A Direct Proof of the Bichteler-Dellacherie Theorem and Connections to Arbitrage

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(joint with Mathias Beiglböck and Walter Schachermayer)

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We have a filtered probability space \((\Omega, F, (F_t)_{0 \leq t \leq T}, \mathbb{P})\) satisfying the usual conditions and a real-valued, càdlàg, adapted process \(S = (S_t)_{0 \leq t \leq T}\).
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*\(S\) is a good integrator if and only if \(S\) is a semimartingale.*
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\[\text{If a locally bounded process } S \text{ satisfies NFLVR for simple integrands, then } S \text{ is a semimartingale.}\]
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\begin{align*}
S \text{ is a good integrator} & \quad \Rightarrow \quad \text{weak-NFLVR} & \quad \Leftarrow \quad \text{NFLVR} \\
\text{NFLVR} & \quad \Downarrow \quad S \text{ is a semimartingale}
\end{align*}
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**Theorem (DS 94)**

If a locally bounded process \(S\) satisfies NFLVR for simple integrands, then \(S\) is a semimartingale.
A simple integrand is a stochastic process of the form

\[ H_t = \sum_{j=1}^{n} h_j \mathbb{1}_{[\tau_{j-1}, \tau_j]}(t) \]

where \(0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n = T\) are stopping times and \(h_j \in L^\infty(\Omega, \mathcal{F}_{\tau_{j-1}}, \mathbb{P})\). Denote by \(SI\) the vector space of simple integrands.
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For each $S$, we may well-define the integration operator

$$I : SI \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P})$$

$$\sum_{j=1}^{n} h_j \mathbb{1}_{[\tau_{j-1}, \tau_j]} \mapsto \sum_{j=1}^{n} h_j(S_{\tau_j} - S_{\tau_{j-1}}) =: (H \cdot S)_T.$$
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$S$ is a good integrator if $I$ is continuous i.e. if $\|H^n\|_\infty \to 0$, then $(H^n \cdot S)_T \to 0$ in probability.
S is a good integrator if for every sequence \((H^n)_{n=1}^\infty\) satisfying \(\|H^n\|_\infty \to 0\), we have \((H^n \cdot S)_T \to 0\) in probability.
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Theorem (Main Theorem)

Let $S$ be locally bounded process. TFAE:

- $S$ satisfies NFLVRLI.
- $S$ is a semimartingale.
Assume $S_0 = 0$, $T = 1$. 
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- Assume $S_0 = 0, \ T = 1$.
- Being a semimartingale is a local property, hence assume $S$ is bounded. WLOG, $S \leq 1$.
- Consider $\mathcal{D}_n = \{0, \frac{1}{2^n}, \ldots, \frac{2^n-1}{2^n}, 1\}$ and $S^n$ sampled on $\mathcal{D}_n$. 
  
  Apply discrete Doob-Meyer to obtain $S_n = M_n + A_n$, where $(M_n^j)_{2^n} = 0$ is a martingale and $(A_n^j)_{2^n} = 0$ is predictable.
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Lemma

Assume NFLVRLI. For $\varepsilon > 0$, there exist a constant $C > 0$ and a sequence of $\{\frac{j}{2^n}\}_{j=1}^{2^n} \cup \{\infty\}$-valued stopping times $(\varrho_n)_{n=1}^{\infty}$ such that $\mathbb{P}(\varrho_n < \infty) < \varepsilon$ and

$$TV(A^n,\varrho_n) = \sum_{j=1}^{2^n(\varrho_n \wedge 1)} \left| A^n_{\frac{j}{2^n}} - A^n_{\frac{j-1}{2^n}} \right| \leq C,$$  \hspace{1cm} (1)

$$\|M^n_{1,\varrho_n}\|_{L^2(\Omega)}^2 = \|M^n_{\varrho_n \wedge 1}\|_{L^2(\Omega)}^2 \leq C.$$  \hspace{1cm} (2)
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$$\| M_1^{n, \varrho_n} \|_{L^2(\Omega)}^2 = \| M_{\varrho_n \wedge 1}^n \|_{L^2(\Omega)}^2 \leq C. \quad (2)$$

Idea:

$$H_t^n = -2 \sum_{j=1}^{2^n} S_j \mathbb{1}_{[\frac{j-1}{2^n}, \frac{j}{2^n}]}(t) \Rightarrow \| M_1^{n, \varrho_n} \|_{L^2(\Omega)}^2 \leq (H^n \cdot S)_T.$$
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Idea:

$H^n_t = -2 \sum_{j=1}^{2^n} S_{\frac{j-1}{2^n}} \mathbb{1}_{[\frac{j-1}{2^n}, \frac{j}{2^n}]}(t) \Rightarrow \| M^n_{1, \varrho_n} \|_{L^2(\Omega)}^2 \leq (H^n \cdot S)_T.$

$H^n_t = \sum_{j=1}^{2^n} \text{sign} \left( A^n_{\frac{j}{2^n}} - A^n_{\frac{j-1}{2^n}} \right) \mathbb{1}_{[\frac{j-1}{2^n}, \frac{j}{2^n}]}(t) \Rightarrow TV(A^n) \leq (H^n \cdot S)_T.$
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- $A^n \to A$ and $M^n \to M$ on $[0, \varrho]$. 

$S = A + M$ on $[0, \varrho]$. Since $\epsilon$ was arbitrary, $S$ is a semimartingale.

Lemma (Komlos $L^2$-version)

Let $(f_n)_{n \geq 1}$ be a sequence of measurable functions on a probability space $(\Omega, \mathcal{F}, P)$ such that $\sup_{n \geq 1} \|f_n\|_2 < \infty$. Then, there exist functions $g_n \in \text{conv}(f_n, f_n+1, \ldots)$ such that $(g_n)_{n \geq 1}$ converges almost surely and in $\|\cdot\|_{L^2(\Omega)}$.

$R_T = 1_{[0,\varrho]} \to R_T$ using Komlos. $R_T$ is a good enough substitute for $\varrho$.

Using Komlos again, $A^n \to A$ and $M^n \to M$ on $[0,\varrho]$. 
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- \(R_T^n = 1_{[0, \varrho_n]} \to R_T\) using Komlos.
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