Option Pricing and the Cost of Risk, via capital reserve and convex risk measures

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Outline

Pricing and hedging in an incomplete market

In an incomplete market a perfect hedge is not possible.
There are risks which cannot be hedged by continuous trading.

Questions

- How can we price and hedge derivatives in an incomplete market?
- How should we handle the residual risk?

The market model

Let the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be given where $T > 0$ denotes a fixed time horizon. The discounted price process is described as a $\mathbb{R}$-valued semimartingale $S = (S_t)_{t\in[0,T]}$ additional we have a set of trading strategies given by $\Pi(x)$ and a derivative $F \in \mathcal{F}_T$ which we want to price and hedge.

Pricing and hedging (x, π):

- Initial capital x.
- Trading strategy $\pi \in \Pi(x)$, such that the value of our portfolio at time T is

$$X_T^{\pi,x} := x + \int_0^T \pi_t dS_t.$$

Different pricing methods in incomplete markets

Some methods are:

- Superhedging: $\mathbb{P}(X_T^{\pi,x} \geq F)$
- Mean-variance optimal: $\mathbb{E}_{\mathbb{P}}|X_T^{\pi,x} - F|^2$
- Utility indifference pricing:
  $u(x,F) := \sup_{\pi\in\Pi(x)} \mathbb{E}_{\mathbb{P}}[U(X_T^{\pi,x} + F)]$

  $$\begin{array}{ll} \text{Buyers indifferent price:} & p\colon u(x,0) = u(x-p,F) \\ \text{Sellers indifferent price:} & s\colon u(x,0) = u(x+s,-F) \end{array}$$
- Minimization of risk:

  $$\begin{array}{ll} \text{Buyer:} & \inf_{\pi\in\Pi(x)} \rho(F - X_T^{\pi,x}) \\ \text{Seller:} & \inf_{\pi\in\Pi(x)} \rho(X_T^{\pi,x} - F) \end{array}$$
- ...

Trader and regulator

Model pricing and hedging of a derivative as a trade-off between trader and regulator.

- The regulator requires the traders to cover the *residual risk* by an additional bank account Z, which earns a smaller rate of return than the standard deposit bank account. The additional bank account serves as a *capital reserve* and contains the minimal amount of money which depends on the risk of the trader's portfolio.
- The trader knows the risk measure of the regulator and tries to minimize the price.

Therefore, pricing an option consist of two parts: the cost of a hedging strategy that reduces the risk and capital reserve.

Market model with capital reserve

Pricing hedging with a capital reserve (discounted):

- Capital reserve: $dZ_t = \tilde{r} Z_t \, dt$ with $\tilde{r} < 0$.
- Portfolio:

$$\begin{aligned} Y_T^{\pi,\theta,x} :=& x + \int_0^T \pi_t \, dS_t + \int_0^T \theta_t \, dZ_t \\ =& X_T^{\pi,x} + \int_0^T \theta_t \, dZ_t . \end{aligned}$$

Here θ_t represents the wealth invested into the capital reserve at time t.

Risk measure as a capital reserve

- The trader wants to price the derivative F.
- The regulator requires that the trader covers his hedging error at time 0. The capital reserve is modeled via a risk measure.

$$\theta := \rho(X_T^{\pi,x} - F)$$

θ is constant over time.

Capital reserve

Two step optimization to price a derivative F:

- Optimal hedging strategy π which minimizes the total risk for a given x.
- Trader wishes to minimize the price of the derivative.

The price of the derivative F is given by:

$$\inf_{x \in \mathbb{R}^+} \left\{ x + \inf_{\pi \in \Pi(x)} \rho(X_T^{\pi,x} - F) \cdot (1 - e^{\tilde{r}T}) \right\}.$$

First step

Optimal hedging strategy: Minimization of the total risk

For a given x:

$$\inf_{\pi \in \Pi(x)} \rho(X_T^{\pi,x} - F).$$

Done by *Toussaint, Sircar* (2009) for $X_T^{\pi,x}, F \in L^2$.

Convex risk measures: First approach

Artzner, Delbaen, Eber, Heath (1999) for coherent / convex risk measures.

Definition

A convex risk measure is a mapping $\rho : L^\infty \to \mathbb{R}$ satisfying the following properties for all $X, Y \in L^\infty$:

- Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- Translation invariance: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.
- Convexity: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$, for $0 \leq \lambda \leq 1$.

Convex risk measures: First approach

Dual representation

Suppose $\rho : L^\infty \to \mathbb{R}$ is a convex risk measure and ρ has the Fatou property, i.e. for any bounded sequence (X_n) which converges $\mathbb{P}$-a.s. to some X, $\rho(X) \leq \liminf \rho(X_n)$, then ρ has the following dual representation

$$\rho(X) = \sup_{\mathbb{P}\in\mathcal{P}} \left\{ \mathbb{E}_\mathbb{P}[-X] - \alpha_\rho(\mathbb{P}) \right\}$$

Jouini, Schachermayer, Touzi (2006) proofed that convex risk measures on L^∞ which a law invariant have the Fatou property.

Shortcomings of L^∞

Bounded financial positions are neither ideal for hedging and general payoffs nor realistic

- Most models are unbounded (Black-Scholes, . . .)
- Call Options $F = (S_T - K)^+ \notin L^\infty$
- Buy-and-hold strategy $aS_T + b \notin L^\infty$
- Risk measures defined on L^∞ are always finite

Extend convex risk measures to L^p, $p \in [1, \infty]$ and allow the measured risk to have the value $+\infty$.

Convex risk measures on L^p-spaces

Biagini, Frittelli (2009).

Definition

A L^p-convex risk measure $p \in [0, \infty]$ is a mapping $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ satisfying the following properties:

- Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- Translation invariance: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.
- Convexity: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$, for $0 \leq \lambda \leq 1$.
- Normality: $\rho(0) = 0$.

Convex risk measures on L^p-spaces

Dual representation

Suppose $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ is a convex risk measure. Assume ρ is proper and lower semicontinuous w.r.t. $\|\cdot\|_p$, then ρ admits the following dual representation

$$\rho(X) = \sup_{\mathbb{P}\in\mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[-X] - \alpha_\rho(\mathbb{P}) \right\}$$

Inf-convolution

Barrieu, El Karoui (2005) for L^∞, *Toussaint, Sircar* (2009) for L^2, *Arai* (2010) for L^Φ.

Definition

Let ρ be a L^p-convex risk measure and ϕ a functional on $L^p \to \mathbb{R} \cup \{\infty\}$. We define the inf-convolution of ρ and ϕ as

$$\rho \Box \phi(X) := \inf_{Y \in L^p} \{\rho(X - Y) + \phi(Y)\} = \inf_{Y \in L^p} \{\rho(Y) + \phi(X - Y)\}$$

Inf-convolution

Dual representation

Suppose that ρ is a L^p-convex risk measure. Assume that ϕ is convex, proper and lower semi-continuous with $\mathrm{dom}(\phi) \subset L^p$, $-\mathrm{dom}(\rho) \cap \mathrm{dom}(\phi) \neq \emptyset$ and $\mathrm{dom}(\phi)$ is weakly compact.
Then the inf-convolution $\rho \Box \phi$ is a convex risk measure and admits the dual representation

$$\rho \Box \phi(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[-X] - \alpha_{\rho \Box \phi}(\mathbb{P}) \right\}$$

with penalty function

$$\alpha_{\rho \Box \phi}(\mathbb{P}) = \alpha_{\rho}(\mathbb{P}) + \alpha_{\phi}(\mathbb{P}),$$

Indicator function

Let C be a non-empty convex closed subset of L^p and ϕ be an indicator function of C, meaning

$$\phi(X) := \delta_C(X) = \begin{cases} 0, & \text{for } X \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then ϕ is a proper convex, lower semi-continuous functional.

The penalty function is given by the support function ψ on $-C$

$$\alpha_\phi(\mathbb{P}) = \psi_{-C}(\mathbb{P}) := \sup_{X \in -C} \mathbb{E}_{\mathbb{P}}[X].$$

We want to minimize: $\inf_{x \in \Pi(x)} \rho(X_T^{\pi,x} - F)$.

$$\delta_{\Pi(x)}(X_T^{\pi,x}) = \begin{cases} 0, & \text{if } \exists \pi \in \Pi(x) \text{ s.t. } x + \int_0^T \pi_t dS_t = X_T^{\pi,x}, \\ +\infty, & \text{otherwise.} \end{cases}$$

This can be written as a special case of an inf-convolution of ρ and the indicator function δ on the convex set $\Pi(x)$

$$\inf_{\pi \in \Pi(x)} \rho(X_T^{\pi,x} - F) = \inf_{X \in L^p} \left\{ \rho(X_T^{\pi,x} - F) + \delta_{\Pi(x)}(X_T^{\pi,x}) \right\}$$
$$= \rho \Box \delta_{-\Pi(x)}(-F).$$

Need that $\left\{X_T^{\pi,x}, \pi \in \Pi(x)\right\}$ is closed and convex for the L^p norm.
Solution depends on the

Second step

The problem

$$\inf_{x\in\mathbb{R}^+}\left\{x+\inf_{\pi\in\Pi(x)}\rho(X_T^{\pi,x}-F)\cdot(1-e^{\tilde{r}T})\right\}.$$

- Translation invariance of the risk measure should help.
- Depends on the set of hedging strategies $\Pi(x)$.

Concluding remarks

Next steps:

- Get some results!
- Dynamic formulation like *Schweizer, Klöppel (2007)* for indifference pricing. In this case use the superhedging portfolio as a benchmark $F_t := \operatorname*{ess.sup}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[F|\mathcal{F}_t]$ for a European style derivative and for an American style derivative with payoff F_t at t.

$$\inf_{x\in\mathbb{R}} \left\{ x + \inf_{\pi\in\Pi(x)} \int_0^T \rho(X_t^{\pi,x} - F_t) dZ_t \right\}$$