Work Effort, Consumption and Portfolio Choice: When the Occupational Decision Matters

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Motivation and Framework

- Observation: Highly-qualified individuals have often the choice between different career paths;
- Decision problem between two career paths:
  - Mid-level management position in a large company with a rather high salary
  - Executive position within a smaller listed company with less salary and the possibility to influence the company's performance
- Modelling of the optimization and decision problem:
  - Studied from the point of view of a highly-qualified individual in a smaller company with the option to join a larger company
  - The individual can invest in the financial market including the share of the smaller listed company
  - Stochastic control problem

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Framework

Utility-maximizing Individual

- The individual receives a constant salary rate δ proportional to her wealth.
  - Gain in utility from a higher salary rate.
- The individual's initial wealth V_0 is invested in the money market account, a diversified market portfolio, and own company shares.
- The value of her own company's stock is influenced via work effort:
  - Gain in utility from the increased value of her direct shareholding.
  - Loss in utility for her work effort $\rightarrow$ disutility term.
- The individual consumes at a continuous rate k_t proportional to her wealth.
  - Gain in utility by the ability to consume.

Characterization of the Individual

- Utility function of wealth
- Utility function of consumption with time preference ρ
- Disutility function associated with time preference $\tilde{\rho}$ and work effectiveness parameters
  - Inverse work productivity κ
  - Disutility stress α

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- Money Market Account:

$$\mathrm{d}B_t = r\, B_t\, \mathrm{d}t\,, \quad B_0 = 1\,, \tag{1}$$

- Market Portfolio:

$$\mathrm{d}P_t = P_t\,(\mu^P\,\mathrm{d}t + \sigma^P\,\mathrm{d}W_t^P)\,, \quad P_0 \in \mathbb{R}^+\,, \tag{2}$$

- Company's share price process is a controlled diffusion with SDE

$$\mathrm{d}S_t^\lambda = S_t^\lambda\,\left([r + \lambda_t\,\sigma]\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \beta\left[\frac{\mathrm{d}P_t}{P_t} - r\mathrm{d}t\right]\right)\,, \quad S_0 \in \mathbb{R}^+\,, \tag{3}$$

where the Sharpe ratio $\lambda_t = (\mu_t - r)/\sigma$ is controlled by the individual.

Individual influences the own company's share price.

$\triangleq$ Gain in utility from the increased value of her direct shareholding.

Remark

W^P and W are two independent standard Brownian motions, but the instantaneous correlation between S_t^λ and P_t is $\rho_t = \beta\,\sigma^P/\sqrt{\sigma^2 + (\beta\sigma^P)}$.

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Highly-qualified individual

- Endowed with initial wealth $V_0 > 0$.
- Salary rate δ proportional to her current wealth.
- Seeks to maximize total utility for a given time horizon $T > 0$ by controlling
  - the portfolio holdings $\pi^{\boldsymbol{P}}$ and $\pi^{\boldsymbol{S}}$,
  - the consumption k ,
  - the work effort λ .

$\Rightarrow$ All controls are collected in the vector process $u = (\pi^{\boldsymbol{P}}, \pi^{\boldsymbol{S}}, k, \lambda)$.

For a fixed salary rate, control strategy $u = (\pi^P, \pi^S, k, \lambda)$ and initial wealth $V_0 > 0$, the wealth process is given by:

$$\mathrm{d}V_t^u = V_t^u \left[(1 - \pi_t^P - \pi_t^S)\, \mathrm{d}B_t / B_t + \pi_t^P \mathrm{d}P_t / P_t + \pi_t^S \mathrm{d}S_t^\lambda / S_t^\lambda + \delta\, \mathrm{dt} - \mathrm{k_t}\, \mathrm{dt} \right] . \quad (4)$$

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Utility of Wealth and Consumption

- The utility from final wealth at time T is represented by a utility function U_1.
- The utility from consumption over the period $[t, T]$ is represented by a utility function U_2.

Work Effort Choice and Disutility

The individual's instanteneous disutility of work effort is reprensented by a Markovian disutility rate (cost function) $C(t, v, \lambda_t)$ for control strategy (λ_t), where $c : [0, T] \times \mathbb{R}^+ \times [r, \infty) \times \mathbb{R}^+ \to \mathbb{R}_0^+$.

$\Rightarrow$ The *optimal investment and consumption control decision including work effort* is the solution of

$$\Phi(t, v) = \sup_{u \in A(t,v)} \mathbb{E}^{t,v} \left[U_1(V_T^u) + \int_t^T U_2(s, V_s^u, k_s) \mathrm{d}s - \int_t^T C(s, V_s^u, \lambda_s) \, \mathrm{d}s \right] , \tag{5}$$

where $(t, v) \in [0, T] \times \mathbb{R}^+$.

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where $(t, v) \in [0, T] \times \mathbb{R}^+$. (5)

$$0 = \sup_{u\in\mathbb{R}^2\times[0,\infty)^2} \Phi_t(t,v) + \Phi_v(t,v)\, v\,(r + \pi^S\,\lambda\,\sigma + [\pi^P + \beta\pi^S](\mu^P - r) + \delta - k_t)$$
$$+ \frac{1}{2}\,\Phi_{vv}(t,v)\,v^2\,([\pi^S\,\sigma]^2 + [\pi^P\,\sigma^P + \beta\pi^S\sigma_P]^2) + U_2(t,k_t,v) - C(t,v,\lambda)$$
$$\text{where } (t,v)\in[0,T)\times\mathbb{R}^+, \text{ and } U_1(v) = \Phi(T,v)\,, \quad \text{for } v\in\mathbb{R}^+\,. \tag{6}$$

$\Rightarrow$ Maximizers $\pi^{P^\star}$, $\pi^{S^\star}$, $\lambda^\star$ and $k^\star$ of (6) by establishing the FOCs:

$$\begin{aligned} \pi^{P^\star}(t,v) &= -\frac{(\mu^P - r)}{v(\sigma^P)^2}\,\frac{\Phi_v(t,v)}{\Phi_{vv}(t,v)} - \beta\pi^{S^\star}(t,v)\quad, \\ \pi^{S^\star}(t,v) &= -\frac{\lambda^\star(t,v)}{v\sigma}\,\frac{\Phi_v(t,v)}{\Phi_{vv}(t,v)}\quad, \end{aligned} \tag{7}$$

where $\lambda^\star$ is the solution of the implicit equation

$$\lambda\,\frac{\Phi_v^2(t,v)}{\Phi_{vv}(t,v)} + \frac{\partial C}{\partial\lambda}(t,v,\lambda) = 0 \quad \text{for all } (t,v)\in[0,T]\times\mathbb{R}^+\,, \tag{8}$$

and $k^\star$ is the solution of the equation

$$\frac{\partial U_2}{\partial k}(t,k,v) - v\,\Phi_v(t,v) = 0. \tag{9}$$

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Substituting the maximizers (7) in the HJB (6) then yields:

$$\begin{aligned}\Phi_t(t,v) + \Phi_v(t,v)\,v\,\left(r+\delta-k^\star(t,v)\right) - \frac{1}{2}(\lambda^\star(t,v))^2\,\frac{\Phi_v^2(t,v)}{\Phi_{vv}(t,v)}\\ - \frac{1}{2}(\lambda_P)^2\,\frac{\Phi_v^2(t,v)}{\Phi_{vv}(t,v)} + U_2(t,k^\star(t,v)) - C(t,v,\lambda^\star(t,v)) = 0\,,\end{aligned} \tag{10}$$

where $\lambda_P := \dfrac{\mu_P - r}{\sigma^P}$.

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Goal:

Solve equation (10) for a special choice of the utility function of wealth, the utility function of consumption and the disutility function.

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Utility and Disutility Functions

The utility function U_1 of wealth satisfies:

$$U_1(v) = K \log(v)\,, \qquad \text{for } v \in \mathbb{R}^+\,, \tag{11}$$

for a constant $K > 0$.

The utility function U_2 of consumption satisfies:

$$U_2(t,k,v) = e^{-\rho\, t} \log(v\, k), \qquad \text{for } (t,v,k) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}_0^+\,, \tag{12}$$

where $\rho \in \mathbb{R}^+$ is the time preference of consumption.

And the disutility of control (i.e. work effort) C satisfies:

$$C(t,v,\lambda) = e^{-\tilde{\rho}\, t} \kappa \frac{\lambda^\alpha}{\alpha}\,, \qquad \text{for } (t,v,\lambda) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}_0^+\,, \tag{13}$$

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Deriving the Solution

Knowing the utility and disutility functions now, we can solve the FOCs (8) and (9):

$$\lambda^\star = \left(\frac{e^{\tilde{\rho} t}}{\kappa} \frac{\Phi_v^2}{-\Phi_{vv}} \right)^{\frac{1}{\alpha - 2}} \quad \text{and} \quad k^\star = \frac{e^{-\rho\, t}}{v \Phi_v} .$$

Substituting this into (10) yields the following simplified equation:

$$0 = \Phi_t + \Phi_v \, v \, (r + \delta) + \frac{1}{2} \frac{\Phi_v^2}{-\Phi_{vv}} \left(\lambda^P\right)^2 + \frac{\alpha - 2}{2\,\alpha} \kappa^{-\frac{2}{\alpha - 2}} \left(\frac{\Phi_v^2}{-\Phi_{vv}} \right)^{\frac{\alpha}{\alpha - 2}} - e^{-\rho\, t} - \rho\, t\, e^{-\rho\, t} - e^{-\rho\, t} \log(\Phi_v) . \tag{14}$$

Now, the solution Φ can be derived by assuming an ansatz of the form

$$\Phi(t, v) = \log(v)\, f(t) + g(t) \quad \text{with } f(T) = 1 \ \text{ and } \ g(T) = 0 .$$

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Substituting this approach in (14) produces a easily solvable ODE, which yields the following **solutions**:

$$\pi^{P\star}(t,v) = \frac{\mu^P - r}{(\sigma^P)^2} - \beta\,\pi^{S\star}(t,v)\,, \quad \text{and} \quad \pi^{S\star}(t,v) = \frac{\lambda^\star(t,v)}{\sigma}\,,$$
$$\lambda^\star(t,v) = \left(\frac{e^{\tilde\rho\, t}}{\kappa} f(t)\right)^{\frac{1}{\alpha-2}}\,, \quad \text{and} \quad k^\star(t,v) = \frac{e^{-\rho\, t}}{f(t)}\,, \tag{15}$$

and value function

$$\phi(t,v) = f(t)\,\log(v) + g(t)\,,$$

with

$$f(t) = \begin{cases} K + \frac{e^{-\rho\, t} - e^{-\rho\, T}}{\rho}\,, & \text{for } \rho \neq 0\,, \\ K + T - t\,, & \text{for } \rho = 0\,, \end{cases} \tag{16}$$

and

$$g(t) = \left(r + \delta + \frac{1}{2}\lambda_P^2\right)\int_t^T f(s)\,\mathrm{d}s + \frac{\alpha-2}{2\,\alpha}\int_t^T \left(\frac{e^{\tilde\rho\, s}}{\kappa}\right)^{\frac{2}{\alpha-2}} f(s)^{\frac{\alpha}{\alpha-2}}\,\mathrm{d}s$$
$$- \int_t^T (1+\rho\, s)\, e^{-\rho\, s}\,\mathrm{d}s - \int_t^T e^{-\rho\, s}\,\log(f(s))\,\mathrm{d}s\,. \tag{17}$$

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Career Path 1: Job Offer from the smaller listed Company

- Contract offered by the principal at $t = 0$ with a constant salary rate δ.
- Ability of controlling the Sharpe ratio by spending work effort $\rightarrow$ higher utility from an increased expected return.

$\Rightarrow$ Value function:

$$\Phi(0, v, \delta) = \left(K + \frac{1 - e^{-\rho T}}{\rho}\right) \log(v) + \frac{\alpha - 2}{2\,\alpha} \int_0^T \left(\frac{e^{\tilde{\rho}\, s}}{\kappa}\right)^{\frac{2}{\alpha - 2}} f(s)^{\frac{\alpha}{\alpha - 2}}\, \mathrm{d}s$$
$$+ \left(r + \delta + \frac{1}{2}\,\lambda_P^2\right)\left(K\,T + \frac{1}{\rho^2}\left[1 - e^{-\rho\,T}(1 + \rho T)\right]\right) - \frac{1}{\rho}\left(1 - e^{-\rho\,T}\right)$$
$$+ T\,e^{-\rho\,T} + K\,\log(K) - \log\left(K + \frac{1}{\rho}\left[1 - e^{-\rho\,T}\right]\right)\left(K + \frac{1}{\rho}\left[1 - e^{-\rho\,T}\right]\right).$$

Career Path 2: Outside Option

- Contract offered with a constant salary rate δ_0 from a larger company.
- No ability of controlling the Sharpe ratio!

$\Rightarrow$ Value function:

$$\Phi^0(0, v, \delta_0) = \Phi(0, v, \delta_0) - \frac{\alpha - 2}{2\,\alpha} \int_t^T \left(\frac{e^{\tilde{\rho}\, s}}{\kappa}\right)^{\frac{2}{\alpha - 2}} f(s)^{\frac{\alpha}{\alpha - 2}}\, \mathrm{d}s\,.$$

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$$+ \left(r + \delta + \frac{1}{2}\,\lambda_P^2\right)\left(K\,T + \frac{1}{\rho^2}\left[1 - e^{-\rho\,T}(1 + \rho\,T)\right]\right) - \frac{1}{\rho}\left(1 - e^{-\rho\,T}\right)$$
$$+ T\,e^{-\rho\,T} + K\,\log(K) - \log\left(K + \frac{1}{\rho}\left[1 - e^{-\rho\,T}\right]\right)\left(K + \frac{1}{\rho}\left[1 - e^{-\rho\,T}\right]\right).$$

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- Contract offered with a constant salary rate δ_0 from a larger company.
- No ability of controlling the Sharpe ratio!

$\Rightarrow$ Value function:

$$\Phi^0(0, v, \delta_0) = \Phi(0, v, \delta_0) - \frac{\alpha - 2}{2\,\alpha} \int_t^T \left(\frac{e^{\tilde{\rho}\, s}}{\kappa}\right)^{\frac{2}{\alpha-2}} f(s)^{\frac{\alpha}{\alpha-2}}\, \mathrm{d}s\,.$$

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Appropriate Salary Rate

Participation Constraint

Utility by accepting the contract > utility of the outside option, i.e. $\Phi(\delta) > \Phi^0(\delta_0)$.

Requiring that $\Phi(0, v, \delta) = \Phi^0(0, v, \delta_0)$, we get the minimal appropriate salary rate:

$$\delta = \begin{cases} \delta_0 - \frac{(\alpha-2)}{2\,\alpha} \frac{\int_0^T \left(\frac{e^{\tilde{\rho}\, s}}{\kappa}\right)^{\frac{2}{\alpha-2}} f(s)^{\frac{\alpha}{\alpha-2}}\, \mathrm{d}s}{K\,T + \frac{1}{\rho^2}\left[1 - e^{-\rho\, T}(1+\rho\, T)\right]}\,, & \text{for } \rho \neq 0\,, \\ \delta_0 - \frac{(\alpha-2)}{2\,\alpha} \frac{\int_0^T \left(\frac{e^{\tilde{\rho}\, s}}{\kappa}\right)^{\frac{2}{\alpha-2}} f(s)^{\frac{\alpha}{\alpha-2}}\, \mathrm{d}s}{K\,T + \frac{1}{2}\,T^2}\,, & \text{for } \rho = 0\,. \end{cases}$$

- If the principal of the smaller listed company offers at least the salary rate δ, then the individual accepts the contract.
- Note: $\delta \leq \delta_0$ (due to the ability of improve the smaller listed company's performance).

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Optimal Effort $\lambda^\star$ w.r.t. $1/\kappa$ and t

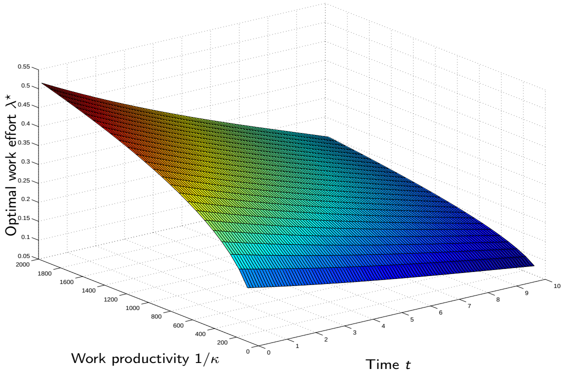

Figure: Optimal work effort $\lambda^\star$ w.r.t. work productivity $1/\kappa$ and time t for fixed disutility stress $\alpha = 5$, time preferences $\rho = 0.11$ and $\tilde{\rho} = -0.09$ and time horizon $T = 10$ years.

Optimal Effort $\lambda^\star$ w.r.t. α and t

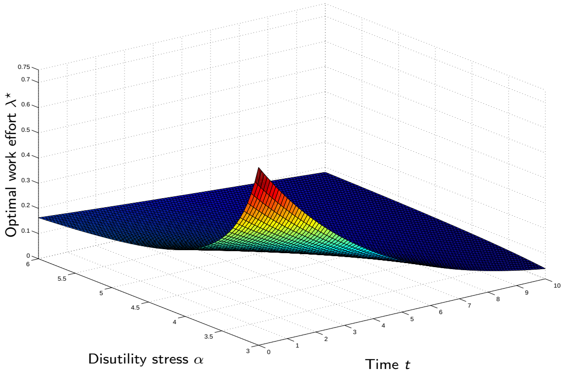

Figure: Optimal work effort $\lambda^\star$ w.r.t. disutility stress α and time t for fixed work productivity $1/\tilde{\kappa} = 100$, time preferences $\rho = 0.11$ and $\tilde{\rho} = -0.09$ and time horizon $T = 10$ years.

Appropriate Salary Rate δ w.r.t. α and $1/\kappa$

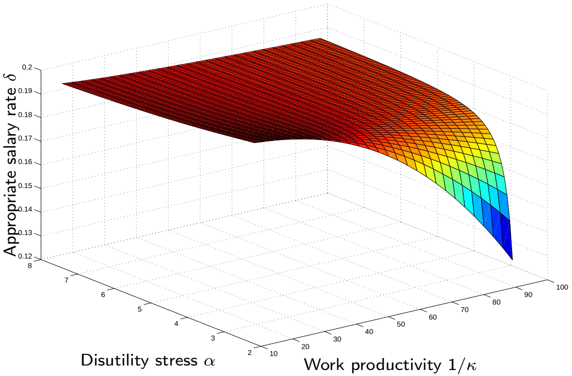

Figure: Appropriate salary rate δ w.r.t. disutility stress α and work productivity $1/\kappa$ given outside salary rate $\delta_0 = 0.2$, time horizon $T = 5$ years and time preferences $\rho = 0.11$, $\tilde{\rho} = -0.09$, respectively.

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Towards optimal option portfolios:

- Pay the individual calls on the own-company's stock (or ESOs).
- Individual invests in the options instead of the own-company's shares.
- Derive optimal option portfolios for this investment problem.

Towards the "constrained indvidual":

- Develop dynamic "game" with company determining the individual's own-company shareholding and the individual controlling work effort, the left investment decisions and the consumption rate;
- $\rightarrow$ Economic equilibrium game with company taking first step (Stackelberg game).
- Determine optimal mixed compensation (cash, shares, and options);

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