Work Effort, Consumption and Portfolio Choice: When the Occupational Decision Matters

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2. Set-Up
   - Financial Market
   - Controls and Wealth Process
   - Stochastic Control Problem

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   - HJB Equation
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Motivation and Framework

- Observation: Highly-qualified individuals have often the choice between different career paths;

- Decision problem between two career paths:
  - Mid-level management position in a large company with a rather high salary
  - Executive position within a smaller listed company with less salary and the possibility to influence the company’s performance

- Modelling of the optimization and decision problem:
  - Studied from the point of view of a highly-qualified individual in a smaller company with the option to join a larger company
  - The individual can invest in the financial market including the share of the smaller listed company
  - Stochastic control problem
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### Framework

#### Utility-maximizing Individual

- The individual receives a constant salary rate $\delta$ proportional to her wealth.
  - **Gain** in utility from a higher salary rate.
- The individual’s initial wealth $V_0$ is invested in the money market account, a diversified market portfolio, and own company shares.
- The value of her own company’s stock is influenced via work effort:
  - **Gain** in utility from the increased value of her direct shareholding.
  - **Loss** in utility for her work effort $\rightarrow$ disutility term.
- The individual consumes at a continuous rate $k_t$ proportional to her wealth.
  - **Gain** in utility by the ability to consume.

#### Characterization of the Individual

- Utility function of wealth
- Utility function of consumption with time preference $\rho$
- Disutility function associated with time preference $\tilde{\rho}$ and work effectiveness parameters
  - Inverse work productivity $\kappa$
  - Disutility stress $\alpha$
Framework

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Money Market Account:

\[ dB_t = r B_t \, dt, \quad B_0 = 1, \]  

Market Portfolio:

\[ dP_t = P_t \left( \mu^P \, dt + \sigma^P \, dW^P_t \right), \quad P_0 \in \mathbb{R}^+, \]  

Company’s share price process is a controlled diffusion with SDE

\[ dS^\lambda_t = S^\lambda_t \left( [r + \lambda_t \sigma] \, dt + \sigma \, dW_t + \beta \left[ \frac{dP_t}{P_t} - r \, dt \right] \right), \quad S_0 \in \mathbb{R}^+, \]

where the Sharpe ratio \( \lambda_t = (\mu_t - r)/\sigma \) is controlled by the individual.

Individual influences the own company’s share price.

\( \hat{=} \) Gain in utility from the increased value of her direct shareholding.

Remark

\( W^P \) and \( W \) are two independent standard Brownian motions, but the instantaneous correlation between \( S^\lambda_t \) and \( P_t \) is

\[ \rho_t = \beta \sigma^P / \sqrt{\sigma^2 + (\beta \sigma^P)}. \]
Money Market Account:

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\[ dS_t^\lambda = S_t^\lambda \left( [r + \lambda_t \sigma] \, dt + \sigma \, dW_t + \beta \left[ \frac{dP_t}{P_t} - r \, dt \right] \right) , \quad S_0 \in \mathbb{R}^+ , \]  

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Highly-qualified individual

- Endowed with initial wealth $V_0 > 0$.
- Salary rate $\delta$ proportional to her current wealth.
- Seeks to maximize total utility for a given time horizon $T > 0$ by controlling
  - the portfolio holdings $\pi^P$ and $\pi^S$,
  - the consumption $k$,
  - the work effort $\lambda$.

$\Rightarrow$ All controls are collected in the vector process $u = (\pi^P, \pi^S, k, \lambda)$.

For a fixed salary rate, control strategy $u = (\pi^P, \pi^S, k, \lambda)$ and initial wealth $V_0 > 0$, the wealth process is given by:

$$dV_t^u = V_t^u [(1 - \pi_t^P - \pi_t^S) dB_t/B_t + \pi_t^P dP_t/P_t + \pi_t^S dS_t^\lambda/S_t^\lambda + \delta dt - k_t dt]. \quad (4)$$
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\frac{dV_t^u}{V_t^u} = \left[ (1 - \pi_t^P - \pi_t^S) \frac{dB_t}{B_t} + \pi_t^P \frac{dP_t}{P_t} + \pi_t^S \frac{dS_t^\lambda}{S_t^\lambda} + \delta dt - k_t dt \right].
$$

(4)
Utility of Wealth and Consumption

- The utility from final wealth at time $T$ is represented by a utility function $U_1$.
- The utility from consumption over the period $[t, T]$ is represented by a utility function $U_2$.

Work Effort Choice and Disutility

The individual’s instantaneous disutility of work effort is represented by a Markovian disutility rate (cost function) $C(t, v, \lambda_t)$ for control strategy $(\lambda_t)$, where $c : [0, T] \times \mathbb{R}^+ \times [r, \infty) \times \mathbb{R}^+ \rightarrow \mathbb{R}_0^+$. (5)

$\Rightarrow$ The optimal investment and consumption control decision including work effort is the solution of

$$
\Phi(t, v) = \sup_{u \in A(t,v)} \mathbb{E}^{t,v} \left[ U_1(V^u_T) + \int_t^T U_2(s, V^u_s, k_s) ds - \int_t^T C(s, V^u_s, \lambda_s) ds \right],
$$

where $(t, v) \in [0, T] \times \mathbb{R}^+$. (5)
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$$

where $(t, v) \in [0, T] \times \mathbb{R}^+$. 

(5)

Sascha Desmettre  The Occupational Decision
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Illustration of Results

Outlook
0 = \sup_{u \in \mathbb{R}^2 \times [0, \infty)^2} \Phi_t(t, v) + \Phi_v(t, v) v (r + \pi^S \lambda \sigma + [\pi^P + \beta \pi^S](\mu^P - r) + \delta - k_t) \\
+ \frac{1}{2} \Phi_{vv}(t, v) v^2 ([\pi^S \sigma]^2 + [\pi^P \sigma^P + \beta \pi^S \sigma_P]^2) + U_2(t, k_t, v) - C(t, v, \lambda)

where \((t, v) \in [0, T) \times \mathbb{R}^+\), and \(U_1(v) = \Phi(T, v)\), for \(v \in \mathbb{R}^+\).

\(\Rightarrow\) Maximizers \(\pi^{P*}, \pi^{S*}, \lambda^*\) and \(k^*\) of (6) by establishing the FOCs:

\[
\pi^{P*}(t, v) = -\frac{\mu^P - r}{v \sigma^2} \frac{\Phi_v(t, v)}{\Phi_{vv}(t, v)} - \beta \pi^{S*}(t, v),
\]

\[
\pi^{S*}(t, v) = -\frac{\lambda^*(t, v)}{v \sigma} \frac{\Phi_v(t, v)}{\Phi_{vv}(t, v)}
\]

where \(\lambda^*\) is the solution of the implicit equation

\[
\lambda \frac{\Phi^2_v(t, v)}{\Phi_{vv}(t, v)} + \frac{\partial C}{\partial \lambda}(t, v, \lambda) = 0 \quad \text{for all } (t, v) \in [0, T] \times \mathbb{R}^+,
\]

and \(k^*\) is the solution of the equation

\[
\frac{\partial U_2}{\partial k}(t, k, v) - v \Phi_v(t, v) = 0.
\]

Sascha Desmettre  The Occupational Decision
\[ 0 = \sup_{u \in \mathbb{R}^2 \times [0, \infty)^2} \Phi_t(t, v) + \Phi_v(t, v) v (r + \pi_S \lambda \sigma + [\pi_P + \beta \pi_S](\mu^P - r) + \delta - k_t) \]
\[ + \frac{1}{2} \Phi_{vv}(t, v) v^2 ([\pi_S \sigma]^2 + [\pi_P \sigma^P + \beta \pi_S \sigma_P]^2) + U_2(t, k_t, v) - C(t, v, \lambda) \]
where \((t, v) \in [0, T) \times \mathbb{R}^+, \) and \(U_1(v) = \Phi(T, v), \) for \(v \in \mathbb{R}^+.\) (6)

\[ \Rightarrow \] Maximizers \(\pi_P^*, \pi_S^*, \lambda^* \) and \(k^*\) of (6) by establishing the FOCs:
\[ \pi_P^*(t, v) = -\frac{1}{v(\sigma^P)^2} \frac{\Phi_v(t, v)}{\Phi_{vv}(t, v)} - \beta \pi_S^*(t, v) , \]
\[ \pi_S^*(t, v) = -\frac{\lambda^*(t, v)}{v \sigma} \frac{\Phi_v(t, v)}{\Phi_{vv}(t, v)} , \] (7)

where \(\lambda^*\) is the solution of the implicit equation
\[ \lambda \frac{\Phi_v^2(t, v)}{\Phi_{vv}(t, v)} + \frac{\partial C}{\partial \lambda}(t, v, \lambda) = 0 \quad \text{for all} \quad (t, v) \in [0, T] \times \mathbb{R}^+, \] (8)

and \(k^*\) is the solution of the equation
\[ \frac{\partial U_2}{\partial k}(t, k, v) - v \Phi_v(t, v) = 0. \] (9)
Substituting the maximizers (7) in the HJB (6) then yields:

$$
\Phi_t(t, v) + \Phi_v(t, v) v \left( r + \delta - k^*(t, v) \right) - \frac{1}{2} \left( \lambda^*(t, v) \right)^2 \frac{\Phi^2_v(t, v)}{\Phi_{vv}(t, v)} 
$$

$$
- \frac{1}{2} \left( \lambda_P \right)^2 \frac{\Phi^2_v(t, v)}{\Phi_{vv}(t, v)} + U_2(t, k^*(t, v)) - C(t, v, \lambda^*(t, v)) = 0,
$$

where $\lambda_P := \frac{\mu_P - r}{\sigma_P}$.

→

**Goal:**

Solve equation (10) for a special choice of the utility function of wealth, the utility function of consumption and the disutility function.
Substituting the maximizers (7) in the HJB (6) then yields:

\[
\Phi_t(t, \nu) + \Phi_\nu(t, \nu) \nu (r + \delta - k^*(t, \nu)) - \frac{1}{2} (\lambda^*(t, \nu))^2 \frac{\Phi^2_\nu(t, \nu)}{\Phi_{\nu\nu}(t, \nu)} \\
- \frac{1}{2} (\lambda_P)^2 \frac{\Phi^2_\nu(t, \nu)}{\Phi_{\nu\nu}(t, \nu)} + U_2(t, k^*(t, \nu)) - C(t, \nu, \lambda^*(t, \nu)) = 0,
\]

where \(\lambda_P := \frac{\mu_P - r}{\sigma_P}\).

\[\rightarrow\]

**Goal:**

Solve equation (10) for a special choice of the utility function of wealth, the utility function of consumption and the disutility function.
Utility and Disutility Functions

The utility function $U_1$ of wealth satisfies:

$$U_1(v) = K \log(v), \quad \text{for } v \in \mathbb{R}^+,$$

for a constant $K > 0$.

The utility function $U_2$ of consumption satisfies:

$$U_2(t, k, v) = e^{-\rho t} \log(v k), \quad \text{for } (t, v, k) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}_0^+,$$

where $\rho \in \mathbb{R}^+$ is the time preference of consumption.

And the disutility of control (i.e. work effort) $C$ satisfies:

$$C(t, v, \lambda) = e^{-\tilde{\rho} t} \kappa \frac{\lambda^\alpha}{\alpha}, \quad \text{for } (t, v, \lambda) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}_0^+,$$

where $\kappa =$ inverse work productivity and $\alpha =$ disutility stress and $\tilde{\rho} \in \mathbb{R}^-$ is the time preference for the work effort.
Utility and Disutility Functions

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where $\kappa = \text{inverse work productivity}$ and $\alpha = \text{disutility stress}$ and $\tilde{\rho} \in \mathbb{R}^-$ is the time preference for the work effort.
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where $\kappa = \text{inverse work productivity}$ and $\alpha = \text{disutility stress}$ and $\tilde{\rho} \in \mathbb{R}^-$ is the time preference for the work effort.
Knowing the utility and disutility functions now, we can solve the FOCs (8) and (9):

\[ \lambda^* = \left( \frac{e^{\tilde{\rho}t} \Phi_v^2}{\kappa - \Phi_{vv}} \right)^{\frac{1}{\alpha - 2}} \quad \text{and} \quad k^* = \frac{e^{-\rho t}}{\nu \Phi_v}. \]

Substituting this into (10) yields the following simplified equation:

\[ 0 = \Phi_t + \Phi_v \nu (r + \delta) + \frac{1}{2} \frac{\Phi_v^2}{\kappa - \Phi_{vv}} \left( \lambda^* \right)^2 + \frac{\alpha - 2}{2\alpha} \kappa - \frac{2}{\alpha - 2} \left( \frac{\Phi_v^2}{\kappa - \Phi_{vv}} \right)^{\frac{\alpha}{\alpha - 2}} - e^{-\rho t} - \rho t e^{-\rho t} - e^{-\rho t} \log(\Phi_v). \]

Now, the solution \( \Phi \) can be derived by assuming an ansatz of the form

\[ \Phi(t, \nu) = \log(\nu) f(t) + g(t) \quad \text{with} \quad f(T) = 1 \quad \text{and} \quad g(T) = 0. \]
Deriving the Solution

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Substituting this into (10) yields the following simplified equation:

\[ 0 = \Phi_t + \Phi_v \nu (r + \delta) + \frac{1}{2} \frac{\Phi_v^2}{-\Phi_{vv}} \left( \lambda^* \right)^2 + \frac{\alpha - 2}{2 \alpha} \kappa - \frac{2}{\alpha-2} \left( \frac{\Phi_v^2}{-\Phi_{vv}} \right)^{\frac{\alpha}{\alpha-2}} \]
\[ - e^{-\rho t} - \rho t e^{-\rho t} - e^{-\rho t} \log(\Phi_v). \] (14)

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and  

\[ k^* = \frac{e^{-\rho t}}{v \Phi_v}. \]

Substituting this into (10) yields the following simplified equation:

\[ 0 = \Phi_t + \Phi_v \nu (r + \delta) + \frac{1}{2} \frac{\Phi_v^2}{\kappa - \Phi_{vv}} (\lambda^P)^2 + \frac{\alpha - 2}{2 \alpha} \kappa \Phi_v^2 \left( \frac{\Phi_v^2}{\alpha-2} \right) \]

\[ - e^{-\rho t} - \rho t e^{-\rho t} - e^{-\rho t} \log(\Phi_v). \]  

(14)

Now, the solution \( \Phi \) can be derived by assuming an ansatz of the form

\[ \Phi(t, \nu) = \log(\nu) f(t) + g(t) \quad \text{with} \quad f(T) = 1 \quad \text{and} \quad g(T) = 0. \]
Substituting this approach in (14) produces a easily solvable ODE, which yields the following solutions:

\[ \pi^P(t, v) = \frac{\mu^P - r}{(\sigma^P)^2} - \beta \pi^S(t, v), \quad \text{and} \quad \pi^S(t, v) = \frac{\lambda^*(t, v)}{\sigma}, \]

(15)

\[ \lambda^*(t, v) = \left( \frac{e^{\tilde{\rho} t}}{\kappa} f(t) \right)^{\frac{1}{\alpha-2}}, \quad \text{and} \quad k^*(t, v) = \frac{e^{-\rho t}}{f(t)}, \]

and value function

\[ \phi(t, v) = f(t) \log(v) + g(t), \]

with

\[ f(t) = \begin{cases} 
K + \frac{e^{-\rho t} - e^{-\rho T}}{\rho}, & \text{for } \rho \neq 0, \\
K + T - t, & \text{for } \rho = 0,
\end{cases} \]

(16)

and

\[ g(t) = \left( r + \delta + \frac{1}{2} \lambda_P^2 \right) \int_t^T f(s) \, ds + \frac{\alpha - 2}{2 \alpha} \int_t^T \left( \frac{e^{\tilde{\rho} s}}{\kappa} \right)^{\frac{2}{\alpha-2}} f(s) \frac{\alpha}{\alpha-2} \, ds \]

\[ - \int_t^T (1 + \rho s) e^{-\rho s} \, ds - \int_t^T e^{-\rho s} \log(f(s)) \, ds. \]

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Career Path 1: Job Offer from the smaller listed Company

- Contract offered by the principal at \( t = 0 \) with a constant salary rate \( \delta \).
- Ability of controlling the Sharpe ratio by spending work effort \( \rightarrow \) higher utility from an increased expected return.

\( \Rightarrow \) Value function:

\[
\Phi(0, v, \delta) = \left( K + \frac{1 - e^{-\rho T}}{\rho} \right) \log(v) + \frac{\alpha - 2}{2\alpha} \int_0^T \left( \frac{e^{\bar{\rho} s}}{\kappa} \right)^{\frac{2}{\alpha - 2}} f(s) \frac{\alpha}{\alpha - 2} ds \\
+ \left( r + \delta + \frac{1}{2} \lambda^2_{\bar{\rho}} \right) \left( K T + \frac{1}{\rho^2} \left[ 1 - e^{-\rho T} (1 + \rho T) \right] \right) - \frac{1}{\rho} \left( 1 - e^{-\rho T} \right) \\
+ T e^{-\rho T} + K \log(K) - \log \left( K + \frac{1}{\rho} \left[ 1 - e^{-\rho T} \right] \right) \left( K + \frac{1}{\rho} \left[ 1 - e^{-\rho T} \right] \right).
\]

Career Path 2: Outside Option

- Contract offered with a constant salary rate \( \delta_0 \) from a larger company.
- No ability of controlling the Sharpe ratio!

\( \Rightarrow \) Value function:

\[
\Phi^0(0, v, \delta_0) = \Phi(0, v, \delta_0) - \frac{\alpha - 2}{2\alpha} \int_t^T \left( \frac{e^{\bar{\rho} s}}{\kappa} \right)^{\frac{2}{\alpha - 2}} f(s) \frac{\alpha}{\alpha - 2} ds.
\]
Career Path 1: Job Offer from the smaller listed Company

- Contract offered by the principal at \( t = 0 \) with a constant salary rate \( \delta \).
- Ability of controlling the Sharpe ratio by spending work effort \( \rightarrow \) higher utility from an increased expected return.

\[ \Phi(0, v, \delta) = \left( K + \frac{1 - e^{-\rho T}}{\rho} \right) \log(v) + \frac{\alpha - 2}{2 \alpha} \int_0^T \left( \frac{e^{\tilde{s}_\rho s}}{\kappa} \right)^{\frac{2}{\alpha-2}} f(s) \frac{\alpha}{\alpha-2} \, ds \]

\[ + \left( r + \delta + \frac{1}{2} \lambda^2 \rho \right) \left( K T + \frac{1}{\rho^2} \left[ 1 - e^{-\rho T} (1 + \rho T) \right] \right) \frac{1}{\rho} \left( 1 - e^{-\rho T} \right) \]

\[ + T e^{-\rho T} + K \log(K) - \log \left( K + \frac{1}{\rho} \left[ 1 - e^{-\rho T} \right] \right) \left( K + \frac{1}{\rho} \left[ 1 - e^{-\rho T} \right] \right) . \]

Career Path 2: Outside Option

- Contract offered with a constant salary rate \( \delta_0 \) from a larger company.
- No ability of controlling the Sharpe ratio!

\[ \Phi^0(0, v, \delta_0) = \Phi(0, v, \delta_0) - \frac{\alpha - 2}{2 \alpha} \int_t^T \left( \frac{e^{\tilde{s}_\rho s}}{\kappa} \right)^{\frac{2}{\alpha-2}} f(s) \frac{\alpha}{\alpha-2} \, ds . \]
Appropriate Salary Rate

Participation Constraint

Utility by accepting the contract $>\text{utility of the outside option, i.e. } \Phi(\delta) > \Phi^0(\delta_0)$.

Requiring that $\Phi(0, \nu, \delta) = \Phi^0(0, \nu, \delta_0)$, we get the minimal appropriate salary rate:

$$\delta = \begin{cases} 
\delta_0 - \frac{(\alpha-2)}{2 \alpha} \left( \frac{\int_0^T \left( \frac{e^{\tilde{\rho} s}}{\kappa} \right)^{\frac{2}{\alpha-2}} f(s) \frac{\alpha}{\alpha-2} \, ds}{K T + \frac{1}{\rho^2} \left[1 - e^{-\tilde{\rho} T(1+\rho T)}\right]} \right), & \text{for } \rho \neq 0, \\
\delta_0 - \frac{(\alpha-2)}{2 \alpha} \left( \frac{\int_0^T \left( \frac{e^{\tilde{\rho} s}}{\kappa} \right)^{\frac{2}{\alpha-2}} f(s) \frac{\alpha}{\alpha-2} \, ds}{K T + \frac{1}{2} T^2} \right), & \text{for } \rho = 0. 
\end{cases}$$

- If the principal of the smaller listed company offers at least the salary rate $\delta$, then the individual accepts the contract.
- Note: $\delta \leq \delta_0$ (due to the ability of improve the smaller listed company’s performance).
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Optimal Effort $\lambda^*$ w.r.t. $1/\kappa$ and $t$

Figure: Optimal work effort $\lambda^*$ w.r.t. work productivity $1/\kappa$ and time $t$ for fixed disutility stress $\alpha = 5$, time preferences $\rho = 0.11$ and $\bar{\rho} = -0.09$ and time horizon $T = 10$ years.
Optimal Effort $\lambda^*$ w.r.t. $\alpha$ and $t$

**Figure:** Optimal work effort $\lambda^*$ w.r.t. disutility stress $\alpha$ and time $t$ for fixed work productivity $1/\tilde{\kappa} = 100$, time preferences $\rho = 0.11$ and $\tilde{\rho} = -0.09$ and time horizon $T = 10$ years.
Figure: Appropriate salary rate $\delta$ w.r.t. disutility stress $\alpha$ and work productivity $1/\kappa$ given outside salary rate $\delta_0 = 0.2$, time horizon $T = 5$ years and time preferences $\rho = 0.11$, $\bar{\rho} = -0.09$, respectively.
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Towards optimal option portfolios:

- Pay the individual calls on the own-company’s stock (or ESOs).
- Individual invests in the options instead of the own-company’s shares.
- Derive optimal option portfolios for this investment problem.

Towards the “constrained individual”:

- Develop dynamic “game” with company determining the individual’s own-company shareholding and the individual controlling work effort, the left investment decisions and the consumption rate;
- Economic equilibrium game with company taking first step (Stackelberg game).
- Determine optimal mixed compensation (cash, shares, and options);
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