Infinite dimensional stochastic calculus via regularization with some financial perspectives

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Let W be the real Brownian motion equipped with its canonical filtration $(\mathcal{F}_t)$.
$\langle W \rangle_t = t$.

- If $h \in L^2(\Omega)$, the martingale representation theorem states the existence of a predictable process $\xi \in L^2(\Omega \times [0, T])$ such that

$$h = \mathbb{E}[h] + \int_0^T \xi_s dW_s$$

- If $h \in \mathbb{D}^{1,2}$ in the sense of Malliavin, Clark-Ocone formula implies that $\xi_s = \mathbb{E}\left[D^m h | \mathcal{F}_s\right]$, so that

$$h = \mathbb{E}[h] + \int_0^T \mathbb{E}\left[D^m h | \mathcal{F}_s\right] dW_s \qquad (1)$$

where D^m is the Malliavin gradient.

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Examples of processes with finite quadratic variation

1) S is an $(\mathcal{F}_t)$-**semimartingale** with decomposition $S = M + V$, M $(\mathcal{F}_t)$-local martingale and V bounded variation process. So $[S] = [M]$.

2) D is a $(\mathcal{F}_t)$-**Dirichlet** process with decomposition $D = M + A$, M $(\mathcal{F}_t)$-local martingale and A an $(\mathcal{F}_t)$-adapted zero quadratic variation process. $[D] = [M]$. Föllmer (1981).

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3) D is a $(\mathcal{F}_t)$-**weak-Dirichlet** process with decomposition $D = M + A$, M $(\mathcal{F}_t)$-local martingale and A such that $[A, N] = 0$ for any continuous $(\mathcal{F}_t)$-local martingale N. Errami-Russo (2003), Gozzi-Russo (2005)

   1. In general D does not have finite quadratic variation
   2. If A is a finite quadratic variation process $[D] = [M] + [A]$
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4) $B^{H,K}$ bifractional Brownian motion with parameters $H \in]0,1[$, $K \in]0,1]$ such that $HK \geq 1/2$

- If $HK > 1/2$, $[B^{H,K}] = 0$.
- If $HK = 1/2$, then
  - $[B^{H,K}]_t = 2^{1-K} t$
  - If $K = 1$ and if $H = 1/2$, $B^{H,K}$ is a Brownian motion
  - If $K \neq 1$, $B^{H,K}$ is not a semimartingale (not even a Dirichlet with respect to its own filtration).

5) Skorohod integrals. If (u_t) is in $L^{1,2}$, under reasonable conditions on Du, $[\int_0^t u_s \delta W_s]_t = \int_0^t u_s^2 ds$.

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6) For fixed $k \geq 1$, Föllmer Wu Yor construct a weak k-order Brownian motion X, which in general is not even Gaussian. X is a **weak k-order Brownian motion** if for every $0 \leq t_1 \leq \cdots \leq t_k < +\infty$, $(X_{t_1}, \cdots, X_{t_k})$ is distributed as $(W_{t_1}, \cdots, W_{t_k})$. If $k \geq 4$ then $[X]_t = t$.

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Notation

Definition

Let $T > 0$ and $X = (X_t)_{t \in [0,T]}$ be a real continuous process prolongated by continuity.
Process $X(\cdot)$ defined by

$$X(\cdot) = \{X_t(u) := X_{t+u}; u \in [-T, 0]\}$$

will be called **window process**.

- $X(\cdot)$ is a $C([-T,0])$-valued stochastic process.
- $C([-T,0])$ is a typical non-reflexive Banach space.

A representation problem

We suppose $X_0 = 0$ and $[X]_t = t$.
The main task will consist in looking for classes of functionals

$$H : C([-T, 0]) \longrightarrow \mathbb{R}$$

such that the r.v.

$$h := H(X_T(\cdot))$$

admits representation

$$h = H_0 + \text{"} \int_0^T \xi_s dX_s \text{"}$$

- Moreover we look for an explicit expression for
  - $H_0 \in \mathbb{R}$
  - ξ adapted process with respect to the canonical filtration of X

Idea

We will obtain the representation formula by expressing $h = H(X_T(\cdot))$ as

$$h = H(X_T(\cdot)) = \lim_{t \uparrow T} u(t, X_t(\cdot))$$

where $u \in C^{1,2}\left([0, T[\times C([-T, 0])\right)$ solves an infinite dimensional PDE, if previous limit exists.

Representation of $h = H(X_T(\cdot))$

Then

$$h = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(s, X_s(\cdot)) d^- X_s \tag{2}$$

where $D^{\delta_0} u(s, \eta) = D\, u(s, \eta)(\{0\})$ is the *projection* of the Fréchet derivative $Du\,(t, \eta)$ on the linear space generated by Dirac measure δ_0, we recall that
$D\, u : [0, T] \times C([-T, 0]) \longrightarrow C^*([-T, 0]) = \mathcal{M}([-T, 0])$.

Forward integral for real valued processes

Definition

Let X (resp. Y) be a continuous (resp. locally integrable) process. Suppose that the random variables

$$\int_0^t Y_s d^- X_s := \lim_{\epsilon \to 0} \int_0^t Y_s \frac{X_{s+\epsilon} - X_s}{\epsilon} ds$$

exists in probability for every $t \in [0, T]$ and the limiting process admits a continuous modification, then the limiting process denoted by $\int_0^{\cdot} Y d^- X$ is called the **(proper) forward integral of** Y **with respect to** X.

Russo-Vallois 1993

Covariation for real valued processes

Definition

The **covariation of** X **and** Y is defined by

$$[X, Y]_t = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t (X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s) ds$$

if the limit exists in the ucp sense with respect to t.
Obviously $[X, Y] = [Y, X]$.
If $X = Y$, X is said to be **finite quadratic variation process** and $[X] := [X, X]$.

Connections with semimartingales

1. Let S^1, S^2 be $(\mathcal{F}_t)$-semimartingales with decomposition $S^i = M^i + V^i$, $i = 1, 2$ where M^i $(\mathcal{F}_t)$-local continuous martingale and V^i continuous bounded variation processes. Then
   - $[S^i]$ classical bracket and $[S^i] = \langle M^i \rangle$.
   - $[S^1, S^2]$ classical bracket and $[S^1, S^2] = \langle M^1, M^2 \rangle$.
   - If S semimartingale and Y cadlag and predictable

$$\int_0^{\cdot} Y d^- S = \int_0^{\cdot} Y dS \quad \text{(Itô)}$$

Itô formula for finite quadratic variation processes

Theorem

Let $F : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $F \in C^{1,2}([0, T[\times\mathbb{R})$ and X be a finite quadratic variation process. Then

$$\int_0^t \partial_x F(s, X_s) d^- X_s$$

exists in the ucp sense and equals

$$F(t, X_t) - F(0, X_0) - \int_0^t \partial_s F(s, X_s) ds - \frac{1}{2} \int_0^t \partial_{x\,x} F(s, X_s) d[X]_s$$

An infinite dimensional framework

We fix now in a general (infinite dimensional) framework. Let

- B general Banach space
- $\mathbb{X}$ a B-valued process
- $F : B \longrightarrow \mathbb{R}$ be of class C^2 in Fréchet sense.

An Ito formula for B-valued processes

We would like to have an Itô type expansion of $F(\mathbb{X})$, available also for $B = C([-T, 0])$-valued processes, as window processes, i.e. when $\mathbb{X} = X(\cdot)$.
The literature does not apply: several problems appear even in the simple case $W(\cdot)$!

Fréchet derivative and tensor product of Banach spaces

$F : B \longrightarrow \mathbb{R}$ be of class C^2 in Fréchet sense, then

- $DF : B \longrightarrow L(B; \mathbb{R}) := B^*$;
- $D^2F : B \longrightarrow L(B; B^*) \cong \mathcal{B}(B \times B) \cong (B\hat{\otimes}_\pi B)^*$

where

- $\mathcal{B}(B, B)$ Banach space of real valued bounded bilinear forms on $B \times B$
- $(B\hat{\otimes}_\pi B)^*$ dual of the tensor projective tensor product of B with B.
- $B\hat{\otimes}_\pi B$ fails to be Hilbert even if B is a Hilbert space (is not even a reflexive space).

A first attempt to an Itô type expansion of $F(\mathbb{X})$

$$F(\mathbb{X}_t) = F(\mathbb{X}_0) +'' \int_0^t {}_{B^*}\langle DF(\mathbb{X}_s), d\mathbb{X}_s\rangle_B {}''+$$

$$+ \frac{1}{2}'' \int_0^t {}_{(B\hat{\otimes}_\pi B)^*}\langle D^2F(\mathbb{X}_s), d[\mathbb{X}]_s\rangle_{B\hat{\otimes}_\pi B} {}''$$

A formal proof

$$\int_0^t \frac{F(\mathbb{X}_{s+\epsilon}) - F(\mathbb{X}_s)}{\epsilon} ds \xrightarrow[\epsilon \to 0]{ucp} F(\mathbb{X}_t) - F(\mathbb{X}_0)$$

By a Taylor's expansion the left-hand side equals the sum of

$$\int_0^t {}_{B^*}\langle DF(\mathbb{X}_s), \frac{\mathbb{X}_{s+\epsilon} - \mathbb{X}_s}{\epsilon} \rangle_B ds +$$
$$\int_0^t {}_{(B\hat{\otimes}_\pi B)^*}\langle D^2 F(\mathbb{X}_s), \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)^{\otimes 2}}{\epsilon} \rangle_{B\hat{\otimes}_\pi B} ds + R(\epsilon, t)$$

Stochastic calculus via regularization for Banach valued processes

We will define

- a stochastic integral for B^*-valued integrand with respect to B-valued integrators, which are not necessarily semimartingale.
- a new concept of quadratic variation which generalizes the tensor quadratic variation and which involves a Banach subspace χ of $(B\hat{\otimes}_\pi B)^*$. It will be called χ-quadratic variation of $\mathbb{X}$.

Forward integral for Banach valued processes

Definition

Let $\mathbb{X}$ and $\mathbb{Y}$ be respectively a B-valued and a B^*-valued continuous stochastic processes.
If the process defined for every fixed $t \in [0, T]$ by

$$\int_0^t {}_{B^*}\langle \mathbb{Y}_s, d^-\mathbb{X}_s \rangle_B := \lim_{\epsilon \to 0} \int_0^t {}_{B^*}\langle \mathbb{Y}(s), \frac{\mathbb{X}(s+\epsilon) - \mathbb{X}(s)}{\epsilon} \rangle_B ds$$

in probability admits a continuous version, then process

$$\left(\int_0^t {}_{B^*}\langle \mathbb{Y}_s, d^-\mathbb{X}_s \rangle_B \right)_{t \in [0,T]}$$

will be called **forward stochastic integral of $\mathbb{Y}$ with respect to $\mathbb{X}$**.

Definition of Chi-subspace

Definition

A Banach subspace χ continuously injected into $(B\hat{\otimes}_\pi B)^*$ will be called a **Chi-subspace of** $(B\hat{\otimes}_\pi B)^*$.
In particular it holds

$$\|\cdot\|_\chi \geq \|\cdot\|_{(B\hat{\otimes}_\pi B)^*}.$$

Notion of χ-quadratic variation

Let

- $\mathbb{X}$ be a B-valued continuous process,
- χ a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$,
- $\mathscr{C}([0,T])$ space of real continuous processes equipped with the ucp topology.
- $[\mathbb{X}]^\epsilon$ be the application

$$[\mathbb{X}]^\epsilon : \chi \longrightarrow \mathscr{C}([0,T])$$

defined by

$$\phi \mapsto \left(\int_0^t {}_{\chi}\langle \phi, \frac{J\left((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2 \right)}{\epsilon} \rangle_{\chi^*} \, ds \right)_{t\in[0,T]}$$

where $J : B\hat{\otimes}_\pi B \to (B\hat{\otimes}_\pi B)^{**}$ is the canonical injection a Banach space and its bidual, in the sequel will be omitted.

Definition of Chi-quadratic variation

Definition

$\mathbb{X}$ **admits a χ-quadratic variation** if

H1 For all $(\epsilon_n) \downarrow 0$ it exists a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \frac{\left\| (\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s)^{\otimes 2} \right\|_{\chi^*}}{\epsilon_{n_k}} ds \quad < \infty \quad a.s.$$

H2 There exists $[\mathbb{X}] : \chi \longrightarrow \mathscr{C}([0, T])$ such that

$$[\mathbb{X}]^{\epsilon}(\phi) \xrightarrow[\epsilon \to 0]{ucp} [\mathbb{X}](\phi) \quad \forall\, \phi \in \chi$$

H3 There is a χ^*-valued bounded variation process $\widetilde{[\mathbb{X}]}$, such that $\widetilde{[\mathbb{X}]}_t(\phi) = [\mathbb{X}](\phi)_t$ a.s. for all $\phi \in \chi$.
For every fixed $\phi \in \chi$, processes $\widetilde{[\mathbb{X}]}_t(\phi)$ and $[\mathbb{X}](\phi)_t$ are indistinguishable.

Global quadratic variation concept

Definition

We say that $\mathbb{X}$ admits a **global quadratic variation (g.q.v.)** if it admits a χ-quadratic variation with $\chi = (B\hat{\otimes}_\pi B)^*$.

Infinite dimensional Itô's formula

Let B a separable Banach space

Theorem (Itô's formula)

Let $\mathbb{X}$ a B-valued continuous process admitting a χ-quadratic variation.
Let $F : [0, T] \times B \longrightarrow \mathbb{R}$ be $C^{1,2}$ Fréchet such that

$$D^2F : [0, T] \times B \longrightarrow \chi \subset (B \hat{\otimes}_\pi B)^* \quad \text{continuously}$$

Then for every $t \in [0, T]$ the forward integral

$$\int_0^t {}_{B^*}\langle DF(s, \mathbb{X}_s), d^-\mathbb{X}_s \rangle_B$$

exists and following formula holds.

Ito's formula

$$F(t,\mathbb{X}_t) = F(0,\mathbb{X}_0) + \int_0^t \partial_s F(s,\mathbb{X}_s)ds + \\ + \int_0^t {}_{B^*}\langle DF(s,\mathbb{X}_s), d^-\mathbb{X}_s\rangle_B + \\ + \frac{1}{2}\int_0^t {}_{\chi}\langle D^2F(s,\mathbb{X}_s), d\widetilde{[\mathbb{X}]}_s\rangle_{\chi^*}$$

Window processes

- We fix attention now on $B = C([-T, 0])$-valued window processes.
- X continuous real valued process and $X(\cdot)$ its window process.
- $\mathbb{X} = X(\cdot)$

Evaluations of χ-quadratic variation for window processes

- If X has Hölder continuous paths of parameter $\gamma > 1/2$, then $X(\cdot)$ has a zero g.q.v.
  For instance:
  - $X = B^H$ fractional Brownian motion with parameter $H > 1/2$.
  - $X = B^{H,K}$ bifractional Brownian motion with parameters $H \in]0,1[$, $K \in]0,1]$ s.t. $HK > 1/2$.
- $W(\cdot)$ does not admit a g.q.v.

Some examples of Chi-subspaces

- χ Chi-subspace of $(B\hat{\otimes}_\pi B)^*$ with $B = C([-T,0])$. For instance:
  - $\mathcal{M}([-T,0]^2)$ equipped with the total variation norm.
  - $L^2([-T,0]^2)$.
  - $\mathcal{D}_{0,0} = \{\mu(dx,dy) = \lambda\, \delta_0(dx) \otimes \delta_0(dy)\}$.
  - $\left(\mathcal{D}_0 \oplus L^2\right) \hat{\otimes}_h^2 = \mathcal{D}_{0,0} \oplus L^2([-T,0])\hat{\otimes}_h D_0 \oplus D_0\hat{\otimes}_h L^2([-T,0]) \oplus L^2([-T,0]^2)$.
  - $Diag := \{\mu(dx,dy) = g(x)\delta_y(dx)dy; g \in L^\infty([-T,0])\}$.

Evaluations of χ-quadratic variation for window processes

- $W(\cdot)$ does not admit a $\mathcal{M}([-T,0]^2)$-quadratic variation.
- If X is a real finite quadratic variation process, then
  - $X(\cdot)$ has zero $L^2([-T,0]^2)$-quadratic variation.
  - $X(\cdot)$ has $\mathcal{D}_{0,0}$-quadratic variation

    $$[X(\cdot)] : \mathcal{D}_{0,0} \longrightarrow \mathscr{C}[0,T] \, , \qquad [X(\cdot)]_t(\mu) = \mu(\{0,0\})[X]_t$$

  - $X(\cdot)$ has $\left(\mathcal{D}_0 \oplus L^2\right) \hat{\otimes}_h^2$-quadratic variation

    $$[X(\cdot)] : \left(\mathcal{D}_0 \oplus L^2\right) \hat{\otimes}_h^2 \longrightarrow \mathscr{C}[0,T] \, , \qquad [X(\cdot)]_t(\mu) = \mu(\{0,0\})[X]_t$$

  - $X(\cdot)$ has *Diag*-quadratic variation

    $$[X(\cdot)] : Diag \longrightarrow \mathscr{C}[0,T] \, , \qquad [X(\cdot)]_t(\mu) = \int_0^t g(-x)[X]_{t-x} dx$$

    where $\mu(dx,dy) = g(x)\delta_y(dx)dy$.

Robustness of Black-Scholes formula

Let (S_t) be the price of a financial asset of the type

$$S_t = \exp(\sigma W_t - \frac{\sigma^2}{2} t), \quad \sigma > 0 .$$

Let $h = \tilde{f}(S_T) = f(W_T)$ where $f(y) = \tilde{f}\left(\exp(\sigma y - \frac{\sigma^2}{2} T)\right)$.
Let $\tilde{u} : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ solving

$$\begin{cases} \partial_t \tilde{u}(t,x) + \frac{1}{2} \partial_{xx} \tilde{u}(t,x) = 0 & \\ \tilde{u}(T,x) = \tilde{f}(x) & x \in \mathbb{R} \end{cases}$$

Applying classical Itô formula we obtain

$$h = \tilde{u}(0, S_0) + \int_0^T \partial_x \tilde{u}(s, S_s) dS_s = u(0, W_0) + \int_0^T \partial_x u(s, W_s) dW_s$$

for a suitable $u : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$.

Does one have a similar formula if W is replaced by a finite quadratic variation X such that $[X]_t = t$? The answer is YES.

Let X such that $[X]_t = t$

A1 $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous and polynomial growth

A2 $v \in C^{1,2}([0,T[\times\mathbb{R}) \cap C^0([0,T] \times \mathbb{R})$ such that

$$\begin{cases} \partial_t v(t,x) + \frac{1}{2}\partial_{xx} v(t,x) = 0 \\ v(T,x) = f(x) \end{cases}$$

Then

$$h := f(X_T) = v(0, X_0) + \underbrace{\int_0^T \partial_x v(s, X_s) d^- X_s}_{\text{improper forward integral}}$$

Schoenmakers-Kloeden (1999) Coviello-Russo (2006) Bender-Sottinen-Valkeila (2008)

Natural question

Is it possible to express generalization of it where the option is path dependent? As first step we revisit the toy model.

The toy model revisited

Proposition

We set $B = C([-T, 0])$ and $\eta \in B$ and we define

- $H : B \longrightarrow \mathbb{R}$, by $H(\eta) := f(\eta(0))$
- $u : [0, T] \times B \longrightarrow \mathbb{R}$, by $u(t, \eta) := v(t, \eta(0))$

Then

$$u \in C^{1,2}([0, T[\times B; \mathbb{R}) \cap C^0([0, T] \times B; \mathbb{R})$$

and solves

$$\begin{cases} \partial_t u(t, \eta) + \frac{1}{2}\langle D^2 u(t, \eta) \ , \ \mathbb{1}_D \rangle = 0 \\ u(T, \eta) = H(\eta) \end{cases}$$

Proof.

- $u(T,\eta) = v(T,\eta(0)) = f(\eta(0)) = H(\eta)$
- $\partial_t u\,(t,\eta) = \partial_t v\,(t,\eta(0))$
- $Du\,(t,\eta) = \partial_x v\,(t,\eta(0))\,\delta_0$
- $D^2 u\,(t,\eta) = \partial^2_{x\,x} v\,(t,\eta(0))\,\delta_0 \otimes \delta_0$ $\quad D^2 u(t,\eta) \in \mathcal{D}_{0,0}$
- $\partial_t u\,(t,\eta) + \frac{1}{2} D^2 u\,(t,\eta)(\{0,0\}) = 0$

□

And, let X such that $[X]_t = t$, we have

$$h := H(X_T(\cdot)) = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(s, X_s(\cdot)) d^- X_s$$

Notation

We set $B = C([-T, 0])$ and $\eta \in B$.

- X real continuous stochastic process
- $X_0 = 0$,
- $[X]_t = t$.

A representation problem

The main task will consist in looking for classes of functionals

$$H : B \longrightarrow \mathbb{R}$$

such that the r.v.

$$h := H(X_T(\cdot))$$

admits representation

$$h = H_0 + \int_0^T \xi_s d^- X_s$$

- Moreover we look for an explicit expression for
  - $H_0 \in \mathbb{R}$
  - ξ adapted process with respect to the canonical filtration of X

Idea

Obtain the representation formula by expressing $h = H(X_T(\cdot))$ as

$$h = H(X_T(\cdot)) = \lim_{t \uparrow T} u(t, X_t(\cdot))$$

where $u \in C^{1,2}\left([0, T[\times B\right)$ solves an infinite dimensional PDE, if previous limit exists.

An infinite dimensional PDE

Let $H : B \longrightarrow \mathbb{R}$, in several cases we will show the existence of a function $u : [0, T] \times B \longrightarrow \mathbb{R}$ of class $C^{1,2}\left([0, T[\times B\right) \cap C^0\left([0, T] \times B\right)$ solving

Infinite dimensional PDE

$$\begin{cases} \partial_t u(t,\eta) + '' \int_{-t}^{0} D^{ac} u\left(t,\eta\right) d\eta\, '' + \frac{1}{2}\langle D^2 u\left(t,\eta\right),\ \mathbb{1}_D\rangle = 0 \\ u(T,\eta) = H(\eta) \end{cases} \tag{3}$$

where

- $\mathbb{1}_D(x,y) := \begin{cases} 1 & \text{if } x = y,\ x, y \in [-T, 0] \\ 0 & \text{otherwise} \end{cases}$
- $D^{ac} u\left(t,\eta\right)$ absolute continuous part of measure $Du\left(t,\eta\right)$
- If $x \mapsto D_x^{ac} u\left(t,\eta\right)$ has bounded variation, previous integral is defined by an integration by parts.

Representation of $h = H(X_T(\cdot))$

Then

$$h = H_0 + \int_0^T \xi_s d^- X_s \tag{4}$$

with

- $H_0 = u(0, X_0(\cdot))$
- $\xi_s = D^{\delta_0} u(s, X_s(\cdot))$

A general representation theorem

Theorem

- $H : B \longrightarrow \mathbb{R}$
- $u \in C^{1,2}\left([0, T[\times B\right) \cap C^{0}\left([0, T] \times B\right)$
- $x \mapsto D^{ac}_x u\,(t, \eta)$ has bounded variation
- $D^2 u\,(t, \eta) \in (\mathcal{D}_0 \oplus L^2)\hat{\otimes}^2_h$
- u solves

$$\begin{cases} \partial_t u(t,\eta) + \displaystyle\int_{]-t,0]} D^{ac} u(t,\eta)\, d\eta + \frac{1}{2} D^2 u\,(t,\eta)(\{0,0\}) = 0 \\ u(T,\eta) = H(\eta) \end{cases} \tag{5}$$

then h has representation (4).

The proof follows immediately applying the Itô's formula.

Sufficient conditions to solve (5)

1. When X general process such that $[X]_t = t$.
   - H has a smooth Fréchet dependence on $L^2([-T, 0])$.
   - $h := H(X_T(\cdot)) = f\left(\int_0^T \varphi_1(s) d^- X_s, \ldots, \int_0^T \varphi_n(s) d^- X_s\right)$,
     - $f : \mathbb{R}^n \to \mathbb{R}$ measurable and with linear growth
     - $(\varphi_i) \in C^2([0, T]; \mathbb{R})$
2. When $X = W$ if Clark-Ocone formula does not apply.
   For instance when $h \notin \mathbb{D}^{1,2}$, or $h \notin L^2(\Omega)$ (even not in $L^1(\Omega)$).

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Thank you!!!

A stochastic flow

Definition

For $0 < s < t < T$ and $\eta \in B$ the **stochastic flow** is defined

$$Y_t^{s,\eta}(x) = \begin{cases} \eta(x+t-s) & x \in [-T, s-t] \\ \eta(0) + W_t(x) - W_s & x \in [s-t, 0] \end{cases}$$

where W standard Brownian motion.

Remark

- $(Y_t^{s,\eta})_{0 \leq s \leq t \leq T,\, \eta \in B}$ is a B-valued random field
- $$Y_r^{s,\eta} = Y_r^{t, Y_t^{s,\eta}} \quad \text{for} \quad 0 < s < t < r < T$$

Theorem

Let $H : L^2([-T,0]) \longrightarrow \mathbb{R}$

- $H \in C^3(L^2[-T,0])$ with $D^2H \in L^2([-T,0]^2)$ and D^3H polynomial growth
- $DH(\eta) \in H^1([-T,0])$ and other technical assumptions

$$u(t,\eta) := \mathbb{E}\left[H(Y_T^{t,\eta})\right]$$

Then

- $u \in C^{1,2}([0,T] \times B)$
- u solves (5)

Theorem

Let

- $$H(\eta) := f\left(\int_{[-T,0]} \varphi_1(u+T)d\eta(u), \ldots, \int_{[-T,0]} \varphi_n(u+T)d\eta(u)\right)$$
- $f : \mathbb{R}^n \to \mathbb{R}$ continuous and with linear growth and
- $(\varphi_i) \in C^2([0,T];\mathbb{R})$
- Matrix $\Sigma_t := (\Sigma_t)_{i,j} = \left(\int_t^T \varphi_i(s)\varphi_j(s)ds\right)$, $t \in [0,T]$.

$$\det(\Sigma_t) > 0 \qquad \forall\, t \in]0,T[$$

Remark

Σ_t is the Covariance matrix of Gaussian vector $G := \left(\int_t^T \varphi_1(s)dW_s, \ldots, \int_t^T \varphi_n(s)dW_s\right)$

Theorem

$$u(t,\eta) := \Psi\left(t, \int_{[-t,0]} \varphi_1(s+t)d\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t)d\eta(s)\right)$$

with

$$\Psi(t, y_1, \dots, y_n) = \int_{\mathbb{R}^n} f(z_1, \dots, z_n)\, p(t, z_1-y_1, \dots, z_n-y_n) dz_1 \cdots dz_n$$

and $p \in C^{3,\infty}([0,T] \times \mathbb{R}^n)$ density of Gaussian vector G

Then

- $u \in C^{1,2}([0,T[\times B) \cap C^0([0,T] \times C([-T,0]))$
- u solves (5)

Remark

If $X = W$ an analougous result is true with a weaker condition on f

Let

- f polynomial growth

Then

- $u \in C^{1,2}([0,T[\times B)$
- $$h = u(0, W_0(\cdot)) + \underbrace{\int_0^T D^{\delta_0} u(s, W_s(\cdot)) d^- W_s}_{\text{improper forward integral}}$$
- $u(0, W_0(\cdot)) = \mathbb{E}[h]$
- f Lipschitz then $D^{\delta_0} u(s, W_s(\cdot)) = \mathbb{E}\left[D_s^m h | \mathcal{F}_t\right]$ since $h \in \mathbb{D}^{1,2}$

Theorem

$H : B \longrightarrow \mathbb{R}$

- $$H(\eta) = f\left(\int_{-T}^{0} \eta(s)ds\right)$$
- $f : \mathbb{R} \longrightarrow \mathbb{R}$ Borel subexponential (not necessarily continuous)
- $h = f\left(\int_0^T W_s ds\right) \in L^1(\Omega)$

$$u(t,\eta) = \int_{\mathbb{R}} f\left(\int_{-T}^{0} \eta(r)dr + \eta(0)(T-t) + x\right) p_\sigma(t,x)dx$$

with $\sigma_t = \sqrt{\frac{(T-t)^3}{3}}$

Theorem

Then

- $u \in C^{1,2}([0,T[\times B)$
-

$$h = u(0, W_0(\cdot)) + \underbrace{\int_0^T D^{\delta_0} u(s, W_s(\cdot)) d^- W_s}_{\text{improper forward integral}}$$

- $u(0, W_0(\cdot)) = \mathbb{E}[h]$

Remark

Since $h \notin L^2(\Omega)$, a priori neither Clark-Ocone formula nor its extensions to Wiener distributions apply

A toy model for X real valued

Let X such that $[X]_t = t$

A1 $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous and polynomial growth

A2 $v \in C^{1,2}([0,T[\times\mathbb{R}) \cap C^0([0,T] \times \mathbb{R})$ such that

$$\begin{cases} \partial_t v(t,x) + \frac{1}{2}\partial_{xx} v(t,x) = 0 \\ v(T,x) = f(x) \end{cases}$$

Then

$$h := f(X_T) = v(0, X_0) + \underbrace{\int_0^T \partial_x v(s, X_s) d^- X_s}_{\text{improper forward integral}}$$

Schoenmakers-Kloeden (1999), Coviello-Russo (2006), Bender-Sottinen-Valkeila (2008)

Considerations about previous representation in toy model

- If $X_t = W_t + t\,G$, G non-negative r.v. $\notin L^1(\Omega)$ and $f(x) = x$ then $h = f(X_T) \notin L^1(\Omega)$.
- If $X = W$,
  1. A1 $\Longrightarrow h = f(W_T) \in L^p(\Omega)$, with $p \geq 1$. not new...but...
  2. $\begin{cases} f \text{ subexponential} \\ f(W_T) \in L^1(\Omega) \end{cases} \Longrightarrow$

$$h := f(W_T) = v(0, W_0) + \underbrace{\int_0^T \partial_x v(t, W_t) d^- W_s}_{\text{improper forward integral}}$$

Remark

f not necessarily continuous, $v \notin C^0([0, T] \times R)$

A first motivating example

1)

$$H(\eta) = \left(\int_{-T}^{0} \eta(s) ds \right)^2$$

$$u(t,\eta) := \left(\int_{-T}^{0} \eta(s) ds + \eta(0)(T-t) \right)^2 + \frac{(T-t)^3}{3}$$

solves (3) and h has representation (4).

$$\partial_t u(t,\eta) = -2\eta(0)\left(\int_{-T}^{0}\eta(s)ds + \eta(0)(T-t)\right) - (T-t)^2$$

$$D_{dx}u(t,\eta) = 2\left(\int_{-T}^{0}\eta(s)ds + \eta(0)(T-t)\right)\cdot$$
$$\cdot\left(\mathbb{1}_{[-T,0]}(x)dx + (T-t)\delta_0(dx)\right)$$

$$\begin{aligned} D^2_{dx\,dy}\phi(t,\eta) &= 2\mathbb{1}_{[-T,0]^2}(x,y)dx\,dy+ \\ &+ 2(T-t)\mathbb{1}_{[-T,0]}(x)dx\,\delta_0(dy)+ \\ &+ 2(T-t)\delta_0(dx)\,\mathbb{1}_{[-T,0]}(y)dy+ \\ &+ 2(T-t)^2\delta_0(dx)\,\delta_0(dy) \end{aligned}$$

$D^2u(t,\eta) \in (\mathcal{D}_0 \oplus L^2)\hat{\otimes}_h^2$ and $[X]_t = t$

Remark

If $X = W$

- Forward integral equals Itô integral
- The representation coincides with Clark-Ocone formula
- $H_0 = \mathbb{E}[h]$.

An interesting case

2)

$$H(\eta) = \int_{-T}^{0} \eta(s)^2 ds$$

$$u(t,\eta) := \int_{-T}^{0} \eta^2(s) ds + \eta(0)^2 (T-t) + \frac{(T-t)^2}{2}$$

solves (3) and h has representation (4).

$$\partial_t u(t,\eta) = -\eta^2(0) - (T-t)\,;$$
$$D_{dx} u(t,\eta) = 2\eta(x)dx + 2\eta(0)(T-t)\,\delta_0(dx)$$
$$D^2_{dx\,dy}\phi(t,\eta) = 2\delta_y(dx)\,dy + 2(T-t)\delta_0(dx)\delta_0(dy) = 2\delta_x(dy)\,dx + 2(T$$

- $D^2 u(t,\eta) \in (Diag \oplus \mathcal{D}_{0,0})$ and $[X]_t = t$
- D^{ac} is not of bounded variation

Stability result for $\mathbb{R}^n$ valued processes

In the finite dimensional case it holds.

Theorem

Let X be a $\mathbb{R}^n$-valued process having all its mutual covariations $([X^*, X]_t)_{1 \leq i,j \leq n} = [X^i, X^j]_t$ and F, $G \in C^1(\mathbb{R}^n)$. Then the covariation $[F(X), G(X)]$ exists and is given by

$$[F(X), G(X)]_\cdot = \sum_{i,j=1}^{n} \int_0^\cdot \partial_i F(X) \partial_j G(X) d[X^i, X^j]$$

Setting $n = 2$, $F(x, y) = f(x)$, $G(x, y) = g(y)$, $f, g \in C^1(\mathbb{R})$ we have:

$$[f(X), g(Y)]_\cdot = \int_0^\cdot f'(X) g'(Y) d[X, Y]$$

Stability result for B-valued processes

Previous results admit some generalizations in the infinite dimensional framework.

Theorem

Let X be a B-valued continuous stochastic process admitting a χ-quadratic variation.
Let $F^i, F^j : B \longrightarrow \mathbb{R}$ be C^1 Fréchet such that for $i, j = 1, 2$

$$DF^i(\cdot) \otimes DF^j(\cdot) : B \times B \longrightarrow \chi \subset (B\hat{\otimes}_\pi B)^*$$
$$(x, y) \mapsto DF^i(x) \otimes DF^j(y) \quad \text{continuous}$$

Then $[F^i(X), F^j(X)]$ exists and it is given by

$$[F^i(X), F^j(X)]_\cdot = \int_0^\cdot \langle DF^i(X_s) \otimes DF^j(X_s), d\widetilde{[X]}_s \rangle$$

Stability results involving window Dirichlet processes

Let D a real continuous $(\mathcal{F}_t)-$Dirichlet process,

$$D = M + A,$$

- D a real continuous $(\mathcal{F}_t)-$Dirichlet process, $D = M + A$,
- M an $(\mathcal{F}_t)-$local martingale
- A a zero quadratic variation process with $A_0 = 0$.

Time-homogeneous Stability Theorem

Theorem

Let

- $F : B \longrightarrow \mathbb{R}$ be C^1 Fréchet
- $DF : B \longrightarrow \mathcal{D}_0 \oplus L^2$ continuously

Then $F(D(\cdot))$ is an $(\mathcal{F}_t)$-Dirichlet process with local martingale component equal to

$$\tilde{M}_\cdot = F\big(D_0(\cdot)\big) + \int_0^\cdot D^{\delta_0} F\big(D_s(\cdot)\big)\, dM_s$$

where $D^{\delta_0} F(\eta) := DF(\eta)(\{0\})$.

Stability results involving window weak Dirichlet processes

- D a finite quadratic variation $(\mathcal{F}_t)-$weak Dirichlet process

$$D = M + A$$

- M is the local martingale

Stability Theorem

Theorem

Let

- $F : [0, T] \times B \longrightarrow \mathbb{R}$ be $C^{0,1}$ Fréchet such that
- $DF : [0, T] \times B \longrightarrow \mathcal{D}_0 \oplus L^2$ continuously

Then $F(\cdot, D_\cdot(\cdot))$ is an $(\mathcal{F}_t)$-weak Dirichlet process with martingale part

$$\tilde{M}_t^F = F(0, D_0(\cdot)) + \int_0^t D^{\delta_0} F(s, D_s(\cdot)) dM_s \; .$$