Option Pricing and the Cost of Risk, via capital reserve and convex risk measures

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Outline

1. Introduction
2. Capital reserve model
3. Risk measures and Inf-convolution
In an incomplete market a perfect hedge is not possible. There are risks which cannot be hedged by continuous trading.

Questions

- How can we price and hedge derivatives in an incomplete market?
- How should we handle the residual risk?
Let the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})\) be given where \(T > 0\) denotes a fixed time horizon. The discounted price process is described as a \(\mathbb{R}\)-valued semimartingale \(S = (S_t)_{t \in [0,T]}\) additional we have a set of trading strategies given by \(\Pi(x)\) and a derivative \(F \in \mathcal{F}_T\) which we want to price and hedge.

**Pricing and hedging \((x, \pi)\):**

- Initial capital \(x\).
- Trading strategy \(\pi \in \Pi(x)\), such that the value of our portfolio at time \(T\) is

\[
X_T^{\pi,x} := x + \int_0^T \pi_t dS_t.
\]
Some methods are:

- **Superhedging:** $\mathbb{P}(X_T^\pi,x \geq F)$
- **Mean-variance optimal:** $\mathbb{E}_\mathbb{P}|X_T^\pi,x - F|^2$
- **Utility indifference pricing:**
  
  \[ u(x, F) := \sup_{\pi \in \Pi(x)} \mathbb{E}_\mathbb{P}[U(X_T^\pi,x + F)] \]
  
  Buyers indifferent price: $p: u(x, 0) = u(x - p, F)$
  Sellers indifferent price: $s: u(x, 0) = u(x + s, -F)$

- **Minimization of risk:**
  
  **Buyer:** $\inf_{\pi \in \Pi(x)} \rho(F - X_T^\pi,x)$
  **Seller:** $\inf_{\pi \in \Pi(x)} \rho(X_T^\pi,x - F)$

- ...
Model pricing and hedging of a derivative as a trade-off between trader and regulator.

- The regulator requires the traders to cover the residual risk by an additional bank account $Z$, which earns a smaller rate of return than the standard deposit bank account. The additional bank account serves as a capital reserve and contains the minimal amount of money which depends on the risk of the trader’s portfolio.
- The trader knows the risk measure of the regulator and tries to minimize the price.

Therefore, pricing an option consist of two parts: the cost of a hedging strategy that reduces the risk and capital reserve.
Market model with capital reserve

Pricing hedging with a capital reserve (discounted):

- Capital reserve: $dZ_t = \tilde{r}Z_t \, dt$ with $\tilde{r} < 0$.
- Portfolio:

\[
Y_T^{\pi,\theta,x} := x + \int_0^T \pi_t \, dS_t + \int_0^T \theta_t \, dZ_t
\]

\[
= X_T^{\pi,x} + \int_0^T \theta_t \, dZ_t.
\]

Here $\theta_t$ represents the wealth invested into the capital reserve at time $t$. 
The trader wants to price the derivative \( F \).

The regulator requires that the trader covers his hedging error at time 0. The capital reserve is modeled via a risk measure.

\[
\theta := \rho(X_T^{\pi, x} - F)
\]

\( \theta \) is constant over time.
Two step optimization to price a derivative $F$:

- Optimal hedging strategy $\pi$ which minimizes the total risk for a given $x$.
- Trader wishes to minimize the price of the derivative.

The price of the derivative $F$ is given by:

$$\inf_{x \in \mathbb{R}^+} \left\{ x + \inf_{\pi \in \Pi(x)} \rho(X_T^{\pi,x} - F) \cdot (1 - e^{\tilde{r}T}) \right\}.$$
First step

Optimal hedging strategy: Minimization of the total risk

For a given $x$:

$$\inf_{\pi \in \Pi(x)} \rho(X_T^{\pi,x} - F).$$

Done by *Toussaint, Sircar* (2009) for $X_T^{\pi,x}, F \in L^2$. 

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**Introduction**

**Capital reserve model**

**Risk measures and Inf-convolution**
Artzner, Delbaen, Eber, Heath (1999) for coherent / convex risk measures.

**Definition**

A convex risk measure is a mapping $\rho : L^\infty \to \mathbb{R}$ satisfying the following properties for all $X, Y \in L^\infty$:

- **Monotonicity**: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- **Translation invariance**: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.
- **Convexity**: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$, for $0 \leq \lambda \leq 1$. 
Convex risk measures: First approach

**Dual representation**

Suppose $\rho : L^\infty \to \mathbb{R}$ is a convex risk measure and $\rho$ has the Fatou property, i.e. for any bounded sequence $(X_n)$ which converges $\mathbb{P}$-a.s. to some $X$, $\rho(X) \leq \lim\inf \rho(X_n)$, then $\rho$ has the following dual representation

$$
\rho(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_\mathbb{P}[-X] - \alpha_\rho(\mathbb{P}) \right\}
$$

_Jouini, Schachermayer, Touzi (2006)_ proofed that convex risk measures on $L^\infty$ which a law invariant have the Fatou property.
Shortcomings of $L^\infty$

Bounded financial positions are neither ideal for hedging and general payoffs nor realistic

- Most models are unbounded (Black-Scholes, ...)
- Call Options $F = (S_T - K)^+ \notin L^\infty$
- Buy-and-hold strategy $aS_T + b \notin L^\infty$
- Risk measures defined on $L^\infty$ are always finite

Extend convex risk measures to $L^p$, $p \in [1, \infty]$ and allow the measured risk to have the value $+\infty$. 
Convex risk measures on $L^p$-spaces

*Biagini, Frittelli (2009).*

**Definition**

A $L^p$-convex risk measure $p \in [0, \infty]$ is a mapping $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ satisfying the following properties:

- **Monotonicity:** If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- **Translation invariance:** If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.
- **Convexity:** $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$, for $0 \leq \lambda \leq 1$.
- **Normality:** $\rho(0) = 0$. 
Suppose $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ is a convex risk measure. Assume $\rho$ is proper and lower semicontinuous w.r.t. $\| \cdot \|_p$, then $\rho$ admits the following dual representation

$$
\rho(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_\mathbb{P}[-X] - \alpha_{\rho}(\mathbb{P}) \right\}
$$

**Definition**

Let $\rho$ be a $L^p$-convex risk measure and $\phi$ a functional on $L^p \to \mathbb{R} \cup \{\infty\}$. We define the inf-convolution of $\rho$ and $\phi$ as

$$\rho \square \phi(X) := \inf_{Y \in L^p} \{\rho(X - Y) + \phi(Y)\} = \inf_{Y \in L^p} \{\rho(Y) + \phi(X - Y)\}$$
Suppose that $\rho$ is a $L^p$-convex risk measure. Assume that $\phi$ is convex, proper and lower semi-continuous with $\text{dom}(\phi) \subset L^p$, $-\text{dom}(\rho) \cap \text{dom}(\phi) \neq \emptyset$ and $\text{dom}(\phi)$ is weakly compact. Then the inf-convolution $\rho \Box \phi$ is a convex risk measure and admits the dual representation

$$\rho \Box \phi(X) = \sup_{P \in \mathcal{P}} \left\{ \mathbb{E}_P[-X] - \alpha_{\rho \Box \phi}(P) \right\}$$

with penalty function

$$\alpha_{\rho \Box \phi}(P) = \alpha_{\rho}(P) + \alpha_{\phi}(P),$$
Let $C$ be a non-empty convex closed subset of $L^p$ and $\phi$ be an indicator function of $C$, meaning

$$\phi(X) := \delta_C(X) = \begin{cases} 0, & \text{for } X \in C, \\ +\infty, & \text{otherwise}. \end{cases}$$

Then $\phi$ is a proper convex, lower semi-continuous functional.

The penalty function is given by the support function $\psi$ on $-C$

$$\alpha_\phi(\mathbb{P}) = \psi_{-C}(\mathbb{P}) := \sup_{X \in -C} \mathbb{E}_\mathbb{P}[X].$$
We want to minimize: \( \inf_{x \in \Pi(x)} \rho(\pi_T^x, x - F) \).

\[
\delta_{\Pi(x)}(\pi_T^x) = \begin{cases} 
0, & \text{if } \exists \pi \in \Pi(x) \text{ s.t. } x + \int_0^T \pi_t dS_t = \pi_T^x , \\
+\infty, & \text{otherwise.}
\end{cases}
\]

This can be written as a special case of an inf-convolution of \( \rho \) and the indicator function \( \delta \) on the convex set \( \Pi(x) \)

\[
\inf_{\pi \in \Pi(x)} \rho(\pi_T^x, x - F) = \inf_{X \in L^p} \left\{ \rho(\pi_T^x, x - F) + \delta_{\Pi(x)}(\pi_T^x) \right\}
\]

\[
= \rho \Box -\delta_{\Pi(x)}(-F).
\]

Need that \( \{\pi_T^x, \pi \in \Pi(x)\} \) is closed and convex for the \( L^p \) norm. Solution depends on the
Second step

The problem

\[
\inf_{x \in \mathbb{R}^+} \left\{ x + \inf_{\pi \in \Pi(x)} \rho(X_T^{\pi,x} - F) \cdot (1 - e^{rT}) \right\}.
\]

- Translation invariance of the risk measure should help.
- Depends on the set of hedging strategies \( \Pi(x) \).
Concluding remarks

Next steps:

- Get some results!
- Dynamic formulation like Schweizer, Klöppel (2007) for indifference pricing. In this case use the superhedging portfolio as a benchmark $F_t := \text{ess.sup } \mathbb{E}_Q[F|\mathcal{F}_t]$ for a European style derivative and for an American style derivative with payoff $F_t$ at $t$.

$$\inf_{x \in \mathbb{R}} \left\{ x + \inf_{\pi \in \Pi(x)} \int_0^T \rho(X_t^\pi, x - F_t) dZ_t \right\}$$