Impulse control problem with switching technology

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- **Problem:** The firm owner decides to switch technology in random time.

- **Consequences:**
  - Switching technology $\Rightarrow$ Stopping time of system $\Rightarrow$ Impulse.
  - The system is revived with a new technology.

- **Objective:** To optimize the firm profit.

- **Observation:**
  - Markovian and homogeneous character between two impulse moments.
  - Markovian form of each revival.
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  - Markovian form of each revival.
The impulse control (or the admissible strategy) has the form:

$$\alpha = (\tau_n, \zeta_{n+1}, \Delta_n, n \geq -1).$$

The control variable has three components:

- Impulse moments: $$(\tau_n)_{n \geq -1}$$ an increasing sequence of stopping times which converges to the default time $\tau$ such that $\tau_{-1} = 0$ and $\tau_{n+1} = \tau_n + \tau_0 \circ \phi_{\tau_n}$.
- $\zeta_{n+1}$ the technology choice at time $\tau_n$.
- $\Delta_n$ the jump size.
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- \(\zeta_{n+1}\) the technology choice at time \(\tau_n\).
- \(\Delta_n\) the jump size.
• \((\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})\): a probability space.
• \((\mathcal{F}_t)_{t \geq 0}\): a right continuous complete filtration.
• \((\mathcal{G}_t)_{t > 0}\): a predictable filtration \(\mathcal{G}_t = \vee_{s < t} \mathcal{F}_s, \ \forall \ s < t\).

The càdlàg process \(\xi_t\) indicates the technology at time \(t\):

\[
\xi_t = \xi_0 1_{[0, \tau_0]}(t) + \sum_{n \geq 0} \zeta_{n+1} 1_{[\tau_n, \tau_{n+1}]}(t) + \emptyset 1_{[\tau, +\infty]}(t).
\]

This process takes its values in \(U\), the finite space of possible technologies. \(\zeta_{n+1}\) is a \(\mathcal{G}_{\tau_n}\)-measurable random variable and \((\tau_n)_{n \geq -1}\) is a sequence of \(\mathcal{G}\)-stopping times. Note \(\overline{U} = U \cup \{\emptyset\}\).
The firm value is given by $S_t = \exp Y_t$, $t \geq 0$, where $Y$ is the càdlàg process defined as

$$
Y_t = Y_0 1_{[0, \tau_0]}(t) + \sum_{n \geq 0} \Delta_n 1_{[\tau_n, \tau_{n+1}]}(t)
+ \int_0^t (b(\xi_s) \, ds + \sigma(\xi_s) \, dW_s) + \{-\infty\} 1_{[\tau, +\infty]}(t),
$$

with $\Delta_n$ is the firm log value jump size, a $\mathcal{G}_{\tau_n}$-measurable random variable. Let $\overline{\mathbb{R}}$ be $\mathbb{R} \cup \{-\infty\}$. Denote by $r^\alpha$ the transition probability from $(\zeta_n, Y_{\tau_n^-})$ to $(\zeta_{n+1}, Y_{\tau_n})$:

$$
\mathbb{P}(\zeta_{n+1} = j, Y_{\tau_n} = x + dy \mid \zeta_n = i, Y_{\tau_n^-} = x) = r^\alpha(i, x; j, dy).
$$
For each strategy $\alpha$, the profit is:

$$k(\alpha) = \int_0^T e^{-\beta s} f(\xi_s, Y_s) \, ds - \sum_{n \geq 0} e^{-\beta \tau_n} c(\zeta_n, Y_{\tau_n^-}, \zeta_{n+1}, Y_{\tau_n})$$

(1)

where

- $\beta > 0$ is a discount coefficient.
- The function $f$ represents the firm net profit.
- The function $c$ is the switching technology cost with $c(i, x, i, x) = 0$.

The expected profit of the firm is defined as:

$$K(\alpha) = \mathbb{E}(k(\alpha) | \xi_0 = i, Y_0 = x).$$

(2)
Goal

Find an admissible strategy $\hat{\alpha}$ which maximizes the expected total profit $K(\alpha)$ defined in (2), i.e.:

$$K(\hat{\alpha}) = \text{ess sup}_{\alpha \in D} K(\alpha),$$

(3)

where $D$ is the admissible strategy set.
The maximum conditional profit

Definition

We call maximum conditional profit the family defined a.s. as:

\[ F_{\theta}^{\alpha} = \text{ess} \sup_{\{\mu_t = \alpha_t, t < \theta\}} \mathbb{E}(k(\mu) | G_\theta). \]

Respectively,

\[ F_{\theta}^{\alpha^+} = \text{ess} \sup_{\{\mu_t = \alpha_t, t \leq \theta\}} \mathbb{E}(k(\mu) | F_\theta). \]
Proposition

The maximum conditional profit $F_{\theta}^\alpha$ (resp. $F_{\theta}^{\alpha+}$) is a positive supermartingale, meaning that $F_{\theta}^\alpha$ (resp. $F_{\theta}^{\alpha+}$) is $\mathbb{P}$-integrable and

$$
\mathbb{E}(F_{\theta}^\alpha | \mathcal{G}_\gamma) \leq F_{\gamma}^\alpha \quad \text{(resp. } \mathbb{E}(F_{\theta}^{\alpha+} | \mathcal{F}_\gamma) \leq F_{\gamma}^{\alpha+} \text{ )}.
$$

Corollary (First optimality criterion (N. El Karoui, 1981))

A necessary and sufficient condition for a strategy $\hat{\alpha}$ to be optimal is that the maximum conditional profit $F_{\hat{\alpha}}^{\alpha+}$ is a martingale.
Proposition

The maximum conditional profit $F_\theta^\alpha$ (resp. $F_\theta^\alpha^+$) is a positive supermartingale, meaning that $F_\theta^\alpha$ (resp. $F_\theta^\alpha^+$) is $\mathbb{P}$-integrable and

$$\mathbb{E} (F_\theta^\alpha | G_\gamma) \leq F_\gamma^\alpha \quad \left(\text{resp. } \mathbb{E} (F_\theta^\alpha^+ | F_\gamma) \leq F_\gamma^\alpha^+\right).$$

Corollary (First optimality criterion (N. El Karoui, 1981))

A necessary and sufficient condition for a strategy $\hat{\alpha}$ to be optimal is that the maximum conditional profit $F_{\hat{\alpha}}^+$ is a martingale.
Definition

We call maximum conditional profit after $\theta > 0$ (respectively right after $\theta$ or $\theta = 0$), the family defined a.s. as:

$$W^{\alpha}_\theta = \text{ess sup}_{\{\mu_t = \alpha_t, \forall t < \theta\}} \mathbb{E}[k_\theta(\mu) | G_\theta],$$

where $k_\theta$ is the profit after $\theta$, (respectively, for all $\theta \geq 0$:

$$W^{\alpha+}_\theta = \text{ess sup}_{\{\mu_t = \alpha_t, \forall t \leq \theta\}} \mathbb{E}[k_{\theta+}(\mu) | F_\theta],$$

where $k_{\theta+}$ is the profit right after $\theta$).
Lemma

We have the following equalities:

\[ F_\theta^\alpha = (k(\alpha) - k_\theta(\alpha)) + W_\theta^\alpha, \quad \theta > 0. \]

\[ F_\theta^{\alpha^+} = (k(\alpha) - k_{\theta^+}(\alpha)) + W_\theta^{\alpha^+}. \]
Proposition

For any strategy $\alpha$ and $0 < \gamma \leq \theta$, we have a.s.

$$W^\alpha_\gamma \geq \mathbb{E}[k_\gamma(\alpha) - k_\theta(\alpha) + W^\alpha_\theta | G_\gamma].$$ (4)

Respectively, for $0 \leq \gamma \leq \theta$, we get a.s.

$$W^{\alpha+}_\gamma \geq \mathbb{E}[k_{\gamma+}(\alpha) - k_{\theta+}(\alpha) + W^{\alpha+}_\theta | F_\gamma].$$ (5)

Moreover, $\hat{\alpha}$ is optimal if and only if equality (5) holds for every couple $(\gamma, \theta)$. 

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We introduce $M = \bigcup_{(i,x) \in U \times \mathbb{R}} M(i,x)$ where $M(i,x)$ verifies:

\[
\left\{ \begin{array}{l}
M(i,x) = \left\{ r^\alpha(i,x;\ldots), \delta_{i,x}; \alpha \in D \right\} \quad \text{if } (i, x) \neq (\emptyset, \Delta) \\
M(\emptyset,\Delta) = \delta(\emptyset,\Delta)
\end{array} \right.
\]

We recall that $r^\alpha$ is a transition probability from couple $(\zeta_n, Y_{\tau_n^-})$ to $(\zeta_{n+1}, Y_{\tau_n})$.

**Hypothesis**

The set $M$ is weakly closed, weakly compact and separable.
Theorem: Second optimality criterion

For any strategy $\alpha$ we have a.s. the following inequalities:

$$W_{\alpha}^{\tau_n} \geq \mathbb{E} \left( \int_{\tau_n}^{\tau_{n+1}} e^{-\beta s} f(\xi_s, Y_s) \, ds - e^{-\beta \tau_0} c(\xi_0, Y_{\tau_0^-}, \zeta_1, Y_{\tau_0}) \bigg| \mathcal{F}_0 \right)$$

$$+ \mathbb{E} \left( W_{\alpha}^{\tau_0^-} \bigg| \mathcal{F}_0 \right)$$

$$W_{\tau_n}^{\alpha} \geq -e^{-\beta \tau_n} \int_{U \times \mathbb{R}} c(\zeta_n, Y_{\tau_n^-}, i, x) r^{\alpha}(., i, dx) + \mathbb{E} \left( W_{\tau_n}^{\alpha} \bigg| G_{\tau_n} \right)$$

Moreover, the strategy $\hat{\alpha}$ is optimal if and only if equality has place simultaneously in the following three inequalities.
Corollary

We have the following equalities:

\[ W^\alpha_{\tau_n} = e^{-\beta \tau_n} \rho(\zeta_n, Y_{\tau_n}), \quad \forall \ n \geq 0, \]

where \( \rho(i, x) = \text{ess sup}_{\mu \in \mathcal{D}} \mathbb{E}_{\{i, x\}}(k(\mu)). \)

\[ W^{\alpha+}_{\tau_n} = e^{-\beta \tau_n} \rho^+(\zeta_{n+1}, Y_{\tau_n}), \quad \forall \ n \geq -1, \]

where \( \rho^+(i, x) = \text{ess sup}_{\{\mu \in \mathcal{D}, \zeta_1^\mu \neq \emptyset\}} \mathbb{E}_{\{i, x\}}(k(\mu)). \)
Proposition (Lepeltier-Marchal, 1984)

For any strategy $\alpha$ and any $n \geq 0$, we have

$$W_{\tau_n}^\alpha = e^{-\beta \tau_n} m\rho^+(\zeta_n, Y_{\tau_n^-}) \text{ a.s.}$$

where $m\rho^+$ is the operator defined by

$$(i, x) \rightarrow \text{ess sup} \int_{\overline{U} \times \mathbb{R}} \nu(i, x; j, dy) \left( -c(i, x, j, y) + \rho^+(j, y) \right).$$

Moreover, the value function $\rho(i, x)$ is equal to $m\rho^+(i, x)$. 
Proposition

The application $\rho^+$ does not depend on the strategy $\alpha$ and satisfies the following equation:

$$\rho^+(i, x) = \text{ess sup}_{T > 0, T \in \mathbb{R}_{-1}} \mathbb{E}_{\{i, x\}} \left( \int_0^T (i, x, .) e^{-\beta s} f(i, Y_s) \, ds \right)$$

$$+ e^{-\beta T((i, x), .)} m \rho^+(i, Y_{T^-(i, Y_s,.)}),$$

where $\mathbb{R}_{-1}$: Set of measurable applications $T$ on $\overline{U} \times \overline{\mathbb{R}} \times \Omega$ such that $T((i, x), .)$ is a $\mathcal{G}$-stopping time.
Theorem: Optimality criterion

For any strategy $\alpha$, we have the following inequalities:

\[
\rho^+(i, x) \geq \mathbb{E}_{\{i,x\}} \left( \int_0^{\tau_0} e^{-\beta s} f(i, Y_s) \, ds \right.
\]
\[
+ \left. e^{-\beta \tau_0} m \rho^+(i, Y_{\tau_0}) \right).
\] (7)

\[
m \rho^+(\zeta_n, Y_{\tau_n^-}) \geq \int_{\mathcal{U} \times \mathbb{R}} r^\alpha(\zeta_n, Y_{\tau_n^-}, i, dx) \left( - c(\zeta_n, Y_{\tau_n^-}, j, y) \n\right.
\]
\[
+ \left. \rho^+(j, y) \right). \] (8)

\[
e^{-\beta \tau_n} \rho^+(\zeta_{n+1}, Y_{\tau_n}) \geq \mathbb{E}_{\{\zeta_{n+1}, Y_{\tau_n}\}} \left( \int_{\tau_n}^{\tau_{n+1}} e^{-\beta s} f(\zeta_{n+1}, Y_s) \, ds \right.
\]
\[
+ \left. e^{-\beta \tau_{n+1}} m \rho^+(\zeta_{n+1}, Y_{\tau_n^-}) \right). \] (9)

$\hat{\alpha}$ is optimal if and only if equality occurs in (7), (8) and (9).
The impulse set is \( I = \{(i, x) : \rho(i, x) = m^* \rho^+(i, x)\} \),
where for \( M^*_{(i, x)} = M_{(i, x)} - \delta_{(i, x)} \), \( m^* \rho^+ \) is the operator:

\[
(i, x) \rightarrow \text{ess sup} \sup_{\nu \in M^*_{(i, x)}} \int_{U \times \mathbb{R}} \nu(i, x; j, dy) (-c(i, x, j, y) + \rho^+(j, y)).
\]

For any \((i, x)\), we define the time

\[
T^*((i, x), .) = \left\{ \begin{array}{ll}
\inf\{t \geq 0 : e^{-\beta t} \rho(i, Y_t^x) = e^{-\beta t} m^* \rho^+(i, Y_t^x)\} \\
+\infty & \text{if the above set is empty.}
\end{array} \right.
\]

**Lemma**

There exists \( r^* \in M \) which achieves the essential supremum such that for any \((i, x)\) \( \in I \):

\[
m^* \rho^+(i, x) = \int_{U \times \mathbb{R}} r^*(i, x; j, dy) (-c(i, x, j, y) + \rho^+(j, y)).
\]
The impulse set is 
\[ I = \{(i, x) : \rho(i, x) = m^* \rho^+(i, x)\} , \]
where for \( M^*_{(i, x)} = M_{(i, x)} - \delta_{(i, x)} \), \( m^* \rho^+ \) is the operator:

\[ (i, x) \rightarrow \text{ess sup} \sup_{\nu \in M^*_{(i, x)}} \int_{U \times \mathbb{R}} \nu(i, x; j, dy)(-c(i, x, j, y) + \rho^+(j, y)). \]

For any \((i, x)\), we define the time

\[ T^*((i, x), .) = \begin{cases} \inf\{t \geq 0 : e^{-\beta t} \rho(i, Y^x_t) = e^{-\beta t} m^* \rho^+(i, Y^x_t)\} & \text{if the above set is empty.} \\
+\infty & \text{otherwise} \end{cases} \]

**Lemma**

There exists \( r^* \in M \) which achieves the essential supremum such that for any \((i, x) \in I\):

\[ m^* \rho^+(i, x) = \int_{U \times \mathbb{R}} r^*(i, x; j, dy) \left( -c(i, x, j, y) + \rho^+(j, y) \right). \]
Theorem: An optimal strategy

The family $\hat{\alpha} = (\hat{\tau}_n, \zeta_{n+1}, \hat{\Delta}_n)$ defined as:

$$\hat{\tau}_0 := \begin{cases} 
T^*((\xi_0, Y_0), \omega) & \text{on } (\xi_0 \neq \emptyset) \cap (T^*((\xi_0, Y_0), \omega) > 0) \\
+\infty & \text{on } (\xi_0 \neq \emptyset) \cap (T^*((\xi_0, Y_0), \omega) = 0) \\
0 & \text{on } (\xi_0 = \emptyset),
\end{cases}$$

then by recurrence, for all $n \geq 1$:
- $\hat{\tau}_n = \hat{\tau}_{n-1} + T^*((\zeta_n, Y_{\hat{\tau}_{n-1}}), .)$,
- the couple $(\zeta_{n+1}, \hat{\Delta}_n)$ has the following law:

$$\begin{cases} 
r^* (\zeta_n, Y_{\hat{\tau}_n}; ., .) & \text{on } (\xi_0 \neq \emptyset) \cap (0 < T^*((\zeta_n, Y_{\hat{\tau}_n}), \omega) < +\infty) \\
\delta_{\{\emptyset, \Delta\}} & \text{otherwise},
\end{cases}$$

is an admissible strategy that satisfies the optimality equalities.
Use numerical methods to exhibit an optimal solution in a specific example: the transition probability measure is supposed to be: $r^\alpha(i, x, 1-i, y) = p_{i,1-i} \otimes \mathcal{N}(x + m, 1)$. We have $f(i, x) = e^x$ and $c(i, x, 1-i, y) = \exp(a x + b(y-x))$. ⇒ A Bang-Bang solution.

Establish a characterization of the value function and consider it as a solution of Hamilton-Bellman-Jacobi inequalities.
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Establish a characterization of the value function and consider it as a solution of Hamilton-Bellman-Jacobi inequalities.
References

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