, T]$:

$$\begin{cases} X_s = x + \int_t^s b(X_r)dr + \int_t^s \sigma(X_r)dW_r; \\ Y_s = g(X_T) + \int_s^T f(r, X_r, Y_r, Z_r)dr - \int_s^T Z_r dW_r, \end{cases} \tag{61}$$

where $s \in [t, T]$ and the coefficients are assumed to be only continuous and uniformly Lipschitz in the spatial variables (x, y, z).

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where $s \in [t, T]$ and the coefficients are assumed to be only continuous and uniformly Lipschitz in the spatial variables (x, y, z). Denote the solution by $(X^{t,x}, Y^{t,x}, Z^{t,x})$. Then,

- for any $s \in [t, T]$, $Y_s^{t,x}$ is $\mathscr{F}_s^t$-measurable, where $\mathscr{F}_s^t = \sigma\{W_s - W_t;\ t \le s \le T\}$;
- in particular, $u(t, x) \stackrel{\triangle}{=} Y_t^{t,x}$ is a deterministic function (Blumenthal $0 - 1$ law!);

Theorem (Pardoux-Peng, '92; Ma-Protter-Yong '94)

FBMP definition

Assume b, σ, f, and g are Lipschitz, then

- *$u(\cdot,\cdot)$ is continuous, Hölder-1/2 in t and Lipschitz in x;*
- *u is the unique viscosity solution of the quasilinear PDE:*

$$\begin{cases} u_t + \frac{1}{2}\mathrm{tr}\left[u_{xx}\sigma\sigma^T\right] + \langle\, b, u_x \,\rangle + f(t,x,u,\sigma^T u_x) = 0, \\ u(T,x) = g(x), \qquad x \in \mathbb{R}^n. \end{cases} \tag{62}$$

- *Further, under regularity conditions on the coefficients,*

$$u(t,x) = E_{t,x}\left\{ g(X_T) + \int_t^T f(r, X_r, Y_r, Z_r)dr \right\} \tag{63}$$

is a (classical) solution to (62), where (X,Y,Z) solves (61).

- *and the following representation holds*

$$u_x(s, X_s) = Z_s\sigma^{-1}(s, X_s), \quad s \in [t,T], \quad P\text{-a.s.} \tag{64}$$

Possible generalizations

How far can the representations (63) and (64) go?

Possible generalizations

How far can the representations (63) and (64) go?

For example, one may ask:

- What are the minimum conditions on f and g under which (63) and (64) *both* hold (e.g., $g(x) = (x - K)^+$ in finance applications — only Lipschitz!)?
- Will Z always be continuous in light of (64)?
- What if b, σ, f, g are random (I.e., can Four Step Scheme be applied for FBSDEs with *random* coefficients?);
- Is there a Feynman-Kac type solution to an *Stochastic PDE*?
- In the SPDE case, can one define the notion of "*Stochastic Viscosity Solution*")?
- ... ...

A Quick Analysis:

Assume

- $f \equiv 0$ and
- $g \in C^1$.

Then, by representation: $u(t,x) = E_{t,x}\{g(X_T)\}$,

$\Longrightarrow u_x(t,x) = E_{t,x}\{g'(X_T)\nabla X_T\}$,

where ∇X is the solution to the *variational equation* of X:

$$\nabla X_s = 1 + \int_t^s b'(X_r)\nabla X_r dr + \int_t^s \sigma'(X_r)\nabla X_r dW_r. \qquad (65)$$

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Question:

What if g (or f) is not differentiable? (Again, consider $g(x) = (x-K)^+$ — simply Lipschitz!)

A New Tool: Malliavin Calculus/Skorohod Integrals

Fournié-Lasry-Lebuchoux-Lions-Touzi, '97; Ma-Zhang, '00:

$$D_\tau g(X_T) = g'(X_T) D_\tau X_T = g'(X_T)\nabla X_T (\nabla X_\tau)^{-1}\sigma(X_\tau)$$

$$\begin{aligned}
\Longrightarrow u_x(t,x) &= E_{t,x}\{g'(X_T)\nabla X_T\} \\
&= E_{t,x}\left\{\int_t^T D_\tau g(X_T)\frac{\sigma(X_\tau)^{-1}(\nabla X_\tau)}{T-t}d\tau\right\} \\
&= E_{t,x}\left\{g(X_T)\int_t^T \frac{\sigma(X_\tau)^{-1}(\nabla X_\tau)}{T-t}dW_\tau\right\} \\
&= E_{t,x}\left\{g(X_T)N_T^t\right\}.
\end{aligned}$$

where $N_s^t \triangleq \int_t^s \sigma(X_\tau)^{-1}(\nabla X_\tau)dW_\tau/(T-t)$, $0 \le t \le s \le T$.

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\end{aligned}$$

where $N_s^t \triangleq \int_t^s \sigma(X_\tau)^{-1}(\nabla X_\tau)dW_\tau/(T-t)$, $0 \le t \le s \le T$.

Note:

Derivative of g is NOT necessary for u_x!

Theorem (Ma-Zhang, 2000)

Suppose that f and g are uniformly Lipschitz in (x, y, z). Let

$$v(t,x) = E_{t,x}\left\{ g(X_T)N_T^t + \int_s^T f(r,\Theta_r)N_r^t dr \middle| \mathscr{F}_s^t \right\} \sigma(X_s^{t,x}),$$

for $(t,x) \in [0,T) \times \mathbb{R}^d$, where $\Theta_r = (X_r, Y_r, Z_r)$, and

$$N_r^s \triangleq \frac{1}{r-s}(\nabla X_s)^{-1} \int_s^r \sigma^{-1}(X_\tau)\nabla X_\tau dW_\tau, \quad 0 \le t \le s < r \le T.$$

Then, for $(t,x) \in [0,T) \times \mathbb{R}^d$,

- *v is uniformly bounded and continuous;*
- *$Z_s^{t,x} = v(s, X_s^{t,x})\sigma(X_s^{t,x})$, $s \in [t,T)$, P-a.s.;*
- *$u_x(t,x) = v(t,x)$;*
- *If we assume further that $g \in C^1$, then all the above hold true on $[0,T] \times \mathbb{R}^d$, and $v(T,x) = g'(x)$.*

Path Regularity of process Z

Recall that if $\xi \in L^2(\mathscr{F}_T^W; \mathbb{R})$, then by Martingale Representation Theorem, $\exists!$ (predictable) process Z with $E\int_0^T |Z_s|^2 ds < \infty$, s.t.

$$Y_t \stackrel{\triangle}{=} E\{\xi|\mathscr{F}_t\} = \xi - \int_t^T Z_s dW_s, \quad t \in [0,T].$$

Question: What can we say about the path regularity of Z?

Path Regularity of process Z

Recall that if $\xi \in L^2(\mathscr{F}_T^W; \mathbb{R})$, then by Martingale Representation Theorem, $\exists!$ (predictable) process Z with $E\int_0^T |Z_s|^2 ds < \infty$, s.t.

$$Y_t \triangleq E\{\xi|\mathscr{F}_t\} = \xi - \int_t^T Z_s dW_s, \quad t \in [0, T].$$

Question: What can we say about the path regularity of Z?

Answer: Nothing!

Examples:

- $\xi = W_T$. Then $Z_t \equiv 1$, $\forall t \in [0, T]$;
- $\xi = \max_{0 \le t \le T} W_t$. Then by the Clark-Ocone formula, $Z_t = E\{D_t\xi|\mathscr{F}_t\} = E\{1_{[0,\tau]}(t)|\mathscr{F}_t\}$, where D is the Malliavin derivative and τ is the a.s. maximum point of W.
- $\xi = \int_0^T h_s dW_s$, where h is any $\mathbb{F}$-predictable process such that $E\int_0^T |h_s|^2 ds < \infty$, then by uniqueness $Z_t \equiv h_t$, $\forall t$, a.s.

Now consider the FBSDE: Path Regularity

$$\begin{cases} X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \\ Y_t = \xi + \int_t^T f(s, \Theta_s)ds - \int_t^T Z_s dW_s, \;\; t \in [0, T], \end{cases} \tag{66}$$

where $\xi = \Phi(X)_T$, and $\Phi : C([0, T]; \mathbb{R}^d) \mapsto \mathbb{R}$ is a functional.

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where $\xi = \Phi(X)_T$, and $\Phi : C([0, T]; \mathbb{R}^d) \mapsto \mathbb{R}$ is a functional.

- If $\Phi(X)_T = g(X_T)$ and g is Lipschitz, then by Rep. Thm.:
$$Z_t = u_x(t, X_t)\sigma(t, X_t) \quad \Longrightarrow \quad Z \text{ is continuous;}$$
- If $\Phi(X)_T = g(X_{t_0}, ..., X_{t_n})$ and g is Lipschitz, then on each subinterval $[t_{i-1}, t_i)$,
$$Z_s = E\left\{g(X_{t_0}, ..., X_{t_n})N^s_{t_i} + \int_s^T f(\Theta_r)N^s_{r\wedge t_i}dr \Big| \mathscr{F}_s\right\}\sigma(X_s).$$
$\Longrightarrow$ Z is a.s. continuous on each $[t_{i-1}, t_i)$, hence càdlàg .

Now consider the FBSDE: ◂ Path Regularity

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$\Longrightarrow$ Z is a.s. continuous on each $[t_{i-1}, t_i)$, hence càdlàg .

Question: Can we go any further to more general functionals for which the process Z has at least a RCLL (càdlàg) version?

Two Worked-Out Cases :

Theorem (Ma-Zhang,00)

Suppose that f is continuous and uniformly Lipschitz in (x, y, z).

- If Φ satisfies the "functional Lipschitz" condition:

$$|\Phi(\mathbf{x}^1) - \Phi(\mathbf{x}^2)| \le L \sup_{t \le s \le T} |\mathbf{x}^1(s) - \mathbf{x}^2(s)| \qquad (67)$$

  for all $\mathbf{x}^1, \mathbf{x}^2 \in C([0, T]; \mathbb{R}^n)$. Then Z has càdlàg paths.

-

- If Φ satisfies the "Integral Lipschitz" condition:

$$|\Phi(\mathbf{x}^1) - \Phi(\mathbf{x}^2)| \le L \int_0^T |\mathbf{x}^1(t) - \mathbf{x}^2(t)| dt, \qquad (68)$$

  then Z has a.s. continuous paths.

Proof (e.g., the functional Lipschitz case)

For any partition $\pi : 0 = t_0 < t_1 < ... < t_n = T$, define $\psi_\pi : \mathbb{R}^{n+1} \mapsto C([0,T];\mathbb{R})$ and $\varphi_\pi : C([0,T];\mathbb{R}) \mapsto \mathbb{R}^{n+1}$ by

$$[\psi_\pi(x_0,\cdots,x_n)](t) \stackrel{\triangle}{=} \frac{t_{i+1}-t}{t_{i+1}-t_i}x_i + \frac{t-t_i}{t_{i+1}-t_i}x_{i+1}, \;\; t \in [t_i, t_{i+1});$$

$$\varphi_\pi(\mathbf{x}) = (\mathbf{x}_{t_0},\cdots,\mathbf{x}_{t_n}), \qquad \mathbf{x} \in C([0,T]).$$

Define $\Phi_\pi := [\Phi \circ \psi_\pi]$ and mollify (Φ_π, f) to $(g_\pi, f_\pi) \in C^1_b$ s.t.

- Φ_π is uniform Lipschitz; and g_π satisfies

$$\sum_{i=0}^{n} |\partial_{x_i} g_\pi(x) y_i| \le L \max_i |y_i|, \qquad \forall x, y \in \mathbb{R}^{n+1}; \qquad (69)$$

- $g_\pi \circ \varphi_\pi \to \Phi$ pointwisely on $C([0,T];\mathbb{R})$, as $|\pi| \to 0$;
- $f_\pi \to f$ uniformly in all variables, as $|\pi| \to 0$.

- Denote the solution to (66) with $\xi = g_\pi(X_{t_0}, \cdots X_{t_n})$ and $f = f_\pi$ by (X, Y^π, Z^π).
- Let ∇X be the solution of (65), and $(\nabla^i Y^\pi, \nabla^i Z^\pi)$ be the solution of the following BSDE on $[t_{i-1}, t_i)$:

$$
\begin{aligned}
\nabla^i Y_t &= \sum_{j \geq i} \partial_j g \nabla X_{t_j} + \int_t^T \langle \nabla f(r), \nabla \Theta_r^{i,\pi} \rangle \, dr \\
&\quad - \int_t^T \nabla^i Z_r^\pi dW_r, \quad t \in [t_{i-1}, t_i),
\end{aligned}
$$

where $\partial_j g = \partial_{x_j} g(X_{t_0}, \cdots, X_{t_n})$, and

$$
\begin{aligned}
&\nabla f(r) = (\partial_x f(\Theta^\pi(r)), \partial_y f(\Theta^\pi(r)), \partial_z f(\Theta^\pi(r))) \\
&\nabla \Theta_r^{i,\pi} = (\nabla X_r, \nabla^i Y_r^\pi, \nabla^i Z_r^\pi) \\
&\Theta_r^\pi = (X_r, Y_r^\pi, Z_r^\pi).
\end{aligned}
$$

- Define: $\nabla^\pi Y^\pi_t \triangleq \sum_{i=0}^{n} \nabla^i Y^\pi_t \mathbf{1}_{[t_{i-1},t_i)}(t) + \nabla^n Y^\pi_{T-}\mathbf{1}_{\{T\}}(t)$.

  Show that $\{\nabla^\pi Y^\pi\}$ is a family of *quasimartingale* (i.e., RCLL and for all partition $\hat{\pi}$, it holds that

$$\sum_{i=1}^{n} E\left\{\left|E\left\{\nabla^\pi Y^\pi_{t_{i-1}} - \nabla^\pi Y^\pi_{t_i}\middle|\mathscr{F}_{t_{i-1}}\right\}\right|\right\} + E\{|\nabla^\pi Y^\pi_T|\} \leq C.)$$

- By the Meyer-Zheng Theorem (1986) $\nabla^\pi Y^\pi$ converges weakly to a càdlàg process $\widetilde{Z}$ under the so-called *pseudo-path topology* (of Meyer-Zheng).
- Using the stability result of BSDE to show that $\nabla^\pi Y^\pi$ converges to Z in $L^2(\Omega \times [0,T])$, hence a.s. converges to Z in the pseudo-path topology. Identifying the laws of Z and $\widetilde{Z}$ we see that Z is càdlàg , a.s. ■

In almost all of the existing theory of Financial Asset Pricing, the "price" process is assumed to be Markov under the so-called *risk neutral measure*. But by a result of Çinlar-Jacod (1981) states that *all "reasonable" strong Markov martingale processes are solutions of equations of the form:*

$$X_t = y + \int_0^t \sigma(r, X_r) dW_r + \int_0^t \int_{\mathbb{R}} b(r, X_{r-}, z) \widetilde{\mu}(drdz), \qquad (70)$$

where W is a Wiener process $\widetilde{\mu}$ is a compensated Poisson random measure with Lévy measure F.

Consider, for example, the Markov Martingale with $b = b(r, x)z$:

$$X_t = y + \int_0^t \sigma(r, X_r) dW_r + \int_0^t \int_{\mathbb{R}} b(r, X_{r-}) z \widetilde{\mu}(drdz). \qquad (71)$$

Let $\Phi : \Delta \mapsto \mathbb{R}$ be s.t. $E|\Phi(X)|^2 < \infty$, and $M_t \stackrel{\triangle}{=} E\{\Phi(X)|\mathscr{F}_t\}$, $t \geq 0$. By Mart. Rep. Thm, $\exists$ $\mathbb{F}$-predictable process Z s.t.

$$M_t = M_0 + \int_0^t Z_s dX_s + N_t, \tag{72}$$

where N is an $\mathbb{F}$-martingale that is orthogonal to X.

Question:

Under what conditions on Φ will Z have càglàd paths?

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where N is an $\mathbb{F}$-martingale that is orthogonal to X.

Question:

Under what conditions on Φ will Z have càglàd paths?

Answer:

- $\Phi(X) = g(X_{t_0}, X_{t_1}, \cdots, X_{t_n})$, $g \in C_b^1(\mathbb{R}^{n+1})$
  — Jacod-Méléard-Protter (2000)
- $|\Phi(\mathbf{x}_1) - \Phi(\mathbf{x}_2)| \leq L \int_0^T |\mathbf{x}_1(t) - \mathbf{x}_2(t)| dt$, $\mathbf{x}_1, \mathbf{x}_2 \in \Delta$
  — Ma-Protter-Zhang (2000)

Possible Applications in Finance:

- $\Phi(X)_T = \frac{1}{T}\int_0^T X_s ds$;
- $\Phi(X)_T = g(\sup_{0\le t\le T} h(t, X_t))$, where g and $h(t,\cdot)$ are uniformly Lipschitz with a common constant K, and $h(\cdot, x)$ is continuous for all x. (Lookback option)
- $\Phi(X)_T = g(\int_0^T h(s, X_{s-})dX_s)$, where g and $h(t,\cdot)$ are uniformly Lipschitz continuous; h is bounded; and for fixed x, $h(\cdot, x)$ is càglàd .
- $\Phi(X) = g(\Phi_1(X), \cdots, \Phi_n(X))$, where g is Lipschitz and Φ_i's are of any of the forms (i)–(iii). (For example, if $g(x) = (K - x)^+$, then g combined with (i) gives an Asian Option.)

5. Weak Solutions of FBSDEs

Recall the general form of forward-backward SDE:

$$\begin{cases} X_t = x + \int_0^t b(s, \Theta_s)ds + \int_0^t \sigma(s, \Theta_s)dW_s \\ Y_t = g(X)_T + \int_t^T h(s, \Theta_s)ds - \int_t^T Z_s dW_s, \end{cases} \tag{73}$$

Definition of Weak Solution of FBSDEs

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Question:

What can we say about the well-posedness of the FBSDE if the coefficients are only continuous?

Example

(i) Decoupled Case:

$$\begin{cases} X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \\ Y_t = g(X)_T + \int_t^T h(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \end{cases}$$

In this case we can take a weak solution $(\Omega, \mathscr{F}, \mathbb{P}, X, W)$, and obtain the (strong) solution (Y, Z) on the space $(\Omega, \mathscr{F}, \mathbb{P})$.

(ii) Weakly Coupled Case:

$$\begin{cases} X_t = x + \int_0^t [b_0(s, X_s) + b_1(s, \Theta_s)]ds + \int_0^t \sigma(s, X_s)dW_s \\ Y_t = g(X)_T + \int_t^T h(s, \Theta_s)ds - \int_t^T Z_s dW_s, \end{cases}$$

where σ^{-1} and b_1 are bounded — Girsanov(?)

Some Preparations

- A quintuple $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F}, W)$ is called a
  - "*standard set-up*" if $(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{F})$ is a complete, filtered prob. space satisfying the *usual hypotheses* and W is a $\mathbb{F}$-B.M.
  - "*Brownian set-up*" if $\mathbb{F} = \mathbb{F}^W \triangleq \{\mathscr{F}_t^W\}_{t\in[0,T]}$.
- "*Canonical Space*": $\Omega \triangleq \Omega^1 \times \Omega^2$, $\mathscr{F} \triangleq \mathscr{F}_\infty^1 \otimes \mathscr{F}_\infty^2$, where
  - $\Omega^i \triangleq \mathbb{D}([0,\infty); \mathbb{R}^{n_i})$, $i = 1, 2$ — path space of X and Y
  - $\mathscr{F}_t^i \triangleq \sigma\{\omega^i(r \wedge t) : r \geq 0\}$, $i = 1, 2$ ($\mathscr{F}_t \triangleq \mathscr{F}_t^1 \otimes \mathscr{F}_t^2$, $t \geq 0$)
- On a canonical space $(\Omega, \mathscr{F})$, denote $\omega = (\omega^1, \omega^2) \in \Omega$, and
  - $(\mathbf{x}_t(\omega), \mathbf{y}_t(\omega)) \triangleq (\omega^1(t), \omega^2(t))$, the "*canonical process*",
  - $\mathscr{P}(\Omega) =$ all prob. meas. on $(\Omega, \mathscr{F})$, with Prohorov metric.

Existing Literature

- Antonelli and Ma ('03) — (**FBSDE**)

  — Existence via Girsanov, Yamada-Watanabe Theorem,
- Buckdahn, Engelbert, and Rascanu ('04) — (**BSDE**, no "Z")

  — Existence via Meyer-Zheng, Yamada-Watanabe Theorem, ...
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Our Main Purpose:

- Find a "backward" version of the "**Martingale Problem**"
- A more general existence result (multi-dimensional, non-Markovian FBSDEs)
- **Uniquenss** (in law)!!!

Definition (Antonelli-Ma, '03)

A standard set-up $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F}, W)$ *along with a triplet of processes* (X, Y, Z) *defined on the set-up is called a weak solution of (73) if*

- (X, Y, Z) *is* $\mathbb{F}$*-adapted; and* (X, Y) *are continuous,*
- *denoting* $\eta_s = \eta(s, (X)_s, Y_s, Z_s)$ *for* $\eta = b, \sigma, h$*, it holds that*

$$P\left\{\int_0^T (|b_s| + |\sigma_s|^2 + |h_s|^2 + |Z_s|^2)\, ds + |g(X)_T|^2 < \infty\right\} = 1$$

- (X, Y, Z) *verifies (73)* $\mathbb{P}-a.s.$

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- (X, Y, Z) *verifies (73)* $\mathbb{P}-$*a.s.*

Remark

- Similar to the forward SDE, a weak sol. allows the flexibility of probability space, and relaxed the most fundamental requirement for a BSDE, i.e., that the set-up is Brownian.
- The Tsirelson-type examples for forward SDEs would lead to the fact that there do exist weak sol. that are not "strong".

Forward-Backward Martingale Problems (FBMP)

Assume $\sigma = \sigma(t, \mathbf{x}, y)$, and let $(\Omega, \mathscr{F})$ be the canonical space and $(\mathbf{x}, \mathbf{y})$ the canonical processes. Denote

- $a = \sigma\sigma^T$;
- $\hat{f}(t, \mathbf{x}, y, z) = f(t, \mathbf{x}, y, z\sigma(t, \mathbf{x}, y))$, for $f = b,\ h$.

Note:

The general case $\sigma = \sigma(t, \mathbf{x}, y, z)$ can can be treated along the lines of "*Four Step Scheme*":

- find a function $\mathbf{\Phi}$ such that

$$\mathbf{\Phi}(t, \mathbf{x}, y, z) = z\sigma(t, \mathbf{x}, y, \mathbf{\Phi}(t, \mathbf{x}, y, z)),$$

- define the functions $\hat{b}$, $\hat{h}$, and $\hat{\sigma}$ as

$$\hat{f}(t, \mathbf{x}, y, z) = f(t, \mathbf{x}, y, \mathbf{\Phi}(t, \mathbf{x}, y, z)), \qquad f = b, h, \sigma.$$

Definition

$\forall (s,x) \in [0,T] \times \mathbb{R}^n$, a solution to $FBMP_{s,x,T}(b,\sigma,h,g)$ is a pair $(\mathbb{P}, \mathbf{z}) \in \mathscr{P}(\Omega) \otimes L^2_{\mathbf{F}}([0,T] \times \Omega; \mathbb{R}^{m\times n})$ such that

- *Both processes $M_{\mathbf{x}}(t) \stackrel{\triangle}{=} \mathbf{x}_t - \int_s^t \hat{b}(r, (\mathbf{x})_r, \mathbf{y}_r, \mathbf{z}_r) dr$ and $M_{\mathbf{y}}(t) \stackrel{\triangle}{=} \mathbf{y}_t + \int_s^t \hat{h}(r, (\mathbf{x})_r, \mathbf{y}_r, \mathbf{z}_r) dr$ are $\mathbb{P}$-mg's for $t \geq s$;*
- $[M^i_{\mathbf{x}}, M^j_{\mathbf{x}}](t) = \displaystyle\int_s^t a_{ij}(r, (\mathbf{x})_r, \mathbf{y}_r) dr, \quad t \geq s, \quad i,j = 1, \cdots n,$
- $M_{\mathbf{y}}(t) = \int_s^t \mathbf{z}_r dM_{\mathbf{x}}(r)$, $t \geq s$.
- $\mathbb{P}\{\mathbf{x}_s = x\} = 1$ *and* $\mathbb{P}\{\mathbf{y}_T = g(\mathbf{x})_T\} = 1$.

Remark

- *The process $\{\mathbf{z}_t\}$ is different from $\{Z_t\}$ in (73)! In fact, $\{\mathbf{z}_t\} \sim \nabla u$, $Z \sim \sigma^T \nabla u$, where u satisfies PDE (62).*
- *(73) has a weak solution $\Longleftrightarrow$ $FBMP_{t,x,T}(a,b,h,g)$ has a solution with $a = \sigma\sigma^T$.*

FBMP vs. Traditional Martingale Problem:

Assume $f(t, \mathbf{x}, y, z) = f(t, x, y, z)$, $f = b,\ \sigma,\ h,\ g$. Then $(\mathbb{P}, \mathbf{z})$ is a solution to the $\mathrm{FBMP}_{s,x,T}(b, \sigma, h, g) \quad \Longleftrightarrow$

$$\begin{cases} d\mathbf{x}_t = \widehat{b}(t, \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t)dt + dM_{\mathbf{x}}(t), \\ d\mathbf{y}_t = -\widehat{h}(t, \mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t)dt + dM_{\mathbf{y}}(t) = -\widehat{h}(t, \cdots)dt + \mathbf{z}_t dM_{\mathbf{x}}(t). \end{cases}$$

Assume $f(t, \mathbf{x}, y, z) = f(t, x, y, z)$, $f = b, \sigma, h, g$. Then $(\mathbb{P}, \mathbf{z})$ is a solution to the FBMP$_{s,x,T}(b, \sigma, h, g)$ $\iff$

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$\iff$ (By Itô and choice of φ):

$$C[\varphi](t) \stackrel{\triangle}{=} \varphi(\mathbf{x}_t, \mathbf{y}_t) - \varphi(x, \mathbf{y}_0) - \int_0^t \mathscr{L}_{s,\mathbf{x}_s,\mathbf{y}_s,\mathbf{z}_s}\varphi(\mathbf{x}_s, \mathbf{y}_s)ds$$

is a $\mathbb{P}$-martingale for all $\varphi \in C^2(\mathbb{R}^n \times \mathbb{R}^m)$. where

$$\begin{aligned} \mathscr{L}_{t,x,y,z} &\stackrel{\triangle}{=} \frac{1}{2}\mathrm{tr}\,\{AD^2_{x,y}\} + \langle\, \widehat{b}, \nabla_x \,\rangle - \langle\, \widehat{h}, \nabla_y \,\rangle; \\ A(t,x,y,z) &\stackrel{\triangle}{=} [I_n, z]^T a(t,x,y)[I_n, z^T]. \end{aligned}$$

Solvability of FBMPs (Existence)

Main Assumption:

(H1) b, σ, h, and g are bounded and uniformly continuous on $(\mathbf{x}, y, z)$, uniformly in t.

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Theorem

Assume (H1), and that $\exists\{(b_n, \sigma_n, h_n, g_n)\}$, *all satisfying (H1), s.t.*

- *for* $f = b, \sigma, h, g$, $\|f_n - f\|_\infty \le \frac{1}{n}$;
- *FBSDE (73) with* $(b_n, \sigma_n, f_n, g_n)$ *has strong sol.* (X^n, Y^n, Z^n);
- *denoting* $Z_t^{n,\delta} \triangleq \frac{1}{\delta}\int_{0\vee(t-\delta)}^t Z_s^n ds$, *it holds that*

$$\lim_{\delta\to 0}\sup_n E\left\{\int_0^T |Z_t^n - Z_t^{n,\delta}|^2 dt\right\} = 0. \tag{74}$$

Then (73) admits a weak solution.

DeepThinking

Sketch of the Proof.

Step 1. Assume $\Theta^n_t \stackrel{\triangle}{=} ((X^n)_t, Y^n_t, Z^n_t)$ "lives" on a fixed prob. space. Denote

$$B^n_t \stackrel{\triangle}{=} \int_0^t b_n(s, \Theta^n_s)ds; \quad F^n_t \stackrel{\triangle}{=} \int_0^t h_n(s, \Theta^n_s)ds; \quad A^n_t \stackrel{\triangle}{=} \int_0^t Z^n_s ds;$$

$$M^n_t \stackrel{\triangle}{=} \int_0^t \sigma_n(s, \Theta^n_s)dW_s; \quad N^n_t \stackrel{\triangle}{=} \int_0^t Z^n_s dW_s,$$

and $\Sigma^n \stackrel{\triangle}{=} (W, X^n, Y^n, B^n, F^n, A^n, M^n, N^n)$.

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Then

- $\{\Sigma^n\}$ are *quasimartingales* under $\mathbb{P}$ with uniformly bounded *conditional variation*. (e.g., $\forall 0 = t_0 < \cdots < t_m = T$,

$$\mathrm{C.Var}(Y^n) \le \sum_{i=0}^{m-1} E\Big\{ \int_{t_i}^{t_{i+1}} |h_n(t,\Theta^n_t)|dt + |g_n(X^n_T)| \Big\} \le C.)$$

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- by Meyer-Zheng tightness criteria, $\mathbb{P}^n \triangleq P \circ [\Sigma^n]^{-1} \to \mathbb{P} \in \mathscr{P}(\widehat{\Omega})$ weakly, as $n \to \infty$ (possibly along a subsequence), where $\widehat{\Omega} \triangleq \mathbb{D}([0, T]; \mathbb{R}^8)$;

Step 2. By a slight abuse of notations, denote the coordinate processs of $\widehat{\Omega}$ by $\Sigma = (W, \mathbf{x}, \mathbf{y}, B, F, A, M, N)$. Then

- W is a Brownian motion under $\mathbb{P}$;
- B, F (whence $\mathbf{x}$), and M are all continuous;
- M, N are martingales ([Meyer-Zheng, Theorem 11], as $\sup_n E\Big\{ \int_0^T |Z_t^n|^2 dt \Big\} < \infty$);
- A is absolutely continuous w.r.t. dt, $\mathbb{P}$-a.s., and for some $\mathbf{z} \in L^2([0,T] \times \tilde{\Omega})$, it holds that $A_t = \displaystyle\int_0^t \mathbf{z}_s ds$, ([Meyer-Zheng, Theorem 10]).

$$\Longrightarrow \qquad \mathbf{x}_t = \mathbf{x}_0 + B_t + M_t, \quad \mathbf{y}_t = \mathbf{y}_0 - F_t + N_t, \quad \forall t, \quad \mathbb{P}\text{-a.s.}$$

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Hope:

$$B_t = \int_0^t b(s, \Theta_s) ds, \; M_t = \int_0^t \sigma(s, \Theta_s) dW_s, \; F_t = \int_0^t h(s, \Theta_s) ds,$$
$$N_t = \int_0^t \mathbf{z}_s dW_s \; \dots$$

Step 3. Show that $B_t = \int_0^t b(s, \Theta_s)ds$ and $F_t = \int_0^t h(s, \Theta_s)ds$.

Key estimates:

- Denote $Z_t^\delta \triangleq \frac{1}{\delta}[A_t - A_{t-\delta}]$ and $\Theta_s^\delta = ((X)_s, Y_s, Z_s^\delta)$;
- by the uniform continuity of b (on z) $\oplus$ Assumption (74)

$$E^{\mathbb{P}}\Big\{|B_t - \int_0^t b(s, \Theta_s)ds|\Big\} = \lim_{\delta\to 0} E^{\mathbb{P}}\Big\{|B_t - \int_0^t b(s, \Theta_s^\delta)ds|\Big\}$$

$$\le \lim_{\delta\to 0} \lim_n E\Big\{\int_0^T |b(s, \Theta_s^n) - b(s, \Theta_s^{n,\delta})|ds\Big\}$$

$$= \lim_n \lim_{\delta\to 0} E\Big\{\int_0^T |b(s, \Theta_s^n) - b(s, \Theta_s^{n,\delta})|ds\Big\} = 0$$

$$\Longrightarrow E^{\mathbb{P}}\Big\{|B_t - \int_0^t b(s, \Theta_s)ds|\Big\} = 0.$$

- Similar for F.

Step 4. Show that $N_t = \int_0^t \mathbf{z}_s dW_s$, $M_t = \int_0^t \sigma(s, \Theta_s) dW_s$.

Key estimates:

- By Dom. Conv. Thm: $\int_0^T |Z_t - Z_t^\delta|^2 dt \to 0, \quad P - a.s.$
- Let $\pi : 0 = t_0 < \cdots < t_m = T$ be any partition. Show that

$$E^{\mathbb{P}}\Big\{\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} |N_t - \sum_{i=0}^{j-1} Z_{t_i}^\delta [W_{t_{i+1}} - W_{t_i}]|^2 dt\Big\} + \frac{C}{\delta^2} E^P\{I^{\pi,\delta}\}$$

$$\leq \quad C \varlimsup_n E\Big\{\int_0^T |\int_0^t Z_s^n dW_s - \int_0^t Z_s^{n,\delta} dW_s|^2 dt\Big\} + \frac{C|\pi|}{\delta^2}$$

$$\leq \quad C \sup_n E\Big\{\int_0^T |Z_t^n - Z_t^{n,\delta}|^2 dt\Big\} + \frac{C|\pi|}{\delta^2}.$$

Letting $|\pi| \to 0$ and using (74) (**Again**!) $\Longrightarrow$ $\lim_{\delta\to 0} I^\delta = 0$.

- Similarly, $M_t = \int_0^t \sigma(s, \Theta_s) dW_s$.

When will Assumption (74) satisfied?

(H2) b, h, σ, and g are deterministic, Lipschitz, and $\frac{1}{K} I \le \sigma_n \sigma_n^* \le K I$, for some $K > 0$.

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- Let $\{(b_n, \sigma_n, h_n, g_n)\}$ be the molifiers of (b, σ, h, g), and let (X^n, Y^n, Z^n) be the correspondin strong solutions
- In light of the "**Four Step Scheme**", the following relations hold:

$$Y_t^n = u^n(t, X_t^n), \quad Z_t^n = \sigma_n(t, X_t^n, u^n(t, X_t^n))\nabla_x u^n(t, X_t^n),$$

where $u^n(t,x)$ is the (classical) solution to the PDE:

$$\begin{cases} u_t^n + \frac{1}{2}\sigma_n^2 D_{xx}^2 u^n + \nabla_x u^n \cdot b_n(\cdots, \sigma_n \nabla_x u^n) + h_n(\cdots) = 0; \\ u^n(T, x) = g_n(x). \end{cases} \tag{75}$$

Hölder Continuous Case

For simplicity, assume $\underline{b \equiv 0}$ and $\underline{m = d = 1}$.

Key Estimates (MZZ-2005):

If σ, h, and g are C^α, and $u \in C^{1,2}$ is the solution to the PDE (75), then $\exists C > 0$, $\alpha \in (0,1)$, and $C_\varepsilon > 0$ for each $\varepsilon > 0$, s.t.

$$|u_x(t,x)| \le C(T-t)^{\frac{\alpha-1}{2}}; \quad |u_{xx}(t,x)| \le C(T-t)^{\frac{\alpha}{2}-1},$$
$$|u_x(t_1,x) - u_x(t_2,x)| \le C_\varepsilon \sqrt{t_2 - t_1}, \quad 0 \le t_1 < t_2 \le T - \varepsilon.$$

Note: $Z_t^n = [u_x^n \sigma_n](t, X_t^n, u^n(t, X_t^n)) \Longrightarrow \forall \delta, \varepsilon > 0$, $\exists \beta = \beta(\alpha) > 0$, s.t.

$$E\Big\{ \int_0^T |Z_t^n - Z_t^{n,\delta}|^2 dt \Big\} \le C_\varepsilon \delta^\beta + C\varepsilon^\alpha.$$

First letting $\delta \to 0$ and then $\varepsilon \to 0 \Longrightarrow$ Assumption (74) holds.

Uniformly Continuous Case

More complicated, but still possible. Need: gradient Estimate of the form:

$$|u_x(s,x) - u_x(t,y)| \le C[|s-t|^{\frac{\alpha}{2}} + |x-y|^{\alpha}] \text{ (!)} \tag{76}$$

- One dimensional case, use the result of Nash
- Higher dimensional case, need L^p-theory (e.g., Lieberman's book)

Some Facts about "Canonical Weak Solution":

We call the weak solution $(\Omega, \mathscr{F}, \mathbb{P}; \mathbf{F}, W, X, Y, Z)$ constructed via "Four Step Scheme" the "*Canonical Weak Solution*". Then,

- $Y_t = u(t, X_t)$, where u is a viscosity solution of the corresponding PDE.
- By an estimate on u (cf. e.g., Delarue, 2003), for $t < t + \delta \le T_0 < T$,

$$|u(t+\delta, X_{t+\delta}) - u(t, X_t)| \le \frac{C}{(T - T_0)^{\frac{\alpha}{2}}} \left[\delta^{\frac{\alpha}{2}} + |X_{t+\delta} - X_t|^{\alpha}\right].$$

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$$|u(t+\delta, X_{t+\delta}) - u(t, X_t)| \le \frac{C}{(T - T_0)^{\frac{\alpha}{2}}}\Big[\delta^{\frac{\alpha}{2}} + |X_{t+\delta} - X_t|^{\alpha}\Big].$$

- Hence

$$\begin{aligned} E_t^{\mathbb{P}}|Y_{t+\delta} - Y_t|^2 &\le \frac{C}{(T-T_0)^{\alpha}}\Big[\delta^{\alpha} + E_t^{\mathbb{P}}\Big|\int_t^{t+\delta} \sigma(\cdot)dW_s\Big|^{2\alpha}\Big] \\ &\le \frac{C}{(T-T_0)^{\alpha}}\delta^{\alpha}. \end{aligned}$$

Consequently,

$$
\begin{aligned}
E_t^{\mathbb{P}}\Big\{\int_t^{t+\delta}|Z_s|^2 ds\Big\} &= E_t^{\mathbb{P}}\Big\{\Big|Y_{t+\delta}-Y_t+\int_t^{t+\delta}h(\cdots)ds\Big|^2\Big\}\\
&\le \frac{C}{(T-T_0)^\alpha}\delta^\alpha.
\end{aligned}
$$

Finally,

$$
E_t^{\mathbb{P}}\Big\{|Y_{t+\delta}-Y_t|^2\Big\}+E_t^{\mathbb{P}}\Big\{\int_t^{t+\delta}|Z_s|^2 ds\Big\}\le\frac{C}{(T-T_0)^\alpha}\delta^\alpha. \quad (77)
$$

Note:

The estimates (77) will be useful in the discussion of uniqueness!

Uniqueness of FBMP

Main Assumptions:

- $m = 1$ and Markovian type
- b, σ, h, and g are **bounded** and **uniformly continuous** in (x, y, z), and $\sigma\sigma^T \geq cI$, $c > 0$. Thus WLOG may assume $b = 0$ (Girsanov).

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Recall that a weak solution is a pair $(\mathbb{P}, Z)$, where $\mathbb{P}$ is a proba. measure on the canonical space $\Omega = \mathbb{C}([0, T]; \mathbb{R}^n) \times \mathbb{C}([0, T]; \mathbb{R})$ and $Z \in L^2_{\mathbf{F}}([0, T] \times \Omega; \mathbb{P})$, such that $W_t \stackrel{\triangle}{=} \int_0^t \sigma^{-1}(t, \mathbf{x}_t, \mathbf{y}_t) d\mathbf{x}_t$, $t \geq 0$ is a $\mathbb{P}$-Brownian motion.

Definition of Uniqueness:

If $(\mathbb{P}^i, Z^i)$, $i = 1, 2$ are two weak solutions, then the processes $(\mathbf{x}, \mathbf{y}, Z^1)$ and $(\mathbf{x}, \mathbf{y}, Z^2)$ have the same finite dimensional distributions, under $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively.

Definition

Let $K : [0, T] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ be such that $\int_0^T K_t^2 dt < \infty$. We say that a pair $(\mathbb{P}, Z)$ is a "K-weak solution" at $(s, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ if the following hold:

- $W_t \stackrel{\triangle}{=} \int_s^t \sigma^{-1}(r, \mathbf{x}_r, \mathbf{y}_r) d\mathbf{x}_r$ *is a* $\mathbb{P}$*-Brownian motion for* $t \geq s$;
- $\mathbb{P}\{\mathbf{x}_s = x, \mathbf{y}_s = y\} = 1$;
- $\mathbf{y}_t = y - \int_s^t h(r, \mathbf{x}_r, \mathbf{y}_r) dr + \int_s^t Z_r dW_r$, $t \in [s, T]$, $\mathbb{P}$-a.s.;
- $\mathbb{P}\{\mathbf{y}_T = g(\mathbf{x}_T)\} = 1$;
- $|Z_t| \leq K_t$, $\forall t \in (s, T)$, $\mathbb{P}$-a.s.

Objective:

Show that the K-weak solution is unique!

K-Weak Solutions

If σ, h, g are Hölder-α continuous, and $u \in C^{1,2}$ is the classical solution to PDE

$$\begin{cases} u_t + \frac{1}{2} u_{xx} \sigma^2 + h(t, x, u, u_x \sigma) = 0; \\ u(T, x) = g(x). \end{cases} \tag{78}$$

Then, recall that we have proved (MZZ-2005) that $\exists C > 0$, depending only on L, T, and α, such that

$$|u_x(t,x)| \leq C(T-t)^{\frac{\alpha-1}{2}}; \quad |u_{xx}(t,x)| \leq C(T-t)^{\frac{\alpha}{2}-1}.$$

Consequently, if we assume that $K_t \geq C(T-t)^{\frac{\alpha-1}{2}}$, then the class of K-weak solutions is nonempty, and it at least contains the canonical weak solution!

Viscosity

K-Weak Solutions

- Denote $\mathscr{O} \stackrel{\triangle}{=} \{(t,x,y) : \exists K\text{-weak solution at } (t,x,y)\}$.
- Define $\overline{\mathscr{O}}$ =cl$\{\mathscr{O}\}$, and $\underline{u}(t,x) \stackrel{\triangle}{=} \inf\{y : (t,x,y) \in \overline{\mathscr{O}}\}$;
  $\bar{u}(t,x) \stackrel{\triangle}{=} \sup\{y : (t,x,y) \in \overline{\mathscr{O}}\}$.

Important Facts

$\underline{u}$ (resp. $\bar{u}$) is a viscosity super-solution (resp. sub-solution) of (78). Consequently, if the Comparison Theorem (for viscosity solutions) holds for the PDE (78). Then

- $\underline{u} \geq \bar{u} \implies \underline{u} \equiv \bar{u} = u$. (I.e., $\mathscr{O}$ is a singleton for each (t,x), and u is the unique viscosity solution to (78).)
- For any K-weak solution $(\mathbb{P}, Z)$, one shows that $(t, \mathbf{x}_t, \mathbf{y}_t) \in \mathscr{O} \implies \mathbf{y}_t = u(t, \mathbf{x}_t)$ holds $\forall t$, $\mathbb{P}$-a.s., as well. (Compare to the canonical weak solution!)

Uniqueness of K-Weak Solutions

Let $(\mathbb{P}^*, Z^*)$ be any K-weak solution, we want to show that it is "identical" to the canonical K-weak solution.

- $dW_t^* = \sigma^{-1}(t, \mathbf{x}_t, u(t, \mathbf{x}_t))d\mathbf{x}_t$.
- W^* is a BM under $\mathbb{P}^* \Longrightarrow (W^*, \mathbf{x})$ is a weak solution to a forward SDE (!)
- $\mathbb{P}^* \circ (W^*, \mathbf{x})^{-1} = \mathbb{P}^0 \circ (W^0, \mathbf{x})^{-1}$ (uniqueness of FMP)
- since both $\mathbb{P}^*$ and $\mathbb{P}^0$ are K-weak solution, one has $\mathbf{y}_t = u(t, \mathbf{x}_t)$, both $\mathbb{P}^*$ and $\mathbb{P}^0$-a.s. (!)
- $\mathbb{P}^* \circ (W^*, \mathbf{x}, \mathbf{y})^{-1} = \mathbb{P}^0 \circ (W^0, \mathbf{x}, \mathbf{y})^{-1}$,
- $\mathbb{P}^* = \mathbb{P}^0$, and furthermore, $\mathbb{P}^* \circ \langle \mathbf{y}, W^* \rangle^{-1} = \mathbb{P}^0 \circ \langle \mathbf{y}, W^0 \rangle^{-1}$
- $Z^* \sim \mathbf{z}$!

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DONE!

Some Observations:

$\mathbf{1}^\circ$ For a weak solution $(\mathbb{P}, Z)$ and any $\delta > 0$, denoting $\mathbb{P}^\omega_t$ to be the r.c.p.d. of $\mathbb{P}\{\cdot|\mathscr{F}_t\}(\omega)$, define

$$K^{\mathbb{P},Z}(t,\delta,\omega) = E^{\mathbb{P}^\omega_t}\Big\{\int_t^{(t+\delta)\wedge T} |Z_s|^2 ds\Big\}.$$

Some Observations:

1° For a weak solution $(\mathbb{P}, Z)$ and any $\delta > 0$, denoting $\mathbb{P}_t^\omega$ to be the r.c.p.d. of $\mathbb{P}\{\cdot|\mathscr{F}_t\}(\omega)$, define

$$K^{\mathbb{P},Z}(t,\delta,\omega) = E^{\mathbb{P}_t^\omega}\Big\{\int_t^{(t+\delta)\wedge T} |Z_s|^2 ds\Big\}.$$

If g, σ, and h are all Hölder continuous, then for any $\delta > 0$, the canonical weak solution $(\mathbb{P}^0, \mathbf{z})$ satisfies:

$$K^{\mathbb{P}^0,\mathbf{z}}(\delta,\omega) \triangleq \sup_{t\in[0,T]} E^{\mathbb{P}_t^{0,\omega}}\Big\{\int_t^{(t+\delta)\wedge T} |\mathbf{z}_s|^2 ds\Big\} \le C\delta^\alpha, \qquad \mathbb{P}^0\text{-a.s.}$$

Hence

$$\lim_{n\to\infty} E^{\mathbb{P}^n}\{K^{\mathbb{P}^n,Z^n}(t_n, 1/\sqrt{n}, \cdot)\} = 0. \tag{79}$$

2° Assume (H1) and (H2). Recall the estimate (77) for the canonical weak solution:

$$E_t^{\mathbb{P}^0}\Big\{|\mathbf{y}_{t+\delta}-\mathbf{y}_t|^2\Big\}+E_t^{\mathbb{P}^0}\Big\{\int_t^{t+\delta}|\mathbf{z}_s|^2ds\Big\}\le\frac{C}{(T-t-\delta)^\alpha}\delta^\alpha.$$

Then, for any $\delta>0$, $\eta>0$, we have

$$\mathbb{P}_t^{0,\omega}\{|\mathbf{y}_{t+\delta}-\mathbf{y}_t|\ge\eta\}\le\frac{C\delta^\alpha}{(T-t-\delta)^\alpha\eta^\alpha}\stackrel{\triangle}{=}k^0(t,\delta,\eta),\quad\mathbb{P}^0\text{-a.e.}$$

Or, in line of (79):

$$\begin{aligned}K^{\mathbb{P}^o,\mathbf{z}}(t,\delta,\cdot)&=E^{\mathbb{P}_t^{0,\omega}}\Big\{\int_t^{(t+\delta)\wedge T}|\mathbf{z}_r|^2dr\Big\}\le\frac{C\delta^\alpha}{(T-t-\delta)^\alpha}\\&\stackrel{\triangle}{=}k^1(t,\delta),\qquad\mathbb{P}^0\text{-a.e. }\omega.\end{aligned}$$

k-Weak Solutions

Definition

We say that a pair $(\mathbb{P}, Z)$ is a "k-weak solution" (resp. $\tilde{k}$-weak solution) at $(s,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$ if it is a weak solution (or solution to the FBMP) such that the following hold:

- *For any $t \in [s, T)$, $\delta > 0$, and $\eta > 0$,*

$$\mathbb{P}_t^{\omega}\{|\mathbf{y}_t - \mathbf{y}_{(t+\delta)\wedge T}| \geq \eta\} \leq k(t,\delta,\eta), \qquad \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

- *(resp. For any $t \in [t, T)$ and $\delta > 0$,*

$$E^{\mathbb{P}_t^{\omega}}\Big\{\int_t^{(t+\delta)\wedge T} |Z_r|^2 dr\Big\} \leq \tilde{k}(t,\delta), \qquad \mathbb{P}\text{-a.s. } \omega \in \Omega.$$

k-Weak Solutions

Remark:

Clearly, the "k-", and "$\tilde{k}$-solutions" are the modifications of the "K-weak solution", with $k : [0, T) \times (0, T) \times (0, 1) \mapsto \mathbb{R}_+$ (resp. $\tilde{k} : [0, T) \times (0, T) \mapsto \mathbb{R}_+$) now satisfying the following properties:

- $k(t_1, \delta_1, \eta) \le k(t_2, \delta_2, \eta)$, $\forall t_1 \le t_2,\ \delta_1 \le \delta_2$
- $\tilde{k}(t_1, \delta_1) \le \tilde{k}(t_2, \delta_2)) \quad \forall t_1 \le t_2,\ \delta_1 \le \delta_2$;
- $\lim\limits_{\delta \to 0} k(t, \delta, \eta) = \lim\limits_{\delta \to 0} \tilde{k}(t, \delta) = 0, \quad \forall (t, \eta)$;
- $k(t, \delta, \eta) \ge k^0(t, \delta, \eta), \quad \forall t < t + \delta < T$;
- $\tilde{k}(t, \delta) \ge k^1(t, \delta)), \quad \forall t < t + \delta < T$.

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Clearly, the "k-", and "$\tilde{k}$-solutions" are the modifications of the "K-weak solution", with $k : [0, T) \times (0, T) \times (0, 1) \mapsto \mathbb{R}_+$ (resp. $\tilde{k} : [0, T) \times (0, T) \mapsto \mathbb{R}_+$) now satisfying the following properties:

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- $k(t, \delta, \eta) \ge k^0(t, \delta, \eta), \quad \forall t < t + \delta < T$;
- $\tilde{k}(t, \delta) \ge k^1(t, \delta)), \quad \forall t < t + \delta < T$.

Theorem (MZZ-2006)

Both $k-$ and $\tilde{k}-$weak solutions are unique.

Two Possibilities:

- Show that every weak solution is a k ($\tilde{k}$)-weak solution
- Show that every weak solution can be "controlled" by a k ($\tilde{k}$)-weak solution

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Example

Assume that the FBSDE is decoupled. (I.e., $b = b(t,x)$, $\sigma = \sigma(t,x)$.) Let
$\mathscr{O} \stackrel{\triangle}{=} \{(t,x,y) : \exists$ a weak solution on $[t,T]$ s.t. $X_t = x,\ Y_t = y\}$.
Then, one can show that

- $\mathscr{O}(t,x) \stackrel{\triangle}{=} \{y : (t,x,y) \in \mathscr{O}\} = [\underline{Y}_t^{t,x}, \overline{Y}_t^{t,x}]$ is an interval;
- $\forall y \in \mathscr{O}(t,x)$, $\exists$ a $\tilde{k}$-weak solution $(\mathbb{P}, Z)$ starting from (t,x,y).

Main Idea:

- Fix a $(\Omega, \mathscr{F}, P, X, W)$ (forward weak solution) starting from (t, x), and find approximation $f_n \uparrow f$ (resp. $f_n \downarrow f$) to obtain solutions $\overline{Y}$ (resp. $\underline{Y}$) (Lepeltier-San Martin);
- By construction, both $\overline{Y}$ and $\underline{Y}$ are $\tilde{k}$-solutions.
- Show that all weak sol's from (t, x, y) can be "controlled" by $(\overline{Y}, \overline{Z})$ and $(\underline{Y}, \underline{Z})$.

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In general, one needs:

- Comparison Theorem for FBSDEs (only at $t = 0$!)
- More knowledge on the PDE solutions
- ......

7. Backward Stochastic PDEs

- $W = (W^1, \cdots, W^d)$ — a d-dimensional Brownian motion.
- $\{\mathscr{F}_t\} = \{\mathscr{F}_t^W\}$.
- $g : \mathbb{R}^n \times \Omega \mapsto \mathbb{R}$ — a random field such that for fixed x, $g(x, \cdot)$ is $\mathscr{F}_T$-measurable.

Backward SPDE (linear version):

$$\begin{aligned} du(t,x) &= -[\mathscr{L}u + \mathscr{M}q + f](t,x)dt + \langle\, q(t,x), dW_t \,\rangle \\ u(T,x) &= g(x), \qquad 0 \le t \le T, \end{aligned} \tag{80}$$

where, for $\varphi \in C^2$ and $\psi \in C^1$,

$$\begin{aligned} (\mathscr{L}\varphi)(t,x) &= \frac{1}{2}\nabla \cdot (A(t,x)\nabla\varphi) + \langle\, a(t,x), \nabla\varphi \,\rangle + c(t,x)\varphi, \\ (\mathscr{M}\psi)(t,x) &= B(t,x)\nabla\psi + h(t,x)\psi, \end{aligned}$$

and A, B, a, c, h and f are $\mathbb{F}$-prog. measurable random fields.

Main Assumptions

The BSPDE is called

- "Parabolic" if $A - BB^T \geq 0$, $\forall(t,x)$, a.s.
- "Super-parabolic:" if $\exists\delta > 0$, $A - BB^T \geq \delta I$, a.e. (t,x), $\mathbb{P}$-a.s.
- "Degenerate Parabolic:" if it is "Parabolic" $\oplus$ "$\exists G \subseteq [0,T] \times \mathbb{R}^n$, $|G| > 0$, such that $\det[A - BB^T] = 0$, $\forall(t,x) \in G$, a.s."
- satisfies the "Symmetric Condition:" if $\left[B(\partial_{x_i} B^T)\right]^T = B(\partial_{x_i} B^T)$, for a.e. (t,x), $\mathbb{P}$-a.s., $1 \leq i \leq n$.

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Assumptions (H)$_m$:

For fixed x, A, B, a, c, h and f are predictable; and g is $\mathscr{F}_T$-measurable. For fixed (t,ω), they are differentiable in x up to order m, and all the partial derivatives are bounded uniformly in (t,ω), by a constant $K_m > 0$.

Let (u,q) be a pair of random fields satisfying (80) $\forall t$, a.s.

- (u,q) is called an *adapted classical solution* of (80) if
$$\begin{cases} u \in C_{\mathscr{F}}([0,T]; L^2(\Omega; C^2(\overline{B_R}))), \\ q \in L^2_{\mathscr{F}}(0,T; C^1(\overline{B_R}; \mathbb{R}^d)), \end{cases} \qquad \forall R > 0,$$
- (u,q) is called an *adapted strong solution* of (80) if
$$\begin{cases} u \in C_{\mathscr{F}}([0,T]; L^2(\Omega; H^2(B_R))), \\ q \in L^2_{\mathscr{F}}(0,T; H^1(B_R; \mathbb{R}^d)), \end{cases} \qquad \forall R > 0,$$
- (u,q) is called an *adapted weak solution* of (80) if
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such that for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ and all $t \in [0,T]$, it holds that

$$\langle u(t,\cdot), \varphi \rangle - \langle g, \varphi \rangle = \int_t^T \Big\{ -\frac{1}{2} \langle A\nabla u, \nabla\varphi \rangle + \langle a\nabla u + cu, \varphi \rangle$$
$$- \langle Bq, \nabla\varphi \rangle + \langle (h,q), \varphi \rangle + \langle f, \varphi \rangle \Big\} ds - \int_t^T \langle q, \varphi \rangle \, dW_s \rangle .$$

Denote

- $m \geq 0$ — integer, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ — multi-index,
- $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $\partial^\alpha \triangleq \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}$
- If $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ is another multi-index, then

$$\beta \leq \alpha \Longleftrightarrow \beta_i \leq \alpha_i \ \forall 1 \leq i \leq n,$$
$$\beta < \alpha \Longleftrightarrow \beta \leq \alpha, \text{and } |\beta| < |\alpha|.$$

Also, for given (u, q), denote

$$F(t, x; u, q, m) \triangleq \sum_{|\alpha| \leq m} \langle (A - BB^T) \nabla(\partial^\alpha u), \nabla(\partial^\alpha u) \rangle + \sum_{|\alpha| \leq m} |\partial^\alpha q + B^T \nabla(\partial^\alpha u) - h \partial^\alpha u|^2 \geq 0.$$

Main Results

Theorem

Suppose that $A(t,x) = A(t)$, and $(H)_m$ holds for some $m \geq 1$. Then

- *BSPDE (80) has a unique adapted weak solution (u,q).*
- *the following estimate holds:*

$$\max_{t\in[0,T]} \mathbb{E}\|u(t,\cdot)\|^2_{H^m} + \mathbb{E}\int_0^T \|q(t,\cdot)\|^2_{H^{m-1}}dt$$
$$+\mathbb{E}\int_{[0,T]\times\mathbb{R}^d} F(t,x;u,q,m)dxdt$$
$$\leq C\Big\{\|f\|^2_{L^2_{\mathscr{F}}(0,T;H^m)} + \|g\|^2_{L^2_{\mathscr{F}_T}(\Omega;H^m)}\Big\},$$

where $C > 0$ depends only on m, T and K_m.

Theorem

Assume Parabolic and symmetric conditions; and that $(H)_m$ holds for some $m \geq 1$, $f \in L^2_{\mathscr{F}}(0,T;H^m(\mathbb{R}^n))$, $g \in L^2_{\mathscr{F}_T}(\Omega;H^m(\mathbb{R}^n))$. Then BSPDE (80) admits a unique weak solution (u,q), s.t.

$$\max_{t\in[0,T]} \mathbb{E}\|u(t,\cdot)\|^2_{H^m} + \|q\|^2_{L^2([0,T]\times\Omega;H^{m-1})} + \|F\|_{L^1([0,T]\times\mathbb{R}^n\times\Omega)} \leq C\Big\{\|f\|^2_{L^2_{\mathscr{F}}(0,T;H^m)} + \|g\|^2_{L^2_{\mathscr{F}_T}(\Omega;H^m)}\Big\},$$

where the constant $C > 0$ only depends on m, T and K_m, and

$$F = F(t,x;u,q,m) = \sum_{|\alpha|\leq m} \Big\{ \langle (A - BB^T)\nabla(\partial^\alpha u), \nabla(\partial^\alpha u)\rangle + \Big|B^T[\nabla(\partial^\alpha u)] + \partial^\alpha q\Big|^2 \Big\}.$$

Main Ideas

- Take an orthonormal basis $\{\varphi_k\}_{k\geq 1} \subset C_0^\infty(\mathbb{R}^n)$ for the space $H^m \equiv H^m(\mathbb{R}^n)$, whose inner product is denoted by

$$(\varphi, \psi)_m \equiv \int_{\mathbb{R}^n} \sum_{|\alpha|\leq m} (\partial^\alpha \varphi)(\partial^\alpha \psi) dx, \quad \forall \varphi, \psi \in H^m.$$

- Consider the following linear BSDE (not BSPDE):

$$\begin{cases} du^{kj}(t) = \Big\{ -\sum_{i=1}^k \big[(\mathscr{L}\varphi_i, \varphi_j)_m u^{ki}(t) - \langle (\mathscr{M}\varphi_i, \varphi_j)_m, q^{ki}(t) \rangle \big] \\ \qquad\qquad -(f, \varphi_j)_m \Big\} dt + \langle q^{kj}(t), dW(t) \rangle, \\ u^{kj}(T) = (g, \varphi_j)_m, \qquad 1 \leq j \leq k. \end{cases}$$

- Define

$$\begin{cases} u^k(t,x,\omega) = \sum_{j=1}^k u^{kj}(t,\omega)\varphi_j(x), \\ q^k(t,x,\omega) = \sum_{j=1}^k q^{kj}(t,\omega)\varphi_j(x), \end{cases}$$

Then $u^k(t,\cdot,\omega) \in C_0^\infty(\mathbb{R}^n)$, $q^k(t,\cdot,\omega) \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^d)$.

Main Ideas

- Prove the *a priori* estimates hold for (u^k, q^k)'s, and then conclude that they are bounded in the space of $L^\infty \times L^2$
- Hence

$$\begin{cases} u^k \to u, & \text{weak}^* \text{ in } L^\infty_{\mathbb{F}}(0,T; L^2(\Omega; H^\ell)), \quad 0 \le \ell \le m, \\ q^k \to q, & \text{weakly in } L^2_{\mathscr{F}}(0,T; H^\ell)^d, \quad 0 \le \ell \le m-1, \end{cases}$$

and for any $|\alpha| \le m$,

$$\begin{cases} (A - BB^T)^{1/2} D(\partial^\alpha u^k) \to (A - BB^T)^{1/2} D(\partial^\alpha u), \\ B^T[D(\partial^\alpha u^k)] + \partial^\alpha q^k \to B^T[D(\partial^\alpha u)] + \partial^\alpha q, \\ \qquad\qquad \text{weakly in } L^2_{\mathbb{F}}(0,T; H^0). \end{cases}$$

- Taking limits to show that (u, q) satisfies the estimates, with constant $C > 0$ depending only on T, m and K_m.
- Argue that the convergence is strong and (u, q) is a weak solution.

Some Remarks

- The "Symmetry Condition" holds in the following cases:
  - B is symmetric (in this case, it is necessary that $n = d$);
  - $d = n = 1$ (B is a scalar);
  - B is independent of x;
  - $B(t, x) = \varphi(t, x)B_0(t)$, where φ is a scalar random field.

- In Theorem 2, if the symmetric condition on B is replaced by either one of the following conditions: for some $\varepsilon_0 > 0$,

  (i) $A - BB^T \geq \varepsilon_0 BB^T \geq 0$,

  (ii) $A - BB^T \geq \varepsilon_0 \sum_{|\alpha|=1}(\partial^\alpha B)(\partial^\alpha B^T) \geq 0$,

  Then the conclusion of Theorem 2 remains true. Furthermore, if (i) holds, the function F in estimate (4) can be improved to

$$F(t, x; u, q, m) = \sum_{|\alpha| \leq m} \langle A\nabla(\partial^\alpha u), \nabla(\partial^\alpha u) \rangle .$$

Some Direct Consequences

- $m \geq 2 \Longrightarrow$ "weak solution" becomes "strong solution";
- $m > 2 + n/2 \Longrightarrow$ "strong solution" becomes "classical sol.";
- "superparabolic condition" $\Longrightarrow$ "

$$\max_{t\in[0,T]} \mathbb{E}\|u(t,\cdot)\|^2_{H^m} + \mathbb{E}\int_0^T \{\|u(t,\cdot)\|^2_{H^{m+1}} + \|q(t,\cdot)\|^2_{H^m}\}dt$$
$$\leq C\Big\{\|f\|^2_{L^2([0,T]\times\Omega;H^{m-1})} + \|g\|^2_{L^2(\Omega;H^m)}\Big\}.$$

- " Coefficients are all deterministic" $\Longrightarrow q = 0$ and u satisfies

$$\begin{cases} u_t = -\mathscr{L}u - f, & (t,x) \in [0,T] \times \mathbb{R}^n, \\ u\big|_{t=T} = g. \end{cases}$$

For given $\lambda \geq 0$ and $m \geq 1$, we say that the BSPDE $\{\mathscr{L}, \mathscr{M}, f, g, \lambda, m\}$ is *regular* if the following conditions are satisfied:

- Parabolicity condition (2) holds;
- $(\mathrm{H})_m$ holds;
- the "Symmetry Condition" holds for B,
- for $\varphi_\lambda(x) \stackrel{\triangle}{=} e^{-\lambda\langle x \rangle} = e^{-\lambda\sqrt{1+|x|^2}}$, it holds that

$$\varphi_\lambda \cdot f \in L^2_{\mathscr{F}}(0, t; H^m(\mathbb{R}^n)), \quad \varphi_\lambda \cdot g \in L^2_{\mathscr{F}_T}(\Omega; H^m(\mathbb{R}^n)).$$

Since a regular BSPDE $\{\mathscr{L}, \mathscr{M}, f, g, \lambda, m\}$ must have at least a unique adapted weak solution, we denote it by (u, q). If $\bar{A}$, $\bar{B}$, $\bar{a}$, $\bar{h}$, $\bar{c}$ is another set of coefficients that determines the operators $\bar{\mathscr{L}}$ and $\bar{\mathscr{M}}$, we denote the corresponding adapted solution of BSPDE $\{\bar{\mathscr{L}}, \bar{\mathscr{M}}, \bar{f}, \bar{g}, \lambda, m\}$ by $(\bar{u}, \bar{q})$.

Theorem

Assume that for some $\lambda > 0$ and $m \geq 2$, the BSPDEs $\{\mathscr{L}, \mathscr{M}, f, g, \lambda, m\}$ and $\{\bar{\mathscr{L}}, \bar{\mathscr{M}}, \bar{f}, \bar{g}, \lambda, m\}$ are both regular. Let (u, q) and $(\overline{u}, \overline{q})$ be the corresponding adapted strong solutions, respectively. Then for some $\mu > 0$,

$$
\begin{aligned}
& \mathbb{E}\int_{\mathbb{R}^n} \varphi_\lambda(x)\big|[u(t,x) - \overline{u}(t,x)]^-\big|^2 dx \\
\leq\ & e^{\mu(T-t)}\mathbb{E}\int_{\mathbb{R}^n} \varphi_\lambda(x)\big|[g(x) - \overline{g}(x)]^-\big|^2 dx \\
& + E\int_t^T e^{\mu(s-t)}\int_{\mathbb{R}^n} \varphi_\lambda(x)\big|[(\mathscr{L} - \overline{\mathscr{L}})\overline{u}(s,x) + (\mathscr{M} - \overline{\mathscr{M}})\overline{q}(s,x) \\
& + f(s,x) - \overline{f}(s,x)]^-\big|^2 dxds, \quad \forall t \in [0, T].
\end{aligned}
$$

Direct Consequences.

- "$g \ge \overline{g}$" $\oplus$ "$(\mathscr{L} - \overline{\mathscr{L}})\overline{u} + (\mathscr{M} - \overline{\mathscr{M}})\overline{q} + f - \overline{f} \ge 0$" $\Longrightarrow u \ge \overline{u}$.
- "$\mathscr{L} = \overline{\mathscr{L}}$, $\mathscr{M} = \overline{\mathscr{M}}$, $g \ge \overline{g}$, $f \ge \overline{f}$" $\Longrightarrow u \ge \overline{u}$.
- "$g \ge 0$, $f \ge 0$" $\Longrightarrow u \ge 0$.
- "$\overline{A}$, $\overline{B}$, $\overline{a}$, $\overline{h}$ and $\overline{c}$ are independent of x" $\oplus$ "$\overline{f}$ and $\overline{g}$ are convex in x" $\Longrightarrow$ $\overline{u}$ is convex in x.
- "$\overline{A}$, $\overline{B}$, $\overline{a}$, $\overline{h}$, $\overline{c}$, $\overline{f}$, $\overline{g}$ are all deterministic" $\oplus$ "$\overline{u}$ convex in x" $\oplus$ "$\mathscr{M} = \overline{\mathscr{M}}$"

  $\oplus$ "$A(t,x) = \overline{A}(t) + A_0(t,x)$, $c(t,x) = \overline{c}(t) + c_0(t,x)$,
  $f(t,x) = \overline{f}(t,x) + f_0(t,x)$, $g(x) = \overline{g}(x) + g_0(x)$"

  $\oplus$ "$\overline{f} \ge 0$, $\overline{g} \ge 0$, $A_0 \ge 0$, $c_0 \ge 0$, $f_0 \ge 0$, $g_0 \ge 0$"

  $\Longrightarrow u \ge \overline{u}$, where $\overline{u}$ satisfies the PDE

$$\begin{cases} \overline{u}_t = -\overline{\mathscr{L}}u - \overline{f}, & (t,x) \in [0,T] \times \mathbb{R}^n, \\ \overline{u}\big|_{t=T} = \overline{g}. \end{cases}$$

BSPDEs in Stochastic Evolution Equation Form

One can also consider a BSPDE as a BSDE in infinite dimensional space. For example, consider

$$dY_t = -BY_t dt - \psi(t, X_t, Y_t, Z_t)dt + Z_t dW_t, \quad Y_t = g(X_T), \tag{81}$$

where

- W is a cylindrical Wiener process in a Hilbert space $\mathscr{W}$,
- B is the infinitesimal generator of a strongly continuous dissipative compact semigroup $S(t) = e^{Bt}$ in a Hilbert space $\mathscr{K}$, and
- X is a Markov process with infinite dimensional state space $\mathscr{H}$. For example, X could be the solution to the stochastic evolution equation:

$$dX_t = AX_t dt + F(t, X_t)dt + G(t, X_t)dW_t, \quad X_0 = x. \tag{82}$$

Note:

There are differences between the BSPDE studied before and the BSDE in infinite dimensional spaces!

Existing Results:

- Hu-Peng (1991) — Semilinear Backward SEEs
- Pardoux-Rascanu (1999) — Backward stochastic Variational Inequalities
- Fuhrman-Tessitore (2002) — Nonlinear Kolmogorov equations in infinite dimensional spaces
- Confortola (2006) — Dissipative BSDEs in infinite dimensional spaces
- Gurtteris — FBSDEs in infinite dimensional spaces
- Hong-Ma-Zhang — FBSPDEs...

8. BSPDEs vs. FBSDEs

Backward Doubly SDE (BDSDE)

The non-linear Feynman-Kac formula was extended to backward SPDEs via the so-called BDSDE, first by Pardoux-Peng ('95).

Consider the following new probabilistic set-up:

- $(\Omega', \mathscr{F}', \mathbb{P}')$ — another complete probability space;
- B — a (k-dim) Brownian motion;
- $\mathscr{F}^B_{t,T} \stackrel{\triangle}{=} \sigma\{B_s - B_T,\ t \le s \le T\} \vee \mathscr{N}'$, where $\mathscr{N}'$ denotes all $\mathbb{P}'$-null sets in $\mathscr{F}'$. Denote $\mathbb{F}^B_T \stackrel{\triangle}{=} \{\mathscr{F}^B_{t,T}\}_{0 \le t \le T}$.
- $\overline{\Omega} = \Omega \times \Omega'$; $\quad \overline{\mathscr{F}} = \mathscr{F} \otimes \mathscr{F}'$; $\quad \overline{\mathbb{P}} = \mathbb{P} \times \mathbb{P}'$;
- $\overline{\mathscr{F}}_t = \mathscr{F}^W_t \otimes \mathscr{F}^B_{t,T}$, for $0 \le t \le T$.

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- $\overline{\Omega} = \Omega \times \Omega'$; $\quad \overline{\mathscr{F}} = \mathscr{F} \otimes \mathscr{F}'$; $\quad \overline{\mathbb{P}} = \mathbb{P} \times \mathbb{P}'$;
- $\overline{\mathscr{F}}_t = \mathscr{F}^W_t \otimes \mathscr{F}^B_{t,T}$, for $0 \le t \le T$.

Note:

$\overline{\mathbb{F}} \stackrel{\triangle}{=} \{\overline{\mathscr{F}}_t\}_{0 \le t \le T}$ is neither increasing nor decreasing, therefore it is **NOT** a filtration!

Backward Doubly SDE (BDSDE)

- R.v. $\xi(\omega)$, $\omega \in \Omega$ or $\eta(\omega')$, $\omega' \in \Omega'$ is viewed as r.v. in $\overline{\Omega}$ by

$$\xi(\overline{\omega}) = \xi(\omega); \qquad \eta(\overline{\omega}) = \eta(\omega'), \qquad \overline{\omega} \stackrel{\triangle}{=} (\omega, \omega').$$

- Let $\mathscr{M}^2(\overline{\mathbb{F}}, [0,T]; \mathbb{R}^n)$ be the set of n-dim measurable processes $h = \{h_t,\ t \in [0,T]\}$ satisfying

$$\overline{E}\Big\{ \int_0^T |h_t|^2 dt \Big\} < \infty; \text{ and } h_t \in \overline{\mathscr{F}}_t, \text{ for a.e.} t \in [0,T].$$

- For $H \in \mathscr{M}^2(\overline{\mathbb{F}}, [0,T]; \mathbb{R}^n)$ and $j = 1, \cdots, k$, we denote $\int_s^t H_r \downarrow dB_r^j$ to be the *backward stoch. integral* against B^j.

Note:

The "backward integral" can be understood as a Skorohod integral. But if H is $\mathbb{F}^B$-adapted, then it is a "time-reversed" standard Itô integral from t to s, adapted to $\mathbb{F}^B$!

Consider now the following FBSDE: for $(t, x) \in [0, T] \times \mathbb{R}^n$, and $s \in [t, T]$,

$$\begin{aligned} X_s^t(x) &= x + \int_t^s b(X_r^t(x))dr + \int_t^s \sigma(X_r^t(x))dW_r, \qquad (83) \\ Y_s^t(x) &= u_0(X_T^t(x)) + \int_s^T f(r, X_r^t(x), Y_r^t(x), Z_r^t(x))dr \\ &\quad + \int_s^T \langle\, g(r, X_r^t(x), Y_r^t(x), Z_r^t(x)), \downarrow dB_r \,\rangle \qquad (84) \\ &\quad - \int_s^T \langle\, Z_r^t(x), dW_r \,\rangle, \end{aligned}$$

where u_0 is a deterministic function. This is the so-called *backward doubly SDE* proposed by Pardoux-Peng in 1995.

Theorem (Pardoux-Peng)

Under the standard assumptions on the coefficients, for each $(t,x) \in [0,T] \times \mathbb{R}^n$ the BDSDE (83) has a unique solution $(X^t(x), Y^t(x), Z^t(x))$ such that

- $\exists \alpha \in (0, \frac{1}{2})$, $\forall t > 0$, $(s,x) \mapsto X^t_s(x)$ is locally Hölder-$C^{\alpha,\alpha/2}$;
- $\forall q \geq 2$, $\exists M_q > 0$, s.t. for $t \in [0,T]$ and $x, x' \in \mathbb{R}^n$,

$$\mathbb{E}\Big\{\sup_{t\leq r\leq s} |X^t_r(x) - x|^q\Big\} \leq M_q(s-t)(1+|x|^q),$$

$$\mathbb{E}\Big\{\Big[\sup_{t\leq s\leq T} |Y^t_s(x)|^2 + \int_t^T |Z^t_s(x)|^2 ds\Big]^{q/2}\Big\} \leq M_q(1+|x|^q);$$

$$\mathbb{E}\Big\{\sup_{t\leq r\leq s} |(X^t_r(x) - X^t_r(x')) - (x-x')|^q\Big\} \leq M_q(s-t)(|x-x'|^q);$$

- $Y^t_s(x) = Y^r_s(X^t_r(x))$, $Z^t_s(x) = Z^r_s(X^t_r(x))$, a.e. $s \in [0,r]$, a.s.;

We note that, unlike the single BSDE case, if we define

$$u(t,x) \stackrel{\triangle}{=} Y_t^t(x), \qquad (t,x) \in [0,T] \times \mathbb{R}^n,$$

then by the Blumenthal $0-1$ law, this is a random field on the probability space $(\Omega', \mathscr{F}', \mathbb{P}')$, and for each $x \in \mathbb{R}^n$, the mapping $t \mapsto u(t,x)$ is $\mathscr{F}_t^B$-measurable. Namely, with a time-reversal, this is a progressively measurable random field w.r.t. the filtration $\mathbb{F}^B$.

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With the help of Malliavin Calculus, it was first argued in Pardoux-Peng ('94) that, if the coefficients are smooth enough, then the sol. $(X^t(x), Y^t(x), Z^t(x))$ has the following regularity:

- $\sup_{t\le s\le T}\{|X_s^t(x)| + |\nabla_x X_s^t(x)| + |D_{xx}^2 X_s^t(x)|\} \in \cap_{p\ge 1} L^p(\Omega')$
- $(s,t,x) \mapsto Y_s^t(x)$ belongs to $C^{0,0,2}([0,T]^2 \times \mathbb{R}^n)$;
- $(s,t,x) \mapsto Z_s^t(x)$ belongs to $C([0,T]^2 \times \mathbb{R}^n)$, and

$$Z_s^t(x) = \nabla Y_s^t(x)(\nabla X_s^t(x))^{-1}\sigma(X_s^t(x)) \Longrightarrow Z_t^t(x) = u_x(t,x)\sigma(x).$$

Theorem (Pardoux-Peng, '94)

Assume that the coefficients of BDSDE (83) are smooth, and let $(X^t(x), Y^t(x), Z^t(x))$ be the unique solution to (83). Then $u(t,x) \stackrel{\triangle}{=} Y^t_t(x)$ is the unique classical solution to the (backward) SPDE on the space $(\Omega', \mathscr{F}', \mathbb{P}'; \mathbb{F}^B)$:

$$\begin{aligned} du(t,x) &= -\big\{\mathscr{A}u(t,x) + f(t,x,u(t,x),\sigma^*(x)\nabla u(t,x))\big\}dt \\ &\qquad + \langle\, g(t,x,u(t,x),\sigma^*(x)\nabla u(t,x)), \downarrow dB_t\,\rangle, \\ u(T,x) &= u_0(x), \end{aligned} \tag{85}$$

where $\mathscr{A}$ is the second order differential operator:

$$\mathscr{A} = \frac{1}{2}\sum_{i,j=1}^{n}\sum_{\ell=1}^{k}\sigma_{i\ell}(x)\sigma_{j\ell}(x)\partial^2_{x_i x_j} + \sum_{i=1}^{n} b_i(x)\partial_{x_i}.$$

Remark

A more interesting connection between the BDSDEs and SPDEs is when the coefficients are **NOT** smooth. In light of the non-linear Feynman-Kac formula, one would expect that in such a case the random field $u(t,x) = Y_t^t(x)$ should give the "Stochastic Viscosity Solution" to the BSPDE (85). This was done in Buckdahn-Ma (2001-2002).

Consider the following FBSDE with random coefficients: for $t \in [0, T]$,

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t; \\ dY_t = -[\hat{b}_1(t, X_t)Y_t + \widehat{b}_2(t, X_t)Z_t]dt - Z_t dW_t, \\ X_0 = x, \quad Y_T = g(X_T), \end{cases} \tag{86}$$

where b, $\hat{b}_1$, $\hat{b}_2$, and σ are all random fields.

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Objective

- Find square-integrable processes (X, Y, Z) such that they are adapted to $\{\mathscr{F}_t\}$, and satisfies (86) almost surely.
- Determine, if possible, the relations among X, Y, and Z.

Assume sufficient regularity of the coefficients b, σ, $\widehat{b}_1$, $\widehat{b}_2$, and g. In light of "Four Step Scheme" we first solve BSPDE (1) with

$$\begin{aligned} A(t,x) &= \sigma^2(t,x), \quad a(t,x) = b(t,x) + \sigma(t,x)\widehat{b}_2(t,x), \\ c(t,x) &= \widehat{b}_1(t,x), \quad B(t,x) = \sigma(t,x), \quad h(t,x) = -\widehat{b}_2(t,x). \end{aligned}$$

and denote its adapted (classical) solution by (u,q). Then, let X be the solution to the forward SDE in (86), and define

$$Y_t = u(t, X_t, \cdot); \quad Z_t = q(t, X_t, \cdot) + \sigma(t, X_t, \cdot)\nabla u(t, X_t, \cdot),$$

Using Itô-Ventzell Formula, one shows that (X, Y, Z) solves (86)!

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Using Itô-Ventzell Formula, one shows that (X,Y,Z) solves (86)!

Note:

In this case $A(t,x) - BB^T(t,x) = \sigma\sigma^T(t,x) - \sigma\sigma^T(t,x) \equiv 0$, and $B(t,x) \neq 0$ (i.e., $\mathscr{M}$ is unbounded)!

BSPDEs and FBSDEs

Recall that

- $D : L^2(\Omega) \mapsto L^2([0,T] \times \Omega)$ — the Malliavin derivation operator,
- $\mathbb{D}_{1,p}$, $p \geq 2$ — the set of all $\xi \in L^2(\Omega)$ such that

$$\|\xi\|_{1,p} = \|\xi\|_{L^p(\Omega)} + \big\| \|D\xi\|_{L^2([0,T])} \big\|_{L^p(\Omega)} < \infty.$$

Theorem

Under suitable technical conditions, the solutions (X, Y, Z) to FBSDE and (u, q) to BSPDE satisfy the following relations:

- the process $u(\cdot\,, X_\cdot\,, \cdot) \in \mathbb{D}_{1,2}$;
- $D_t u(t, X_t, \cdot) = D_t Y_t = Z_t = q(t, X_t, \cdot) + \sigma(t, \cdot)\nabla u(t, X_t, \cdot)$;
- $q(t, X_t, \cdot) = [D_t u](t, X_t, \cdot)$, $t \in [0, T]$, -a.s. , where $[D_t u](t, X_t(\omega), \omega) \stackrel{\triangle}{=} D_t u(t, x, \omega)\big|_{x = X_t(\omega)}$.

Some Remarks

- The theorem regarding BSPDE and FBSDE can be thought of as a Stochastic Feynman-Kac Formula.
- An immediate application in Finance would be the Stochastic Black-Scholes Formula (Ma-Yong, book)
- The Comparison Theorem could be used to prove the Convexity of the European Contingent Claims and the Robustness of Black-Scholes Formula, along the lines of El Karoui-Jeanblanc-Shreve (1999)
- The well-posedness of BSPDEs with similar type (or Stochastic Feynman-Kac formula) was extended to semilinear case (Hu-Ma-Yong, 2004)
- Quasilinear case (or fully coupled FBSDEs) is still not known so far.

General Quasi-linear/Random Coefficient Cases

Consider the following FBSDE with possibly random coefficients:

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s)ds + \int_0^t \sigma(s, X_s, Y_s)dW_s; \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \end{cases} \tag{87}$$

In the decoupled case, the FBSDE becomes

$$\begin{cases} X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s; \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \end{cases} \tag{88}$$

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Definition

We say that FBSDE (87) is well-posed if it has a unique solution for any initial value (t, x) and $|\nabla_x \theta| \le C$, where $\theta(t, x)$ is the random field determined by $Y_t = \theta(t, X_t)$.

Assume the random field θ is smooth and takes the following form:

$$d\theta(t,x) = \alpha(t,x)dt + \beta(t,x)dW_t.$$

Applying Itô-Ventzell formula we get

$$d\theta(t,X_t) = [\alpha + \theta_x b + \frac{1}{2}\theta_{xx}\sigma^2 + \beta_x\sigma]dt + [\beta + \theta_x\sigma]dW_t.$$

Then formally we should have

$$Y_t = \theta(t,X_t), \quad Z_t = \beta(t,X_t) + \theta_x(t,X_t)\sigma(t,X_t,\theta(t,X_t)), \quad (89)$$

and

$$\alpha + \theta_x b + \frac{1}{2}\theta_{xx}\sigma^2 + \beta_x\sigma + f(\cdot,\theta,\beta + \theta_x\sigma(\cdot,\theta)) = 0.$$

Thus we may consider the following "decoupling" BSPDE

$$\begin{cases} d\theta(t,x) = -[\frac{1}{2}\theta_{xx}\sigma^2 + \beta_x\sigma + u_x b + f]dt + \beta dW_t; \\ \theta(T,x) = g(x). \end{cases} \tag{90}$$

Corresponding to the well-posedness of the FBSDE, we should have

Definition

We say that θ is a weak solution to (90) if θ_x is bounded and there exists β in L^2 such that, for any "good" function φ on $\mathbb{R}$, it holds:

$$d\int_{\mathbb{R}} \theta(t,x)\varphi(x)dx = \int_{\mathbb{R}} [\frac{1}{2}\theta_x(\sigma^2\varphi)_x + \beta(\sigma\varphi)_x - \theta_x b\varphi + f\varphi]dxdt + \int_{\mathbb{R}} \beta\varphi(x)dxdW_t. \tag{91}$$

A Conjecture

Theorem (Ma-Zhang, 2009)

Assume that b, σ, f, g are uniformly Lipschitz continuous in (x, y, z), and b, σ are bounded. Then

(i) If (90) has a weak solution, then FBSDE (73) has a solution defined by (89).

(ii) FBSDE (73) is wellposed if and only if (90) has a unique weak solution.

(iii) (90) has at most one weak solution.

In particular, if the FBSDE is decoupled, then the corresponding BSPDE (90) has a unique weak solution and (89) holds.

Note that in this case (88) is always wellposed. And if b, σ, f, g are smooth enough, then (90) has a unique classical solution and (89) holds.

Lemma

Assume θ is a classical solution to (90). Then for any good positive φ with $K_\varphi \triangleq \sup_x \Big[\big|\frac{\varphi_x(x)}{\varphi(x)}\big| + \big|\frac{\varphi_{xx}(x)}{\varphi(x)}\big|\Big] < \infty$, there exists a constant C_φ depending only on K_φ and the bounds of the coefficients, such that

$$\mathbb{E}\Big\{\sup_t \int_{\mathbb{R}} |\theta^2(t,x)|^2 \varphi(x)dx + \int_0^T \int_{\mathbb{R}} |[\beta + \theta_x \sigma](t,x)|^2 \varphi(x) dxdt\Big\}$$
$$\leq C_\varphi E\Big\{\int_{\mathbb{R}} |g(x)|^2 \varphi(x) dx + \int_0^T \int_{\mathbb{R}} |f(t,x,0,0)|^2 \varphi(x) dxdt\Big\}.$$

Comparison Theorems

Consider the following FBSDEs, with $\Theta^i = (X^i, Y^i, Z^i)$, $i = 1, 2$:

$$\begin{cases} X_t^i = x + \int_0^t b(s, (W^i)_s, \Theta_s^i)ds + \int_0^t \sigma(s, (W^i)_s, X_s^i, Y_s^i)dW_s^i; \\ Y_t^i = g_1((W^i)_T, X_T^i) + \int_t^T f_1(s, (W^i)_s, \Theta_s^i)ds - \int_t^T Z_s^i dW_s^i; \end{cases} \tag{92}$$

Theorem

Assume that

(i) b, σ, f_2, g_2 *are uniformly Lipschitz continuous in* (x, y, z)*;*

(ii) *FBSDE(92)-2 is wellposeded, and* $Y_t^2 \triangleq \theta(t, (W^2)_t, X_t^2)$, *where* θ *is uniformly Lipschitz continuous in* x*;*

(iv) *(92)-1 has a weak solution;*

(v) $f_1(t, (\omega)_t, \xi) \leq f_2(t, (\omega)_t, \xi)$ *and* $g_1((\omega)_T, x) \leq g_2((\omega)_T, x)$, *for any* $\omega \in C[0, T]$ *and any* $\xi = (x, y, z)$.

Then we have $Y_t^1 \leq \theta(t, (W^1)_t, X_t^1)$. *In particular,* $Y_0^1 \leq Y_0^2$.

Fully Nonlinear PDEs and 2BSDEs

Consider the following fully nonlinear parabolic PDE:

$$u_t + H(t, x, u, Du, D^2u) = 0, \qquad u(T, x) = g(x) \tag{93}$$

Finding numerical method for such a PDE is rather challenging, especially in higher dimensional case.

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A Feynman-Kac Formula (Cheridito-Soner-Touzi-Victoir, '06)

- Let $X_t = x + W_t$, and let u be a (smooth) solution to (93).
- Define $Y_t = u(t, X_t)$, $Z_t = Du(t, X_t)$, $\Gamma_t = D^2u(t, X_t)$, $A_t = [Du_t + D^3u](t, X_t)$. Then, applying Itô, one has

$$\begin{aligned} dY_t &= du(t, X_t) = [u_t + \frac{1}{2}D^2u](t, X_t)dt + Du(t, X_t)dW_t \\ dZ_t &= [Du_t + \frac{1}{2}D^3u](t, X_t)dt + D^2u(t, X_t)dW_t \\ &= A_t dt + \Gamma_t dW_t. \end{aligned} \tag{94}$$

Fully Nonlinear PDEs and 2BSDEs

Note that if we use the Stratonovic integral:

$$Z_t \circ dW_t = Z_t dW_t + \frac{1}{2} d \langle Z_t, W_t \rangle = Z_t dW_t + \frac{1}{2} D^2 u(t, X_t) dt,$$

it would be more convenient to write

$$\frac{1}{2} D^2 u(t, X_t) dt + Z_t dW_t = Z_t \circ dW_t,$$

and thus (94) becomes

$$\begin{aligned} Y_t &= g(X_T) + \int_t^T H(s, X_s, u, Du, D^2 u) ds - \int_t^T Z_s \circ dW_s; \\ dZ_t &= A_t dt + \Gamma_t dW_t. \end{aligned} \tag{95}$$

The BSDE (95) is called the Second Order BSDE or simply 2BSDE.

To this point the 2BSDEs in which $\gamma \mapsto H(t,x,y,z,\gamma)$ is convex have found most applications. In particular when H can be written as the following Fenchel-Legendre transform:

$$H(t,x,y,z,\gamma) = \sup_{\underline{a} \le a \le \bar{a}} \{\frac{1}{2}a^2\gamma + f(t,x,y,a)\},$$

the 2BSDE seem to have the potential of becoming a powerful new tool. The subjects where 2BSDEs seem to be useful include:

- Super-hedging problems under liquidity risk
- G-expectations, G-Martingale Representations, and G-BSDEs
- Dynamic Risks under volatility uncertainty
- Stochastic optimization under volatility uncertainty
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- Ask Touzi for more ...

References

F. Antonelli, *Backward-forward stochastic differential equations*, *Ann. Appl. Probab.*, 3 (1993), no. 3, 777–793.

Antonelli, F. and Ma, J. (2003), *Weak solutions of forward-backward SDE's*, *Stochastic Analysis and Applications*, **21**, no. 3, 493–514.

Barrieu, P. and El Karoui, N. (2004), *Optimal derivatives design under dynamic risk measures*, Mathematics of finance, 13–25, Contemp. Math., 351, Amer. Math. Soc., Providence, RI, 2004.

Briand, P. and Hu, Y. (2006), *BSDE with quadratic growth and unbounded terminal value*, *Probab. Theory Relat Fields*, **136**(4), 604–618.

Briand, P. and Hu, Y. (2008), *Quadratic BSDEs with convex generators and unbounded terminal conditions*, *Probab. Theory Relat Fields*, **141**, 543–567.

References

- Briand, P., Lepeltier, J-P., San Martin, J. (2007), *One-dimensional backward stochastic differential equations whose coefficient is monotonic in y and non-Lipschitz in z*, *Bernoulli*, **13**, no. 1, 80–91.

- Buckdahn, R.; Engelbert, H.-J. *A backward stochastic differential equation without strong solution*, *Theory Probab. Appl.* 50 (2006), no. 2, 284–289.

- Buckdahn, R.; Engelbert, H.-J. *On the continuity of weak solutions of backward stochastic differential equations.*, *Theory Probab. Appl.* 52 (2008), no. 1, 152–160

- Buckdahn, R.; Engelbert, H.-J.; Rascanu, A. (2004), *On weak solutions of backward stochastic differential equations*, *Teor. Veroyatn. Primen.*, **49**, no. 1, 70–108.

- Coquet, F., Hu, Y., Mémin, J., and Peng, S. (2002), *Filtration-consistent nonlinear expectations and related*

References

- Delarue, F., (2002) *On the existence and uniqueness of solutions to FBSDEs in a non-degenerate case*, *Stochastic Process. Appl.*, **99**, 209-286.
- Delarue, F.; Guatteri, G. (2006), *Weak existence and uniqueness for forward-backward SDEs. Stochastic Process. Appl.* **116**(12), 1712–1742.
- Hamadene, S.; Lepeltier, J.-P.; Matoussi, A., (1997), *Double barrier backward SDEs with continuous coefficient*, *Backward stochastic differential equations (Paris, 1995–1996)*, 161–175, Pitman Res. Notes Math. Ser., 364. Longman, Harlow.
- Hu, Y. and Ma, J., (2004), *Nonlinear Feynman-Kac formula and discrete-functional-type BSDEs with continuous coefficients*, *Stochastic Process. Appl.*, **112**, no. 1, 23–51.
- Hu, Y., Ma, J., Peng, S., and Yao, S. (2008), *Representation theorems for quadratic $\mathscr{F}$-consistent nonlinear expectations*,

Hu, Y., Ma, J., and Yong, J. (2002), *On semi-linear degenerate backward stochastic partial differential equations*, *Probab. Theory Related Fields*, **123**, no. 3, 381–411.

Hu, Y. and Peng, S. (1995), *Solution of forward-backward stochastic differential equations*, *Prob. Theory and Rel. Fields*, v.103, **2**, 273–283.

Kazamaki, N. (1994), *Continuous exponential martingales and BMO*, Lecture Notes in Math, 1579, Springer-Verlag, Berlin.

Kobylanski, M. (2000), *Backward stochastic differential equations and partial differential equations with quadratic growth. Ann. Probab.* **28**, no. 2, 558–602.

Kobylanski, M.; Lepeltier, J. P.; Quenez, M. C.; Torres, S. (2002), *Reflected BSDE with superlinear quadratic coefficient*, *Probab. Math. Statist.*, **22**, no. 1, Acta Univ. Wratislav. No. 2409, 51–83.

References

- Lepeltier, M. and San Martin, J. (1997), *Backward stochastic differential equations with nonLipschitz coefficients*, *Stat. and Prob. Letters*, v.32, **4**, 425–430.
- Lepeltier, J.-P.; San Martin, J. (1998),*Existence for BSDE with superlinear-quadratic coefficient*, *Stochastics Stochastics Rep.*, **63**, no. 3-4, 227–240.
- Lepeltier, J-P. and San Martin, J., (2002), *On the existence or non-existence of solutions for certain backward stochastic differential equations*, *Bernoulli*, **8**, no. 1, 123–137.
- Ma, J. and Cvitanic, J., (2001), *Reflected forward-backward SDEs and obstacle problems with boundary conditions*, *J. Appl. Math. Stochastic Anal.*, **14**, no. 2, 113–138.
- Ma, J., Protter, P. and Yong, J. (1994), *Solving forward-backward stochastic differential equations explicitly - a four step scheme*, *Probab. Theory Relat. Fields*, **98**, 339-359.

References

- Ma, J. and Yong, J. (1995), *Solvability of forward-backward SDEs and the nodal set of Hamilton-Jaccobi-Bellman Equations*, *Chin. Ann. Math*, **16B** 279–298.
- Ma, J. and Yong, J., (1997), *Adapted solution of a degenerate backward SPDE, with applications*, *Stochastic Process. Appl.*, **70**, no. 1, 59–84.
- Ma, J. and Yong, J. (1999), *On linear, degenerate backward stochastic partial differential equations*, *Probab. Theory Related Fields*, **113**, no. 2, 135–170.
- Ma, J. and Yong, J. (1999), *Forward-Backward Stochastic Differential Equations and Their Applications*, Lecture Notes in Math., **1702**, Springer.
- Ma, J. and Zhang, J. (2002), *Representation theorems for backward stochastic differential equations*, *Ann. Appl. Probab.*, **12**, no. 4, 1390–1418.

References

- Ma, J. and Zhang, J. (2002), *Path regularity for solutions of backward stochastic differential equations*, *Probab. Theory Related Fields*, **122**, no. 2, 163–190.
- Ma, J. and Zhang, J. (2005), *Representations and regularities for solutions to BSDEs with reflections*, *Stochastic Process. Appl.*, **115**, no. 4, 539–569.
- Ma, J., Zhang, J., and Zheng, Z. (2008), *Weak solutions for forward-backward SDEs—a martingale problem approach. Ann. Probab.* **36**, no. 6, 2092–2125.
- E. Pardoux and S. Peng S., (1990), *Adapted solutions of backward stochastic equations*, *System and Control Letters*, **14**, 55-61.
- E. Pardoux and S. Peng, (1992), *Backward stochastic differential equations and quasilinear parabolic partial differential equations*, *Lecture Notes in CIS*, Springer, 176,

- E. Pardoux and S. Tang, (1999), *Forward-backward stochastic differential equations and quasilinear parabolic PDEs*, *Probab. Theory Related Fields*, **114**, no. 2, 123–150.

- S. Peng and Z. Wu, (1999), *Fully coupled forward-backward stochastic differential equations and applications to optimal control*, *SIAM J. Control Optim.*, **37**, no. 3, 825–843.

- J. Yong, (1997), *Finding adapted solutions of forward-backward stochastic differential equations: method of continuation*, *Probab. Theory Related Fields*, **107**, no. 4, 537–572.

- J. Zhang, (2006), *The wellposedness of FBSDEs. Discrete Contin. Dyn. Syst. Ser. B*, **6**, no. 4, 927–940 (electronic).

THANK YOU VERY MUCH!