Impulse control problem with switching technology

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Outline

Problem and objective

- **Problem:** The firm owner decides to switch technology in random time.
- **Consequences:**
  - Switching technology $\Rightarrow$ Stopping time of system $\Rightarrow$ Impulse.
  - The system is revived with a new technology.
- **Objective:** To optimize the firm profit.
- **Observation:**
  - Markovian and homogeneous character between two impulse moments.
  - Markovian form of each revival.

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Tools

The impulse control (or the admissible strategy) has the form:

$$\alpha = (\tau_n, \zeta_{n+1}, \Delta_n, n \geqslant -1).$$

$\Rightarrow$ The control variable has three components:

- Impulse moments: $(\tau_n)_{n \geqslant -1}$ an increasing sequence of stopping times which converges to the default time τ such that $\tau_{-1} = 0$ and $\tau_{n+1} = \tau_n + \tau_0 \circ \phi_{\tau_n}$.
- ζ_{n+1} the technology choice at time τ_n.
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- $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$: a probability space.
- $(\mathcal{F}_t)_{t\geq 0}$: a right continuous complete filtration.
- $(\mathcal{G}_t)_{t>0}$: a predictable filtration $\mathcal{G}_t = \vee_{s<t} \mathcal{F}_s, \ \forall\, s < t$.
- The càdlàg process ξ_t indicates the technology at time t:

$$\xi_t = \xi_0 \, 1_{[0,\tau_0[}(t) + \sum_{n\geq 0} \zeta_{n+1} \, 1_{[\tau_n,\tau_{n+1}[}(t) + \emptyset \, 1_{[\tau,+\infty[}(t).$$

This process takes its values in U, the finite space of possible technologies. ζ_{n+1} is a $\mathcal{G}_{\tau_n}$-measurable random variable and $(\tau_n)_{n\geqslant -1}$ is a sequence of $\mathcal{G}$-stopping times. Note $\overline{U} = U \cup \{\emptyset\}$.

The firm value is given by $S_t = \exp Y_t,\ t \geq 0$, where Y is the càdlàg process defined as

$$Y_t = Y_0\, 1_{[0,\tau_0[}(t) + \sum_{n \geq 0} \Delta_n\, 1_{[\tau_n,\tau_{n+1}[}(t) + \int_0^t (b(\xi_s)\, ds + \sigma(\xi_s)\, dW_s) + \{-\infty\}\, 1_{[\tau,+\infty[}(t),$$

with Δ_n is the firm log value jump size, a $\mathcal{G}_{\tau_n}$-measurable random variable. Let $\overline{\mathbb{R}}$ be $\mathbb{R} \cup \{-\infty\}$. Denote by r^α the transition probability from $(\zeta_n, Y_{\tau_n^-})$ to $(\zeta_{n+1}, Y_{\tau_n})$:

$$\mathbb{P}(\zeta_{n+1} = j, Y_{\tau_n} = x + dy \,\big|\, \zeta_n = i, Y_{\tau_n^-} = x) = r^\alpha(i, x; j, dy).$$

The economic profit function

For each strategy α, the profit is:

$$k(\alpha) = \int_0^T e^{-\beta s} f(\xi_s, Y_s)\, ds - \sum_{n \geq 0} e^{-\beta \tau_n} c(\zeta_n, Y_{\tau_n^-}, \zeta_{n+1}, Y_{\tau_n}) \quad (1)$$

where

- $\beta > 0$ is a discount coefficient.
- The function f represents the firm net profit.
- The function c is the switching technology cost with $c(i, x, i, x) = 0$.

The expected profit of the firm is defined as:

$$K(\alpha) = \mathbb{E}\big(k(\alpha) \big| \xi_0 = i, Y_0 = x\big). \quad (2)$$

Goal

Find an admissible strategy $\widehat{\alpha}$ which maximizes the expected total profit $K(\alpha)$ defined in (2), i.e.:

$$K(\widehat{\alpha}) = ess \operatorname*{sup}_{\alpha \in \underline{D}} K(\alpha), \tag{3}$$

where $\underline{D}$ is the admissible strategy set.

The maximum conditional profit

Definition

We call maximum conditional profit the family defined a.s. as:

$$F_\theta^\alpha = ess \sup_{\{\mu_t=\alpha_t,\, t<\theta\}} \mathbb{E}\left(k(\mu)\right|\mathcal{G}_\theta\left)\right. .$$

Respectively,

$$F_\theta^{\alpha^+} = ess \sup_{\{\mu_t=\alpha_t,\, t\leq\theta\}} \mathbb{E}\left(k(\mu)\right|\mathcal{F}_\theta\left)\right. .$$

Proposition

The maximum conditional profit F^{α}_{θ} *(resp.* $F^{\alpha^+}_{\theta}$*) is a positive supermartingale, meaning that* F^{α}_{θ} *(resp.* $F^{\alpha^+}_{\theta}$*) is* $\mathbb{P}$*-integrable and*

$$\mathbb{E}\left(F^{\alpha}_{\theta} \,|\, \mathcal{G}_{\gamma}\right) \leq F^{\alpha}_{\gamma} \quad \left(\text{resp. } \mathbb{E}\left(F^{\alpha^+}_{\theta} \,|\, \mathcal{F}_{\gamma}\right) \leq F^{\alpha}_{\gamma^+}\right).$$

Corollary (First optimality criterion (N. El Karoui, 1981))

A necessary and sufficient condition for a strategy $\widehat{\alpha}$ *to be optimal is that the maximum conditional profit* $F^{\widehat{\alpha}^+}_{\cdot}$ *is a martingale.*

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Corollary (First optimality criterion (N. El Karoui, 1981))

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Definition

We call maximum conditional profit after $\theta > 0$ (respectively right after θ or $\theta = 0$), the family defined a.s. as :

$$W_\theta^\alpha = ess \sup_{\{\mu_t = \alpha_t, \, \forall\, t < \theta\}} \mathbb{E}\left[k_\theta(\mu)\,|\mathcal{G}_\theta\right],$$

where k_θ is the profit after θ, (respectively, for all $\theta \geq 0$:

$$W_\theta^{\alpha^+} = ess \sup_{\{\mu_t = \alpha_t, \, \forall\, t \leq \theta\}} \mathbb{E}\left[k_{\theta^+}(\mu)\,|\mathcal{F}_\theta\right],$$

where k_{θ^+} is the profit right after θ).

Lemma

We have the following equalities:

$$\begin{aligned} F_\theta^\alpha &= (k(\alpha) - k_\theta(\alpha)) + W_\theta^\alpha, \ \theta > 0. \\ F_\theta^{\alpha^+} &= (k(\alpha) - k_{\theta^+}(\alpha)) + W_\theta^{\alpha^+}. \end{aligned}$$

The dynamic programming principle

Proposition

For any strategy α and $0 < \gamma \leq \theta$, we have a.s.

$$W_\gamma^\alpha \quad \geq \quad \mathbb{E}\Big[k_\gamma(\alpha) - k_\theta(\alpha) + W_\theta^\alpha \,|\, \mathcal{G}_\gamma\Big]. \tag{4}$$

Respectively, for $0 \leq \gamma \leq \theta$, we get a.s.

$$W_\gamma^{\alpha^+} \quad \geq \quad \mathbb{E}\Big[k_{\gamma^+}(\alpha) - k_{\theta^+}(\alpha) + W_\theta^{\alpha^+} \,|\, \mathcal{F}_\gamma\Big]. \tag{5}$$

Moreover, $\widehat{\alpha}$ is optimal if and only if equality (5) holds for every couple (γ, θ).

Notation

We introduce $M = \cup_{(i,x)\in\overline{U}\times\mathbb{R}} M_{(i,x)}$ where $M_{(i,x)}$ verifies:

$$\begin{cases} M_{(i,x)} = \left\{ r^\alpha(i,x;.,.), \delta_{i,x};\ \alpha \in \underline{D} \right\} & \text{if } (i,x) \neq (\emptyset, \Delta) \\ M_{(\emptyset,\Delta)} = \delta_{(\emptyset,\Delta)} & \text{otherwise.} \end{cases}$$

We recall that r^α is a transition probability from couple $(\zeta_n, Y_{\tau_n^-})$ to $(\zeta_{n+1}, Y_{\tau_n})$.

Hypothesis

The set M is weakly closed, weakly compact and separable.

Theorem: Second optimality criterion

For any strategy α we have a.s. the following inequalities:

$$
\begin{aligned}
W_0^{\alpha^+} &\geq \mathbb{E}\left(\int_0^{\tau_0} e^{-\beta s} f(\xi_s, Y_s)\, ds - e^{-\beta\tau_0} c(\xi_0, Y_{\tau_0^-}, \zeta_1, Y_{\tau_0}) \middle|\, \mathcal{F}_0\right) \\
&+ \mathbb{E}(W_{\tau_0}^{\alpha^+} \,|\, \mathcal{F}_0) \\
W_{\tau_n}^{\alpha} &\geq -e^{-\beta\tau_n} \int_{\overline{U}\times\overline{\mathbb{R}}} c(\zeta_n, Y_{\tau_n^-}, i, x) r^{\alpha}(., i, dx) + \mathbb{E}(W_{\tau_n}^{\alpha^+} \,|\, \mathcal{G}_{\tau_n}) \\
W_{\tau_n}^{\alpha^+} &\geq \mathbb{E}\left(\int_{\tau_n}^{\tau_{n+1}} e^{-\beta s} f(\xi_s, Y_s)\, ds \middle|\, \mathcal{F}_{\tau_n}\right) + \mathbb{E}(W_{\tau_{n+1}}^{\alpha} \,|\, \mathcal{F}_{\tau_n})
\end{aligned}
$$

Moreover, the strategy $\widehat{\alpha}$ is optimal if and only if equality has place simultaneously in the following three inequalities.

Corollary

We have the following equalities:

$$W^{\alpha}_{\tau_n} = e^{-\beta\tau_n}\rho(\zeta_n, Y_{\tau_n^-}),\ \forall\ n \geq 0,$$

where $\rho(i,x) = \operatorname{ess\,sup}_{\mu\in\underline{D}} \mathbb{E}_{\{i,x\}}(k(\mu))$.

$$W^{\alpha+}_{\tau_n} = e^{-\beta\tau_n}\rho^+(\zeta_{n+1}, Y_{\tau_n}),\ \forall\ n \geq -1,$$

where $\rho^+(i,x) = \operatorname{ess\,sup}_{\{\mu\in\underline{D},\, \zeta_1^\mu\neq\emptyset\}} \mathbb{E}_{\{i,x\}}(k(\mu))$.

Proposition (Lepeltier-Marchal, 1984)

For any strategy α and any $n \geq 0$, we have

$$W^{\alpha}_{\tau_n} = e^{-\beta\tau_n} m\rho^{+}(\zeta_n, Y_{\tau_n^-}) \quad a.s.$$

where $m\rho^{+}$ is the operator defined by

$$(i,x) \rightarrow ess \sup_{\nu \in M_{(i,x)}} \int_{\overline{U}\times\overline{\mathbb{R}}} \nu(i,x;j,dy)\left(-c(i,x,j,y) + \rho^{+}(j,y)\right).$$

Moreover, the value function $\rho(i,x)$ is equal to $m\rho^{+}(i,x)$.

Proposition

The application ρ^+ does not depend on the strategy α and satisfies the following equation:

$$\begin{aligned} \rho^+(i,x) &= ess \sup_{T>0, T\in \underline{R}_{-1}} \mathbb{E}_{\{i,x\}}\Big(\int_0^{T((i,x),.)} e^{-\beta s} f(i, Y_s)\, ds \\ &+ e^{-\beta T((i,x),.)} m\rho^+(i, Y_{T^-((i,x),.)})\Big), \end{aligned} \tag{6}$$

where $\underline{R}_{-1}$: Set of measurable applications T on $\overline{U} \times \overline{\mathbb{R}} \times \Omega$ such that $T((i,x),.)$ is a $\mathcal{G}$-stopping time.

Theorem: Optimality criterion

For any strategy α, we have the following inequalities:

$$\begin{aligned}\rho^+(i,x) &\geq \mathbb{E}_{\{i,x\}}\Big(\int_0^{\tau_0} e^{-\beta s} f(i, Y_s)\, ds \\ &+ e^{-\beta\tau_0} m\rho^+(i, Y_{\tau_0^-})\Big). \end{aligned} \tag{7}$$

$$\begin{aligned}m\rho^+(\zeta_n, Y_{\tau_n^-}) &\geq \int_{\overline{U}\times\overline{\mathbb{R}}} r^\alpha(\zeta_n, Y_{\tau_n^-}, i, dx)\Big(-c(\zeta_n, Y_{\tau_n^-}, j, y) \\ &+ \rho^+(j,y)\Big). \end{aligned} \tag{8}$$

$$\begin{aligned}e^{-\beta\tau_n}\rho^+(\zeta_{n+1}, Y_{\tau_n}) &\geq \mathbb{E}_{\{\zeta_{n+1}, Y_{\tau_n}\}}\Big(\int_{\tau_n}^{\tau_{n+1}} e^{-\beta s} f(\zeta_{n+1}, Y_s)\, ds \\ &+ e^{-\beta\tau_{n+1}} m\rho^+(\zeta_{n+1}, Y_{\tau_{n+1}^-})\Big). \end{aligned} \tag{9}$$

$\widehat{\alpha}$ is optimal if and only if equality occurs in (7), (8) and (9).

The impulse set is $I = \{(i,x) : \rho(i,x) = m^*\rho^+(i,x)\}$,
where for $M^*_{(i,x)} = M_{(i,x)} - \delta_{(i,x)}$, $m^*\rho^+$ is the operator:

$$(i,x) \to ess \sup_{\nu \in M^*_{(i,x)}} \int_{\overline{U}\times\overline{\mathbb{R}}} \nu(i,x;j,dy)(-c(i,x,j,y) + \rho^+(j,y)).$$

For any (i,x), we define the time

$$T^*((i,x),.) = \begin{cases} \inf\{t \geq 0 : e^{-\beta t}\rho(i,Y_t^x) = e^{-\beta t} m^*\rho^+(i,Y_t^x)\}. \\ +\infty \qquad \text{if the above set is empty.} \end{cases}$$

Lemma

There exists $r^* \in M$ which achieves the essential supremum such that for any $(i,x) \in I$:

$$m^*\rho^+(i,x) = \int_{\overline{U}\times\overline{\mathbb{R}}} r^*(i,x;j,dy)\left(-c(i,x,j,y) + \rho^+(j,y)\right).$$

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Theorem: An optimal strategy

The family $\widehat{\alpha} = (\widehat{\tau}_n, \zeta_{n+1}, \widehat{\Delta}_n)$ defined as:

$$\left\{ \begin{array}{l} \widehat{\tau}_0 := \left\{ \begin{array}{lll} T^*((\xi_0, Y_0), \omega) & \text{on} & (\xi_0 \neq \emptyset) \cap (T^*((\xi_0, Y_0), \omega) > 0) \\ +\infty & \text{on} & (\xi_0 \neq \emptyset) \cap (T^*((\xi_0, Y_0), \omega) = 0) \\ 0 & \text{on} & (\xi_0 = \emptyset), \end{array} \right. \\ r^*(\xi_0, Y_{0-}, \zeta_1, Y_0) \ \text{ is a transition probability measure on } \ \mathcal{G}_{\widehat{\tau}_0}, \end{array} \right.$$

then by recurrence, for all $n \geq 1$:

- $\widehat{\tau}_n = \widehat{\tau}_{n-1} + T^*((\zeta_n, Y_{\tau_{n-1}}), .)$,
- the couple $(\zeta_{n+1}, \widehat{\Delta}_n)$ has the following law:

$$\left\{ \begin{array}{lll} r^*(\zeta_n, Y_{\widehat{\tau}_n^-}; ., .) & \text{on} & (\xi_0 \neq \emptyset) \cap (0 < T^*((\zeta_n, Y_{\widehat{\tau}_n^-}), \omega) < +\infty) \\ \delta_{\{\emptyset, \Delta\}} & & \text{otherwise,} \end{array} \right.$$

is an admissible strategy that satisfies the optimality equalities.

- Use numerical methods to exhibit an optimal solution in a specific example: the transition probability measure is supposed to be: $r^{\alpha}(i, x, 1-i, y) = p_{i,1-i} \otimes \mathcal{N}(x+m, 1)$. We have $f(i, x) = e^x$ and $c(i, x, 1-i, y) = \exp\big(a\,x + b\,(y-x)\big)$.
  $\Rightarrow$ A Bang-Bang solution.
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References

- **A. Bensoussan et J.L. Lions:** *Contrôle Impulsionnel et Inéquations quasi-variationnelles*. Dunod, Paris, 1982
- **M.H.A. Davis:** *Markov Models and Optimization*. Chapman et Hall,1993.
- **N. El Karoui:** *Les Aspects Probabilistes du Contrôle Stochastique*. Lecture Notes in Mathematics 876, Springer-Verlag, Berlin, 1981.
- **P.A. Lepeltier et B. Marchal:** *Théorie Générale du Contrôle Impulsionnel Markovien*. SIAM J.Control and optimization, Vol.22, No.4, 1984.