Model and Calibration Risks for the Heston Model

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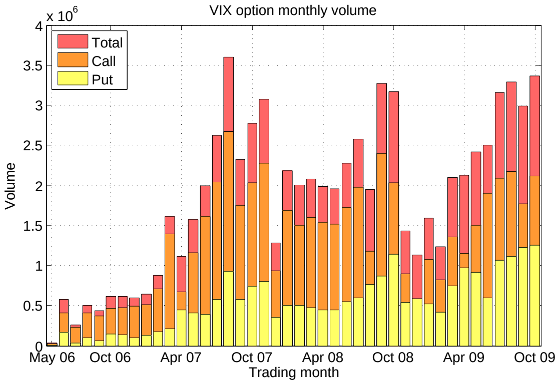
Motivation

- calibrated parameters: unstable and often unreasonable
- emergence of market data for vol derivatives $\Rightarrow$ fix beforehand v_0, η (from time series or market quotes) $\Rightarrow$ calibration on a reduced parameter set



Objectives

1. **market implied estimate** of the long run variance η
2. compare the different estimate of η and v_0
3. compare the **calibration performance** of the Heston model by using
   - a fully free parameter set $\{v_0, \kappa, \eta, \lambda, \rho\}$;
   - a reduced parameter set $\{\kappa, \lambda, \rho\}$, using market data to fix v_0 and η
4. **calibration risk** arising from the different calibration procedures and objective functions: pricing of **exotics**

The Heston model

- Stock price process:

$$\frac{dS_t}{S_t} = (r - q)dt + \sqrt{v_t}dW_t, \quad S_0 \geq 0$$

  Squared volatility process:

$$dv_t = \kappa\left(\eta - v_t\right)dt + \lambda\sqrt{v_t}d\tilde{W}_t, \quad v_0 = \sigma_0^2 \geq 0,$$

  where $W = \{W_t, t \geq 0\}$ and $\tilde{W} = \left\{\tilde{W}_t, t \geq 0\right\}$ are correlated standard BM such that $\mathrm{Cov}\left(dW_t, d\tilde{W}_t\right) = \rho dt$.

- Parameters involved:
  - $v_0 > 0$: initial variance
  - $\kappa > 0$: mean reversion rate
  - $\eta > 0$: long run variance
  - $\lambda > 0$: volatility of variance
  - ρ: correlation

Estimate of v_0

- v_0 = square of the spot price of the VIX index expressed in units:

$$v_0 = \left(\frac{\mathrm{VIX}(t_0)}{100} \right)^2 ;$$

Estimate of η

- **moving window** (MW) estimate:

$$\eta^{\mathrm{MW}} = \frac{1}{T^{\mathrm{VIX}}} \int_{-T^{\mathrm{VIX}}}^{0} \left(\frac{\mathrm{VIX}(t)}{100}\right)^2 dt = \mathrm{mean}_{-T^{\mathrm{VIX}} \le t \le 0} \left(\frac{\mathrm{VIX}(t)}{100}\right)^2 . \tag{1}$$

- **exponentially weighted moving average** (EWMA) estimate:

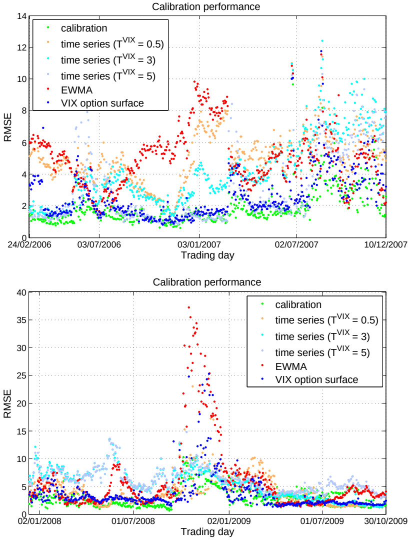
$$\eta^{\mathrm{EWMA}} = (1-\alpha) \sum_{i=1}^{N} \alpha^{N-i} \left(\frac{\mathrm{VIX}(t_i)}{100}\right)^2 \tag{2}$$

- **mark-to-market** (MTM) estimate:

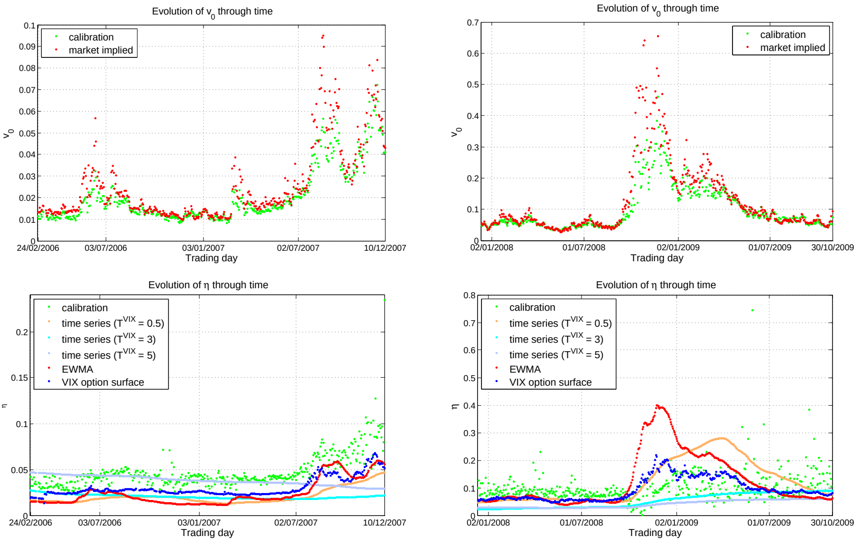
$$P(K,T) - C(K,T) = \exp(-rT)(K - \mathrm{VIX}_T)$$

$$\eta^{\mathrm{MTM}} = \left(\frac{K^{\mathrm{ATM}}}{100}\right)^2 = \left(\frac{\mathrm{LVIX}(t_0)}{100}\right)^2 . \tag{3}$$

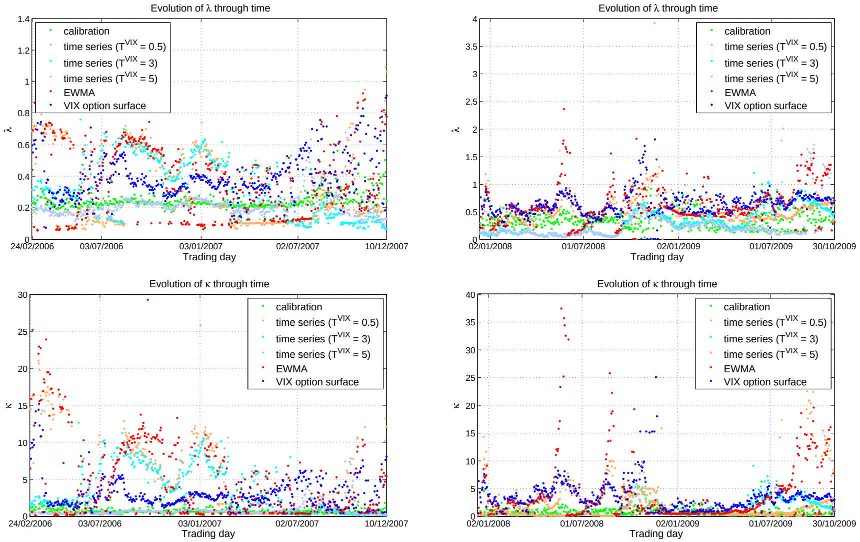
Evolution of the RMSE through time



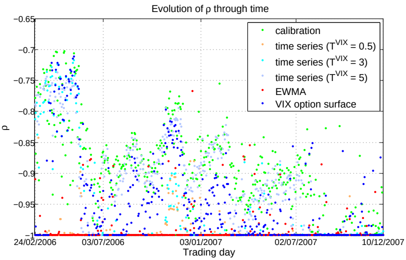
Evolution of the estimate of v_0 and η through time

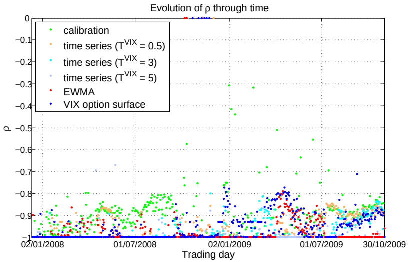


Evolution of the calibrated parameters through time



Evolution of the calibrated parameters through time (Cont.)





Objective functions

Objective functions:

- root mean squared error

$$\text{RMSE} = \sqrt{\sum_{j=1}^{N} \frac{(P_j - \hat{P}_j)^2}{N}}$$

- average absolute error as a percentage of the mean price

$$\text{APE} = \frac{1}{\text{mean}_j \hat{P}_j} \sum_{j=1}^{N} \frac{\left| P_j - \hat{P}_j \right|}{N}$$

- average relative error

$$\text{ARPE} = \frac{1}{N} \sum_{j=1}^{N} \frac{\left| P_j - \hat{P}_j \right|}{\hat{P}_j}.$$

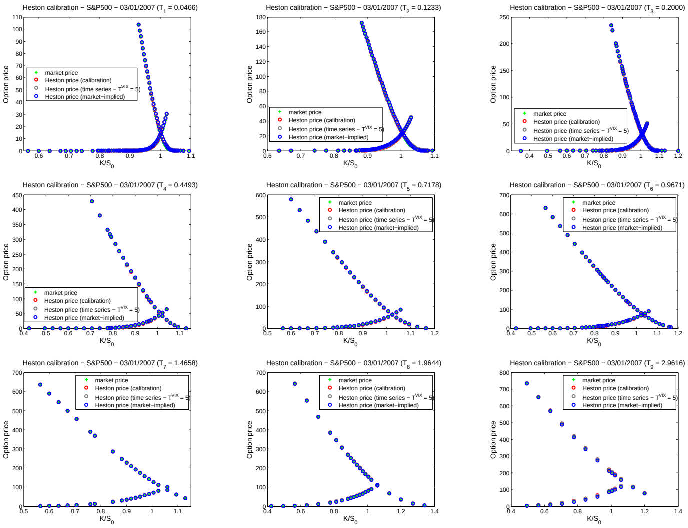
Daily variation of the model parameters

RMSE						
	calibration	MW ($T^{\text{VIX}} = 0.5$)	MW ($T^{\text{VIX}} = 3$)	MW ($T^{\text{VIX}} = 5$)	EWMA	market-implied
average precision	2.439650	4.502293	4.834824	4.654399	5.530451	3.099928
average Δv_0	0.006793	0.009460	0.009460	0.009460	0.009460	0.009460
average $\Delta \eta$	0.022588	0.000567	8.864555e-05	5.796916e-05	0.001152	0.001986
average $\Delta \lambda$	0.080715	0.082411	0.050275	0.045892	0.134313	0.098315
average $\Delta \kappa$	0.413670	1.264941	0.631513	0.570401	2.030938	1.179841
average $\Delta \rho$	0.052782	0.014224	0.012173	0.015022	0.017561	0.038741
ARPE						
	calibration	MW ($T^{\text{VIX}} = 0.5$)	MW ($T^{\text{VIX}} = 3$)	MW ($T^{\text{VIX}} = 5$)	EWMA	market-implied
average precision	0.235445	0.358760	0.359595	0.348677	0.324854	0.305351
average Δv_0	0.009716	0.009460	0.009460	0.009460	0.009460	0.009460
average $\Delta \eta$	0.044100	0.000567	8.864555e-05	5.796916e-05	0.001152	0.001986
average $\Delta \lambda$	0.111124	0.062548	0.049030	0.051594	0.079298	0.082063
average $\Delta \kappa$	0.684585	0.634506	0.578065	0.522025	0.686207	0.691249
average $\Delta \rho$	0.053700	0.035944	0.048639	0.038836	0.034663	0.038442
APE						
	calibration	MW ($T^{\text{VIX}} = 0.5$)	MW ($T^{\text{VIX}} = 3$)	MW ($T^{\text{VIX}} = 5$)	EWMA	market-implied
average precision	0.019201	0.032250	0.035712	0.034686	0.038651	0.022346
average Δv_0	0.007801	0.009460	0.009460	0.009460	0.009460	0.009460
average $\Delta \eta$	0.027242	0.000567	8.864555e-05	5.796916e-05	0.001152	0.001986
average $\Delta \lambda$	0.094447	0.078535	0.059901	0.049769	0.126614	0.101588
average $\Delta \kappa$	0.537634	1.198459	0.931575	0.708374	1.879946	1.139020
average $\Delta \rho$	0.060817	0.030710	0.021987	0.016877	0.037403	0.043229

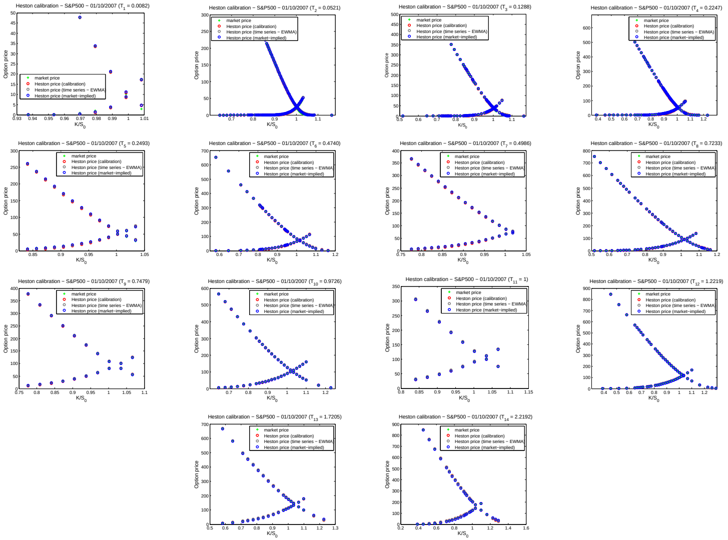
Objective function and calibration time

	calibration	MW ($T^{VIX} = 0.5$)	MW ($T^{VIX} = 3$)	MW ($T^{VIX} = 5$)	EWMA	market-implied
03/01/2007 : the low volatility regime period						
RMSE	1.297874	7.407939	4.555312	**1.268454**	8.600138	1.632820
ARPE	0.234821	0.391997	0.338918	**0.240736**	0.419969	0.288468
APE	0.013068	0.056765	0.034240	**0.012538**	0.065222	0.013605
01/10/2007 : the credit crisis period						
RMSE	2.510712	3.689891	6.782288	5.239480	**2.215681**	2.305956
ARPE	0.256275	0.287134	0.352642	0.308869	**0.242762**	0.270390
APE	0.014204	0.031981	0.047084	0.033527	**0.017981**	0.023710
21/11/2008 : the panic wave period						
RMSE	5.928921	9.583196	**7.720376**	8.268856	23.210741	25.230620
ARPE	0.296433	0.971876	**0.271187**	0.297219	0.611025	0.952419
APE	0.022584	0.046530	**0.034957**	0.037614	0.095117	0.122339
30/10/2009 : the end of the credit crisis						
RMSE	1.648008	3.847030	3.834033	4.033707	4.032063	**2.361227**
ARPE	0.265229	0.339171	0.260288	0.339502	0.339012	**0.262819**
APE	0.016606	0.026462	0.025823	0.028112	0.028055	**0.016775**
Calibration time						
RMSE	136.1719	40.8281	41.1719	22.6406	66.8281	46.1563
ARPE	140.2031	55.2656	53.9063	29.8125	32.9844	56.3906
APE	92.2969	36.1563	37.3438	22.9531	35.1563	35.8750

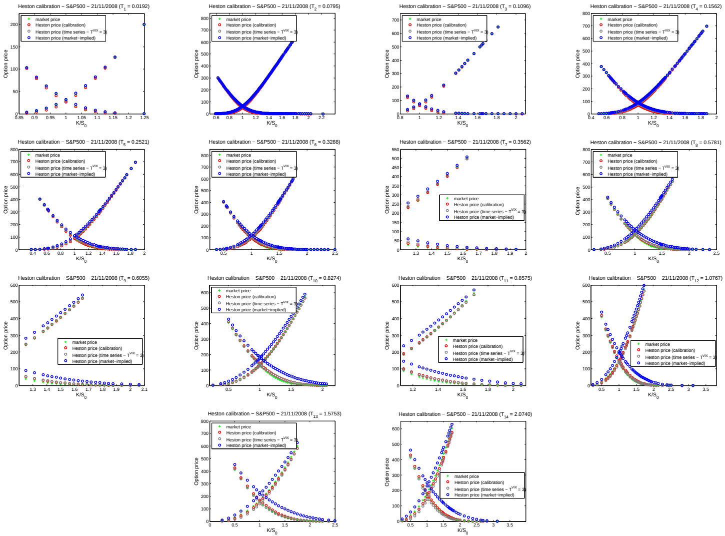
Option surface fit: 03/01/2007



Option surface fit: 01/10/2007



Option surface fit: 21/11/2008



Calibration risk

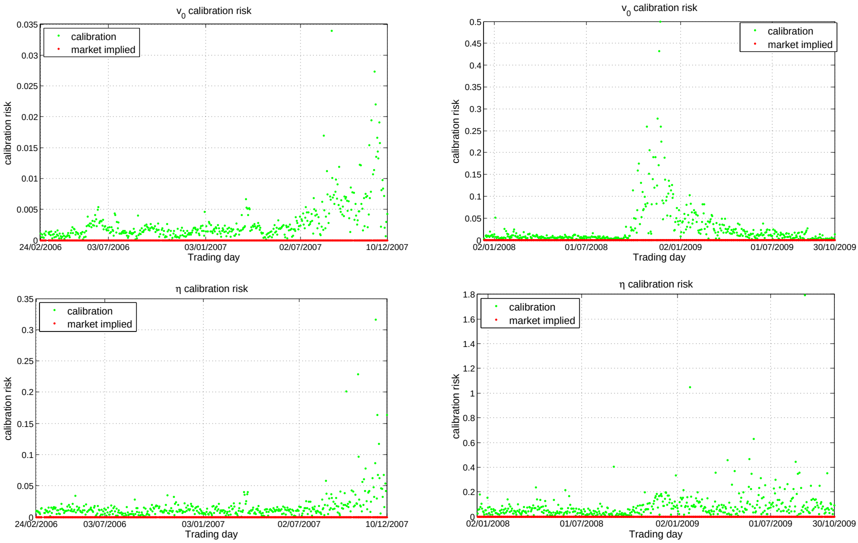
Calibration risk: difference in the value of the calibrated parameters arising from the different specifications of the objective function (Detlefsen and Hardle (2007)).

- maximum absolute value of the difference between the optimal parameter $p^* \in \{v_0^*, \kappa^*, \eta^*, \lambda^*, \rho^*\}$ obtained with the different objective functions:

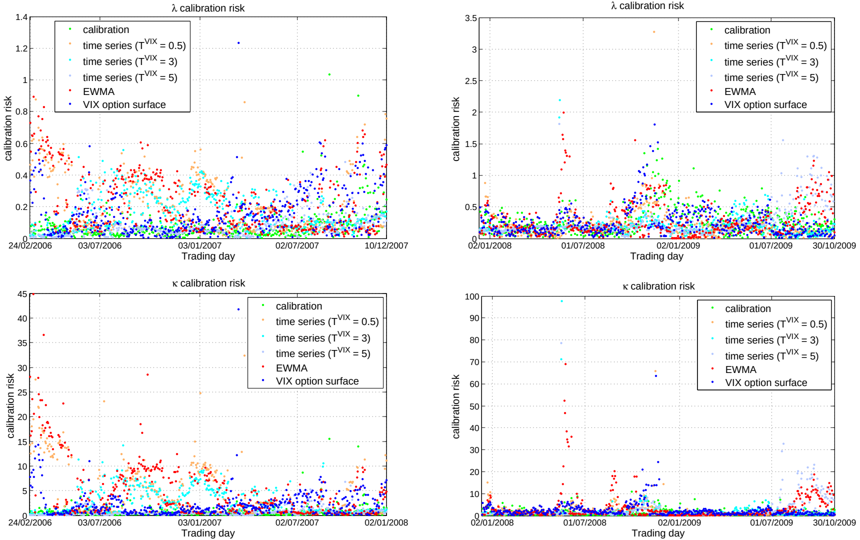
$$\max\left(|p^*_{\text{RMSE}} - p^*_{\text{APE}}|, |p^*_{\text{RMSE}} - p^*_{\text{ARPE}}|, |p^*_{\text{APE}} - p^*_{\text{ARPE}}|\right)$$

- Exotic prices

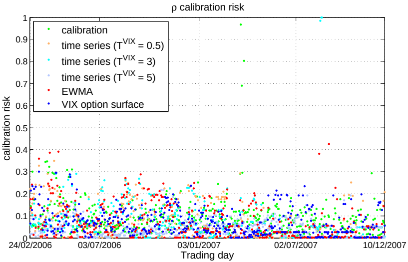
Calibration risk: v_0 and η

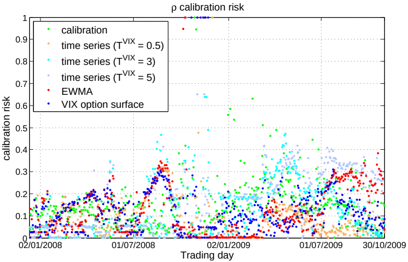


Calibration risk: λ and κ



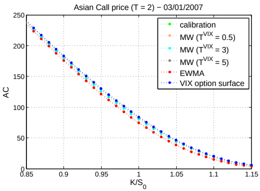
Calibration risk: ρ

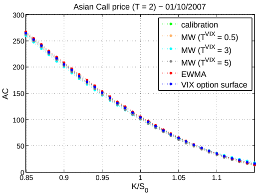


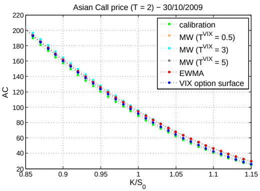


Asian options

$$AC = \exp(-rT)\mathbb{E}_{\mathbb{Q}}[(\text{mean}_{0\leq t\leq T} S_t - K)^+]$$







Lookback options

$$LC = \exp(-rT)\mathbb{E}_{\mathbb{Q}}[(S_T - m_T^S)^+],$$

$$m_t^X = \inf\{X_s, 0 \le s \le t\} \quad \text{and} \quad M_t^X = \sup\{X_s, 0 \le s \le t\}.$$

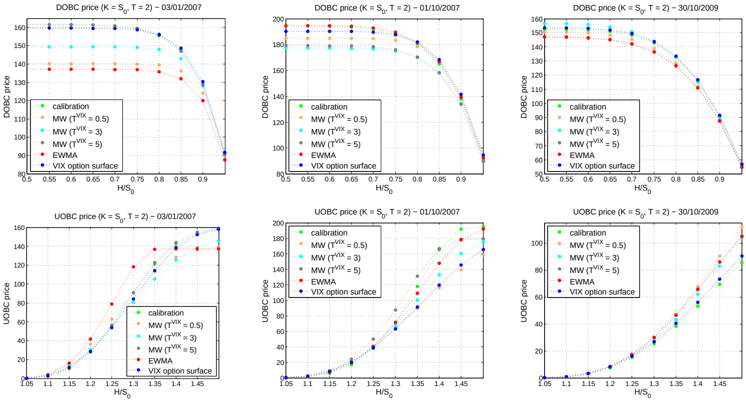
calibration	03/01/2007	01/10/2007	30/10/2009
calibration	270.0157	343.0431	307.3771
MW - $T^{\text{VIX}} = 0.5$	231.8865	328.3097	308.3829
MW - $T^{\text{VIX}} = 3$	249.7935	304.9651	320.9077
MW - $T^{\text{VIX}} = 5$	271.4342	316.5518	299.2174
EWMA	221.4847	349.5726	299.4474
VIX	272.2177	340.0185	312.2275

One-touch barrier option

- knock-out barrier

$$\text{DOBC} = \exp(-rT)\mathbb{E}_{\mathbb{Q}}[(S_T - K)^+ \mathbf{1}(m_T^S > H)]$$

$$\text{UOBC} = \exp(-rT)\mathbb{E}_{\mathbb{Q}}[(S_T - K)^+ \mathbf{1}(M_T^S < H)];$$

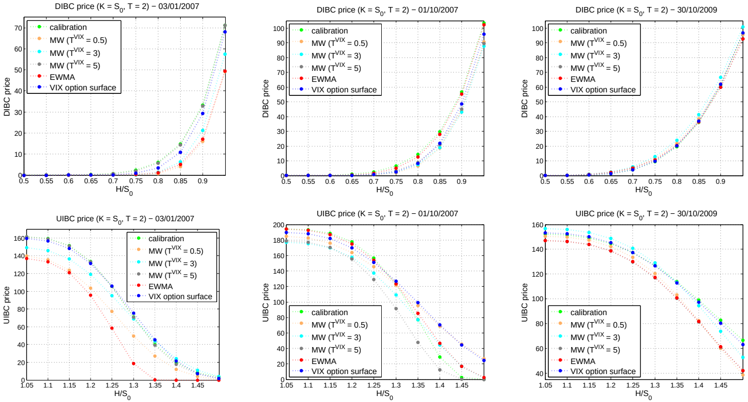


One-touch barrier option (Cont.)

- knock-in barrier

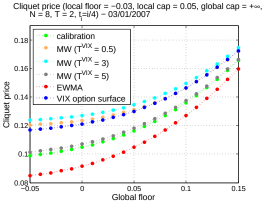
$$\text{DIBC} = \exp(-rT)\mathbb{E}_{\mathbb{Q}}[(S_T - K)^+ \mathbf{1}(m_T^S \leq H)]$$

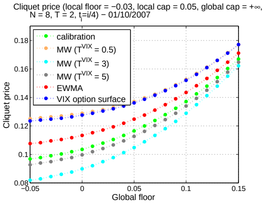
$$\text{UIBC} = \exp(-rT)\mathbb{E}_{\mathbb{Q}}[(S_T - K)^+ \mathbf{1}(M_T^S \geq H)].$$

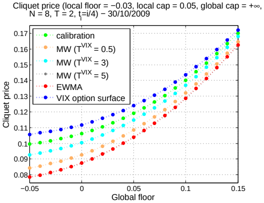


Cliquet option

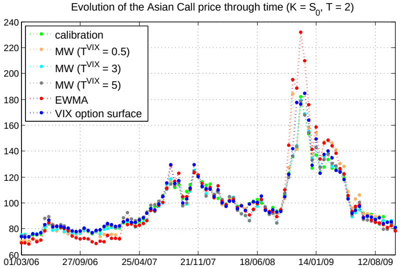
$$\text{Cliquet} = \exp(-rT)\mathbb{E}_{\mathbb{Q}}\left[\min\left(\text{cap}^G, \max\left(\text{floor}^G, \sum_{i=1}^{N}\min\left(\text{cap}^L, \max\left(\text{floor}^L, \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)\right)\right)\right)\right].$$

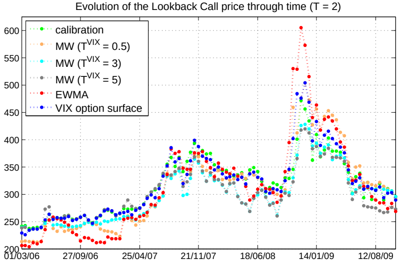


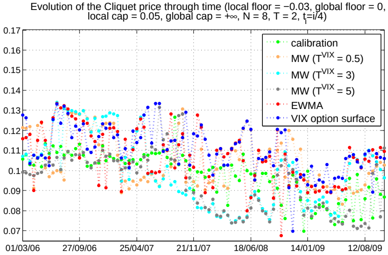




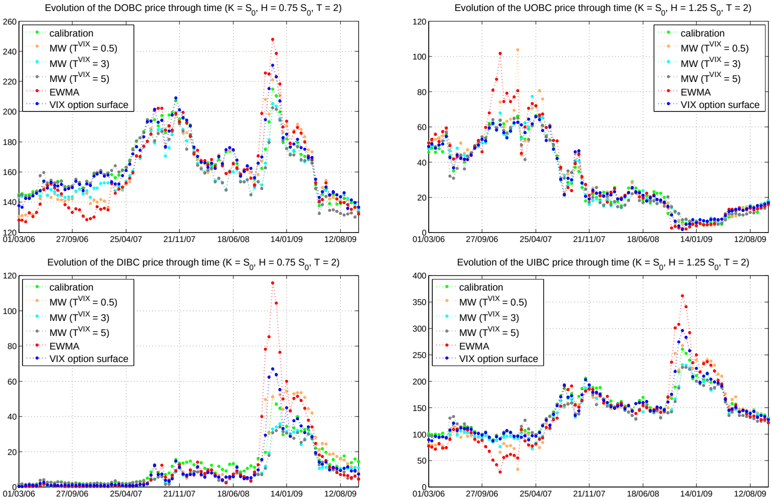
Calibration risk: the choice of the calibration procedure





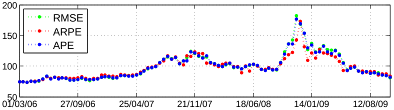


Calibration risk: the choice of the calibration procedure (Cont.)

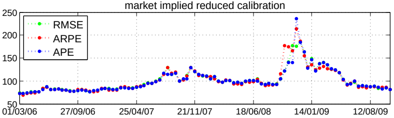


Calibration risk: the choice of the objective function

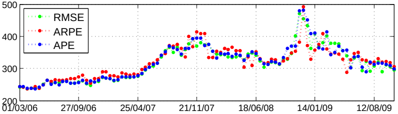
Evolution of the Asian Call price through time ($K = S_0$, $T = 2$) − full calibration



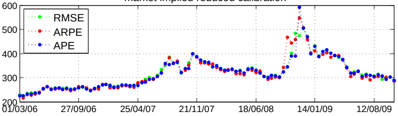
Evolution of the Asian Call price through time ($K = S_0$, $T = 2$) − market implied reduced calibration



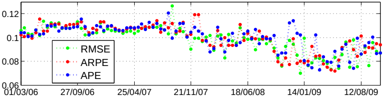
Evolution of the Lookback Call price through time ($T = 2$) − full calibration



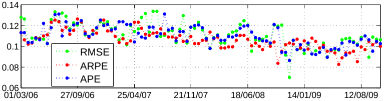
Evolution of the Lookback Call price through time ($T = 2$) − market implied reduced calibration



Evolution of the Cliquet price through time (local floor = −0.03, global floor = 0, local cap = 0.05, global cap = $+\infty$, $N = 8$, $T = 2$, $t_i = i/4$) − full calibration

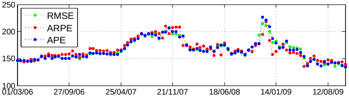


Evolution of the Cliquet price through time (local floor = −0.03, global floor = 0, local cap = 0.05, global cap = $+\infty$, $N = 8$, $T = 2$, $t_i = i/4$) − market implied reduced calibration

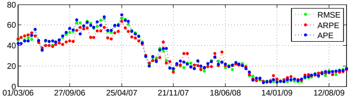


Calibration risk: the choice of the objective function (Cont.)

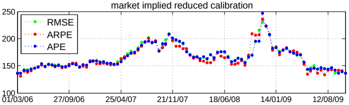
Evolution of the DOBC price through time ($K = S_0$, $H = 0.75\ S_0$, $T = 2$) − full calibration



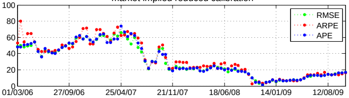
Evolution of the UOBC price through time ($K = S_0$, $H = 0.75\ S_0$, $T = 2$) − full calibration



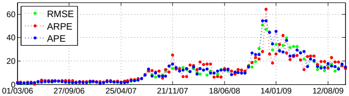
Evolution of the DOBC price through time ($K = S_0$, $H = 0.75\ S_0$, $T = 2$) − market implied reduced calibration



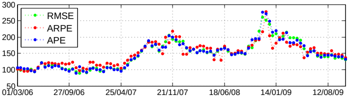
Evolution of the UOBC price through time ($K = S_0$, $H = 0.75\ S_0$, $T = 2$) − market implied reduced calibration



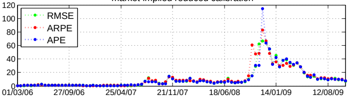
Evolution of the DIBC price through time ($K = S_0$, $H = 0.75\ S_0$, $T = 2$) − full calibration



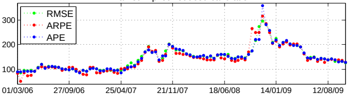
Evolution of the UIBC price through time ($K = S_0$, $H = 0.75\ S_0$, $T = 2$) − full calibration



Evolution of the DIBC price through time ($K = S_0$, $H = 0.75\ S_0$, $T = 2$) − market implied reduced calibration

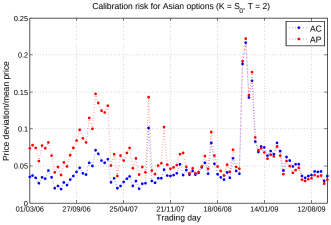


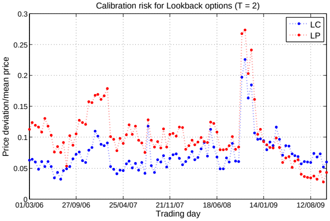
Evolution of the UIBC price through time ($K = S_0$, $H = 0.75\ S_0$, $T = 2$) − market implied reduced calibration

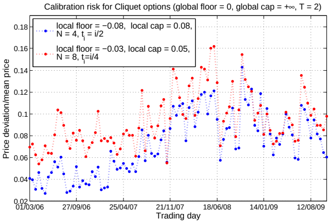


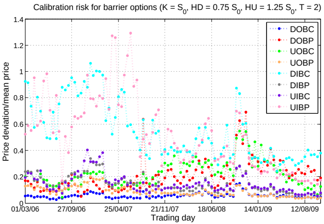
Global model risk

Global model risk measured by $\frac{\text{std(price)}}{\text{mean(price)}}$









Global model risk

calibration	AC	AP	LC	LP	Cliquet 1	Cliquet 2
	Calibration risk arising from the calibration procedure					
RMSE	0.033908	0.048530	0.060000	0.064051	0.085506	0.112690
APE	0.038819	0.054284	0.066131	0.080649	0.071326	0.095247
ARPE	0.066253	0.088669	0.096187	0.140164	0.047125	0.063327
	Calibration risk arising from the objective function					
calibration	0.020271	0.225636	0.520212	0.425640	1.727657	1.726586
MW ($T^{\text{VIX}} = 0.5$)	0.022378	0.252357	0.506769	0.380589	1.727545	1.726371
MW ($T^{\text{VIX}} = 3$)	0.037969	0.256160	0.492920	0.378976	1.727664	1.726532
MW ($T^{\text{VIX}} = 5$)	0.039331	0.232118	0.497650	0.399492	1.727786	1.726739
EWMA	0.014861	0.246158	0.506096	0.381877	1.727508	1.726304
market-implied	0.018896	0.229911	0.513259	0.407050	1.727500	1.726191
	Calibration risk arising from both the calibration procedure and the objective function					
ALL	0.049028	0.213804	0.434454	0.352613	1.453121	1.452149

Global model risk

calibration	DOBC	DOBP	UOBC	UOBP	DIBC	DIBP	UIBC	UIBP
	Calibration risk arising from the calibration procedure							
RMSE	0.039281	0.198357	0.135771	0.067489	0.524566	0.123583	0.087288	0.357493
APE	0.045615	0.185915	0.152006	0.075266	0.528947	0.130241	0.118739	0.465904
ARPE	0.077479	0.2089320	0.254173	0.115150	0.642981	0.182209	0.212589	0.688647
	Calibration risk arising from the objective function							
calibration	0.274359	1.243682	0.814589	0.134273	1.302822	0.199536	0.217512	1.318924
MW ($T^{VIX} = 0.5$)	0.256693	1.205416	0.747263	0.168450	1.374050	0.285221	0.207119	1.318154
MW ($T^{VIX} = 3$)	0.246357	1.153557	0.692015	0.149320	1.486095	0.242757	0.172117	1.427167
MW ($T^{VIX} = 5$)	0.251869	1.173680	0.731435	0.112777	1.470301	0.168577	0.181764	1.442246
EWMA	0.256144	1.196937	0.760032	0.167349	1.377761	0.284868	0.224092	1.306372
market-implied	0.266475	1.240836	0.780326	0.139903	1.390347	0.222046	0.187905	1.296268
	Calibration risk arising from both the calibration procedure and the objective function							
ALL	0.226264	1.011796	0.644223	0.147501	1.174243	0.227528	0.211867	1.135486

Conclusion

- market implied estimate of the long run variance of stochastic volatility models (Put-Call parity of long maturity options on the VIX)
- market implied estimate: same trend as calibrated parameter η but typically lower
- optimal reduced calibration procedure: market implied estimate of η (except during huge investors fear period)
- reduced calibration procedure: significant reduction of calibration computation time and calibration risk (except for κ)
- reduced calibration: more stable η and ρ through time, counterbalanced by lost of stability for κ.
- price of a wide range of exotic options: significantly $\neq$ under the 2 calibration settings (especially for cliquet and barrier options)
- Model risk (choice of calibration procedure) vs calibration risk (specifications of the objective function)
- Even within a particular model, model risk and calibration risk are present.

Thank you for your attention!